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Existence of resolvable H-designs with group sizes 2, 3, 4 and 6

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Abstract In 1987, Hartman showed that the necessary condition $v \equiv 4$ or 8 (mod 12) for the existence of a resolvable SQS(v) is also sufficient for all values of v, with 23 possible exceptions. These last 23 undecided orders were removed by Ji and Zhu in 2005 by introducing the concept of resolvable H-designs. In this paper, we first develop a simple but powerful construction for resolvable H-designs, i.e., a construction of an RH(g^{2n}) from an RH($(2g)^n$), which we call group halving construction. Based on this construction, we provide an alternative existence proof for resolvable SQS(v)s by investigating the existence problem of resolvable H-designs with group size 2. We show that the necessary conditions for the existence of an RH(2^n), namely, $n \equiv 2$ or 4 (mod 6) and $n \geq 4$ are also sufficient. Meanwhile, we provide an alternative existence proof for resolvable H-designs with group size 6. These results are obtained by first establishing an existence result for resolvable H-designs with group size 4, that is, the necessary conditions $n \equiv 1$ or 2 (mod 3) and $n \geq 4$ for the existence of an RH(4^n) are also sufficient for all values of n except possibly $n \in \{73, 149\}$. As a consequence, the general existence problem of an RH(g^n) is solved leaving mainly the case of $g \equiv 0$ (mod 12) open. Finally, we show that the necessary conditions for the existence of a resolvable G-design of type g^n are also sufficient.

Keywords B_4-pairings · Candelabra systems · G-designs · H-designs · H-frames · Resolvable · Steiner quadruple systems

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1 Introduction

A *Steiner quadruple system* of order v, denoted by SQS(v), is an ordered pair $(X, \mathcal{B})$, where X is a set of cardinality v, and $\mathcal{B}$ is a set of 4-subsets of X, called *blocks*, with the property that every 3-subset of X is contained in a unique block. It is well known that an SQS(v) exists if and only if $v \equiv 2$ or 4 (mod 6) [5].

If $(X, \mathcal{B})$ is an SQS(v), then $P \subset \mathcal{B}$ is a *parallel class* if P is itself a partition of X. $(X, \mathcal{B})$ is said to be *resolvable*, denoted by RSQS(v), if $\mathcal{B}$ can be partitioned into $r(v) = \frac{(v-1)(v-2)}{6}$ parts $P_1, P_2, \ldots, P_{r(v)}$, such that each part P_i is a parallel class. In this case, we call $P_1|P_2|\ldots|P_{r(v)}$ a *resolution* of $\mathcal{B}$.

The necessary conditions for the existence of an RSQS(v) are that $v \equiv 4$ or 8 (mod 12) or $v = 1$ or 2. In 1977, the only orders for which an RSQS(v) was known were $v = 2^n$, and the only recursive construction known was the doubling construction (i.e., a construction of an RSQS($2v$) from an RSQS(v)). In 1978, Booth [1] and Greenwell and Lindner [4] provided the first examples with v not a power of two by constructing an RSQS(20) and an RSQS(28). More examples were given by Hartman [6], where he constructed RSQS($q + 1$) for all prime powers $q \equiv 7$ (mod 12) with $q \leq 379$, and RSQS($4p$) for $p \in \{19, 43, 127, 199, 223, 271, 1603\}$ [7].

The main recursive theorems for RSQS(v), i.e., two tripling constructions were provided by Hartman in [8,9], both of which assume some subsystem structures on the input systems. Using the doubling and two tripling constructions together with a large number of initial designs, Hartman [9] proved by induction that the necessary condition $v \equiv 4$ or 8 (mod 12) for the existence of a resolvable SQS(v) is also sufficient for all values of v, with 23 possible exceptions. These last 23 undecided orders were removed by Ji and Zhu [12] by using resolvable H-designs and resolvable candelabra systems (the concept is defined in Sect. 2).

Let v be a non-negative integer, t be a positive integer and K be a set of positive integers. A *group divisible t-design* of order v with block sizes from K, denoted by GDD(t, K, v), is a triple $(X, \mathcal{G}, \mathcal{B})$ such that

(1) X is a set of v elements (called *points*);
(2) $\mathcal{G} = \{G_1, G_2, \ldots\}$ is a set of nonempty subsets (called *groups*) of X which partition X;
(3) $\mathcal{B}$ is a family of transverses (called *blocks*) of $\mathcal{G}$, each of cardinality from K, where a *transverse* is a subset of X intersects any given group in at most one point;
(4) every t-element transverse T of $\mathcal{G}$ is contained in a unique block.

The *type* of the GDD(t, K, v) is defined as the list $(|G||G \in \mathcal{G})$. If a GDD has n_i groups of size g_i, $1 \leq i \leq r$, then we use the notation $g_1^{n_1} g_2^{n_2} \cdots g_r^{n_r}$ to denote the group type. Mills in [14] used H(n, g, k, t) design to denote the GDD(t, k, ng) of type g^n. In this paper, we use H($g_1^{n_1} g_2^{n_2} \cdots g_r^{n_r}$) to denote the GDD($3, 4, \sum n_i g_i$) of type $g_1^{n_1} g_2^{n_2} \cdots g_r^{n_r}$ for short. An H(1^n) is actually an SQS(n).

For the existence of H-designs, Mills [14] showed that for $n > 3, n \neq 5$, an H(g^n) exists if and only if ng is even and $g(n-1)(n-2)$ is divisible by 3, and that for $n = 5$, an H(g^5) exists if g is divisible by 4 or 6. Recently, Ji [11] improved these results by showing that an H(g^5) exists whenever g is even, $g \neq 2$ and $g \not\equiv 10, 26$ (mod 48).

An H(g^n) is said to be *resolvable*, denoted by RH(g^n), if its block set can be partitioned into parallel classes. When $g = 1$, an RH(1^n) is an RSQS(n), which exists for all $n \equiv 4, 8$ (mod 12). Recently, Zhang and Ge [16] established the existence of an RH(6^n) for all even integers $n \geq 4$. We summarize the results as follows:

Theorem 1.1 *The necessary conditions $gn \equiv 0$ (mod 4), $g(n-1)(n-2) \equiv 0$ (mod 3) and $n \geq 4$ for the existence of an RH(g^n) are also sufficient for each $g \in \{1, 6\}$.*

The remainder of this paper is organized as follows. In Sect. 2, we will describe several recursive constructions for resolvable H-designs based on the theory of uniformly resolvable candelabra systems and resolvable H-frames. In particular, we will introduce a simple but powerful construction—group halving construction, as well as a product construction and three tripling constructions. Combining several initial designs together with the recursive methods established in Sect. 2, we give an almost complete solution to the existence problem of an RH(4^n) in Sect. 3. In Sect. 4, by the group halving construction, we show that the necessary conditions $n \equiv 2$ or $4 \pmod 6$ and $n \geq 4$ for the existence of an RH(2^n) are also sufficient. Hence, we provide an alternative existence proof for resolvable SQS(v)s. Meanwhile, we will also provide an alternative existence proof for resolvable H-designs with group size 6. As a consequence, the general existence problem of an RH(g^n) is solved leaving mainly the case of $g \equiv 0 \pmod{12}$ open. Finally, we show that the necessary conditions for the existence of a resolvable G-design of type g^n are also sufficient.

2 Recursive constructions

In this section, we shall describe several recursive constructions for resolvable H-designs. In particular, we will develop a group halving construction and three tripling constructions, which play a key role in the sequel.

2.1 Standard recursive constructions

Lemma 2.1 [12] *There exists an RH(g^4) for any positive integer g.*

Lemma 2.2 [12] (Weighting Construction) *Suppose that there exists an RH(g^n). Then there is an RH$((mg)^n)$ for any positive integer m.*

Let s be a non-negative integer. A *candelabra t-system* (or t-CS) of order v and block sizes from K, denoted by CS(t, K, v), is a quadruple $(X, S, \mathcal{G}, \mathcal{A})$ that satisfies the following properties:

(1) X is a set of v elements;
(2) S is an s-subset (called the *stem* of the *candelabra*) of X;
(3) $\mathcal{G} = \{G_1, G_2, \ldots\}$ is a set of non-empty subsets of $X \backslash S$, which partition $X \backslash S$;
(4) $\mathcal{A}$ is a collection of subsets of X, each of cardinality from K;
(5) every t-subset T of X with $|T \cap (S \cup G_i)| < t$, for all i, is contained in a unique block of $\mathcal{A}$, and no t-subset of $S \cup G_i$, for any i, is contained in any block of $\mathcal{A}$.

The *group type* of a t-CS$(X, S, \mathcal{G}, \mathcal{A})$ is defined as the list $(|G||G \in \mathcal{G} : |S|)$. If a t-CS has n_i groups of size g_i, $1 \leq i \leq r$, and stem size s, then we use the notation $(g_1^{n_1} g_2^{n_2} \cdots g_r^{n_r} : s)$ to denote the group type. A candelabra system with $t = 3$ and $K = \{4\}$ is called a *candelabra quadruple system* and denoted by CQS$(g_1^{n_1} g_2^{n_2} \cdots g_r^{n_r} : s)$.

A CS$(t, K, v)(X, S, \mathcal{G}, \mathcal{A})$ is said to be *resolvable*, denoted by RCS(t, K, v), if the block set $\mathcal{A}$ can be partitioned into several parts, each being a partition on X or a partition on $X \backslash (G \cup S)$ for some $G \in \mathcal{G}$ (called a *partial* parallel class). An RCS(t, K, v) is called *uniform*, denoted by URCS(t, K, v) if all the blocks in each resolution class have the same size. If $K = \{4\}$, it is denoted by RCQS, for which the number of parallel classes on X is $((\sum_{G \in \mathcal{G}} |G|)^2 - \sum_{G \in \mathcal{G}} |G|^2)/6$ and the number of partial parallel classes on $X \backslash (G \cup S)$ is $|G|(|G| + 2|S| - 3)/6$ for each $G \in \mathcal{G}$.

Theorem 2.3 [13] *For each integer $n \geq 2$, there exists an RCQS$(3^{(2^{2n}-1)/3} : 1)$.*

For non-negative integers q, g, k, and t, an $H(q, g, k, t)$ *frame* (as in [10]), denoted by HF(q, g, k, t), is an ordered four-tuple $(X, \mathcal{G}, \mathcal{B}, \mathcal{F})$ with the following properties:

1. X is a set of qg points;
2. $\mathcal{G} = \{G_1, G_2, \ldots, G_q\}$ is an equipartition of X into q groups;
3. $\mathcal{F}$ is a family $\{F_i\}$ of subsets of $\mathcal{G}$ called *holes*, which is closed under intersections. Hence each hole $F_i \in \mathcal{F}$ is of the form $F_i = \{G_{i_1}, G_{i_2}, \ldots, G_{i_s}\}$, and if F_i and F_j are holes then $F_i \cap F_j$ is also a hole. The number of groups in a hole is its size; and
4. $\mathcal{B}$ is a set of k-element transverses of $\mathcal{G}$ with the property that every t-element transverse of $\mathcal{G}$, which is not a t-element transverse of any hole $F_i \in \mathcal{F}$ is contained in precisely one block of $\mathcal{B}$, and no block contains a t-element transverse of any hole.

If an HF$(q, g, 4, 3)$ has n holes of size $m + s$, which intersect on a common hole of size s, then we denote such a design by HF$(m^n : s)$ with group size g, or shortly by HF$_g(m^n : s)$. If an HF$(q, g, 4, 3)$ has only one hole of size s, then we call it an *incomplete H-design* of type $(g^q : g^s)$, denoted by IH$(g^q : g^s)$.

An HF$_g(m^n : s)(X, \mathcal{G}, \mathcal{B}, \mathcal{F})$ with $\mathcal{F} = \{F_i : 0 \le i \le n\}$ and F_0 the common hole of size s is said to be *resolvable*, denoted by RHF$_g(m^n : s)$, if its block set can be partitioned into $(nmg^2(m + 2s - 3) + n(n-1)(mg)^2)/6$ parts with the following properties:

(1) For each hole F_i, $1 \le i \le n$, there are exactly $mg^2(m + 2s - 3)/6$ parts, each being a partition of $X \backslash (\bigcup_{G \in F_i} G)$;
(2) There are $n(n-1)(mg)^2/6$ parts, each being a parallel class on X.

An IH$(g^{m+s} : g^s)(X, \mathcal{G}, \mathcal{B}, F)$ with the only hole F of size s is said to be *resolvable*, denoted by IRH$(g^{m+s} : g^s)$, if its block set can be partitioned into $(m+s-1)(m+s-2)g^2/6$ parts, $(s-1)(s-2)g^2/6$ of which are partitions of $X \backslash (\bigcup_{G \in F} G)$, and $m(m + 2s - 3)g^2/6$ of which are parallel classes on X.

The construction given below is a generalization of the fundamental construction for 3-wise balanced designs.

Theorem 2.4 *Suppose that $(X, S, \Gamma, \mathcal{A})$ is a 3-CS$(m^n : s)$ and $\infty \in S$. Let $K_1 = \{|A| : \infty \in A \in \mathcal{A}\}$ and $K_2 = \{|A| : \infty \notin A \in \mathcal{A}\}$. If there exists an HF$_g(t^{k_1 - 1} : a)$ for each $k_1 \in K_1$ and an $H((gt)^{k_2})$ for each $k_2 \in K_2$, then there exists an HF$_g((tm)^n : t(s-1)+a)$. Furthermore, if the 3-CS$(m^n : s)$ is uniformly resolvable, and each of HF$_g(t^{k_1-1} : a)$ and $H((gt)^{k_2})$ for $k_1 \in K_1$ and $k_2 \in K_2$ is resolvable, then the resultant HF$_g((tm)^n : t(s-1)+a)$ is also resolvable.*

Proof Suppose $(X, S, \Gamma, \mathcal{A})$ is the given URCS$(m^n : s)$, where $\Gamma = \{G_1, \ldots, G_n\}$ and $\mathcal{A}$ has a resolution $\mathcal{A} = (\bigcup_{1 \le i \le n} \mathcal{Q}_i) \bigcup \mathcal{Q}$ with each member of $\mathcal{Q}_i$ being a partition of $X \backslash (G_i \cup S)$ and each member of $\mathcal{Q}$ being a partition of X. Define $G'_{x,j} = \{x\} \times \{j\} \times Z_g$. Let $X' = ((X \backslash \{\infty\}) \times Z_t \times Z_g) \cup (\{\infty\} \times Z_a \times Z_g)$, $\mathcal{G}' = \{G'_{x,j} : x \in X \backslash \{\infty\}, j \in Z_t\} \cup \{G'_{\infty,j} : j \in Z_a\}$, $\mathcal{F} = \{F_i : 0 \le i \le n\}$, where $F_0 = \{G'_{x,j} : x \in S \backslash \{\infty\}, j \in Z_t\} \cup \{G'_{\infty,j} : j \in Z_a\}$ and $F_i = \{G'_{x,j} : x \in G_i, j \in Z_t\} \cup F_0$ for $1 \le i \le n$. We will construct an RHF$_g((tm)^n : t(s-1)+a)$ on X' with group set $\mathcal{G}'$ and hole set $\mathcal{F}$.

For each $B \in \mathcal{A}$ and $\infty \in B$, construct an RHF$_g(t^{|B|-1} : a)$ on $X'_B = ((B \backslash \{\infty\}) \times Z_t \times Z_g) \cup (\{\infty\} \times Z_a \times Z_g)$ with group set $\mathcal{G}'_B = \{G'_{x,j} : x \in B \backslash \{\infty\}, j \in Z_t\} \cup \{G'_{\infty,j} : j \in Z_a\}$ and hole set $\mathcal{F}_B = \{F_x : x \in B\}$, where $F_x = \{G'_{x,j} : j \in Z_t\} \cup F_\infty$ with $F_\infty = \{G'_{\infty,j} : j \in Z_a\}$ being the common hole of size a. Denote its block set by $\mathcal{C}_B$, which has a resolution $\{\mathcal{C}_B(x, j) : x \in B \backslash \{\infty\}, 1 \le j \le tg^2(t + 2a - 3)/6\} \cup \{\mathcal{C}_B(l) : 1 \le l \le$

$(|B|-1)(|B|-2)(tg)^2/6\}$ with each $\mathcal{C}_B(x, j)$ being a partition of $X'_B \setminus (\bigcup_{G \in F_x} G)$ and each $\mathcal{C}_B(l)$ being a parallel class on X'_B.

For each $B \in \mathcal{A}$ and $\infty \notin B$, construct an RH$((gt)^{|B|})$ on $X'_B = B \times Z_t \times Z_g$ with group set $\mathcal{G}'_B = \{\{x\} \times Z_t \times Z_g : x \in B\}$ and block set $\mathcal{C}_B$, which can be partitioned into parallel classes $\mathcal{C}_B(l)$, $1 \le l \le (|B|-1)(|B|-2)(tg)^2/6$.

Then $\mathcal{A}' = \bigcup_{B \in \mathcal{A}} \mathcal{C}_B$ is the block set of the required design. We need to partition the blocks into resolution classes.

For each member $Q \in \mathcal{Q}_i$, $1 \le i \le n$, suppose its block size is k_Q. Then $P_Q(l) = \bigcup_{B \in Q} \mathcal{C}_B(l)$ is a partition of $X' \setminus (\bigcup_{G \in F_i} G)$ for $1 \le l \le (k_Q - 1)(k_Q - 2)(tg)^2/6$.

For each $x \in \bigcup_{G \in F_i} G$, $1 \le i \le n$, $P_{x,j} = \bigcup_{B \in \mathcal{A}, \infty \notin B} \mathcal{C}_B(x, j)$ is a partition of $X' \setminus (\bigcup_{G \in F_i} G)$ for $1 \le j \le tg^2(t + 2a - 3)/6$.

For each member $Q \in \mathcal{Q}$, suppose its block size is k_Q. Then $P'_Q(l) = \bigcup_{B \in Q} \mathcal{C}_B(l)$ is a partition of X' for $1 \le l \le (k_Q - 1)(k_Q - 2)(tg)^2/6$.

Thus we obtain an RHF$_g((tm)^n : t(s-1) + a)$. □

The following theorem is stated in [16].

Theorem 2.5 [16, Lemmas 3.3 and 3.4] *Suppose that there exists an RHF$_g(m^n : s)$. If there exists an IRH$(g^{m+s} : g^s)$, then there exists an IRH$(g^{mn+s} : g^{m+s})$. Furthermore, if there is an RH(g^{m+s}), then there is an RH(g^{mn+s}).*

2.2 Product construction and group halving construction

A regular graph (V, E) of degree k is said to have a *one-factorization* if the edge set E can be partitioned into k parts $E = F_1|F_2|\ldots|F_k$ so that each F_i is a partition of the vertex set V into pairs. The parts F_i are called *one-factors*.

Theorem 2.6 (Product Construction) *If there exist both an RH(g^m) and an RH(g^n), then there exists an RH(g^{mn}) and an IRH$(g^{mn} : g^n)$.*

Proof Let $(X, \mathcal{G}, \mathcal{B})$ be the given RH(g^m), where $\mathcal{G} = \{G_0, \ldots, G_{m-1}\}$. Applying Lemma 2.2, we construct an RH$((ng)^m)$ on $X' = X \times Z_n$ with the group set $\mathcal{G}' = \{G_i \times Z_n : 0 \le i \le m-1\}$ and block set $\mathcal{A}$.

For each i, $0 \le i \le m-1$, construct an RH(g^n) on $G_i \times Z_n$ with group set $\{G_i \times \{l\} : l \in Z_n\}$ and block set $\mathcal{C}_i$, which has a resolution $P_i(k)$, $1 \le k \le (n-1)(n-2)g^2/6$.

Since an RH(g^n) exists, gn is double even. For each i, $0 \le i \le m-1$, let $\mathcal{F}^i = \{F^i_1, \ldots, F^i_{g(n-1)}\}$ be a one-factorization of the complete multiple-graph on $G_i \times Z_n$ with n parts $\{G_i \times \{l\} : l \in Z_n\}$. Let

$$\mathcal{D} = \{\{a, b, c, d\} : \{a, b\} \in F^i_j, \{c, d\} \in F^{i'}_j, 0 \le i \ne i' \le m-1, 1 \le j \le g(n-1)\},$$

then $\mathcal{B}' = \mathcal{A} \cup (\cup_{i=0}^{m-1} \mathcal{C}_i) \cup \mathcal{D}$ is the block set of an H(g^{mn}) on the group set $\mathcal{G}'' = \{G_i \times \{l\} : l \in Z_n, 0 \le i \le m-1\}$. It is clear that $\cup_{i=0}^{m-1} \mathcal{C}_i$ has a resolution $Q(k) = \cup_{i=0}^{m-1} P_i(k)$, $1 \le k \le (n-1)(n-2)g^2/6$. It remains to show that $\mathcal{D}$ can be partitioned into parallel classes.

For each j, $1 \le j \le g(n-1)$, let

$$\mathcal{D}_j = \{\{a, b, c, d\} : \{a, b\} \in F^i_j, \{c, d\} \in F^{i'}_j, 0 \le i \ne i' \le m-1\}, \text{ and}$$

$$D_j = \{\{\{a, b\}, \{c, d\}\} : \{a, b\} \in F^i_j, \{c, d\} \in F^{i'}_j, 0 \le i \ne i' \le m-1\}.$$

If we regard each pair in F^i_j, $0 \le i \le m-1$ as a vertex, we may construct a multi-partite complete graph Γ_j on the vertex set $X'_j = \cup_{i=0}^{m-1} F^i_j$ with partite set $\{F^i_j : 0 \le i \le m-1\}$, where two different vertices connect if and only if they are from different factors F^i_j. Hence, D_j is the edge set of Γ_j. That is to say we obtain a GDD(2, 2, $gnm/2$) of type $(gn/2)^m$ on X'_j with group set $\{F^i_j : 0 \le i \le m-1\}$ and block set D_j.

It is well-known that there always exists a resolvable GDD(2, 2, $gnm/2$) of type $(gn/2)^m$ when $gnm/2$ is even (see [3]). Hence, we can partition the block set D_j of our resulting GDD(2, 2, $gnm/2$) of type $(gn/2)^m$ into parallel classes on X'_j. Therefore, $\mathcal{D}_j$ can also be partitioned in parallel classes of X'. So does $\mathcal{D} = \cup_{1\le j\le g(n-1)}\mathcal{D}_j$. Thus, the desired H$(g^{mn})$ is resolvable.

For each i, $0 \le i \le m-1$, $\mathcal{B}'\backslash\mathcal{C}_i$ is the block set of an incomplete design IRH$(g^{mn} : g^n)$ on X' with group set $\mathcal{G}''$ and hole set $\{G_i \times \{l\} : l \in Z_n\}$. □

With a similar proof to that of Theorem 2.6, we have the following doubling construction. Here, we just need to fill with the trivial design RH(g^2) having no blocks.

Theorem 2.7 (Doubling Construction) *If there exists an RH(g^u), then there exists an RH(g^{2u}) and an IRH$(g^{2u} : g^u)$.*

The following construction for resolvable H-designs is simple but powerful.

Theorem 2.8 (Group Halving Construction) *If there exists an RH$((2g)^n)$, then there exists an RH(g^{2n}).*

Proof Let $(X, \mathcal{G}, \mathcal{B})$ be the given RH$((2g)^n)$ with $\mathcal{G} = \{G_0, \ldots, G_{n-1}\}$. Therefore, gn is even. Halve each group G_i into G_{i0} and G_{i1}, $0 \le i \le n-1$. We will construct an RH(g^{2n}) on the group set $\mathcal{G}' = \{G_{ij} \mid 0 \le i \le n-1, j = 0, 1\}$ as follows.

For each i, $0 \le i \le n-1$, let $\mathcal{F}^i = \{F^i_1, \ldots, F^i_g\}$ be a one-factorization of the bipartite graph on $G_{i0} \cup G_{i1}$. Let

$$\mathcal{D} = \{\{a, b, c, d\} : \{a, b\} \in F^i_j, \{c, d\} \in F^{i'}_j, 0 \le i \ne i' \le n-1, 1 \le j \le g\},$$

then $\mathcal{B}' = \mathcal{B} \cup \mathcal{D}$ is the block set of an H(g^{2n}) on the group set $\mathcal{G}'$. With a similar proof to that of Theorem 2.6, it is clear that $\mathcal{D}$ can be partitioned into parallel classes. This completes the proof. □

2.3 Three tripling constructions

Our first tripling construction is on resolvable H-frames, which is a generalization of the tripling construction for resolvable CQSs developed in [12].

Theorem 2.9 (Tripling Construction I) *Suppose there exists an RHF$_g(n^3 : s)$, then there exists an RHF$_g((3n)^3 : s)$.*

Proof Start with a CQS$(3^3 : 1)$ (as in [12]) on $Z_9 \cup \{\infty\}$ with groups $G_i = \{i, i+3, i+6\}$, $0 \le i \le 2$ and stem $\{\infty\}$, whose block set $\mathcal{B}$ is generated by the following 9 base blocks under the automorphism group $\langle(0\ 3\ 6)(1\ 4\ 7)(2\ 5\ 8)(\infty)\rangle$.

$$\begin{array}{llll}
\mathcal{A}_\infty : & \{0, 1, 2, \infty\}, & \{0, 4, 8, \infty\}, & \{0, 5, 7, \infty\}, \\
\mathcal{A}_1 : & \{1, 3, 2, 6\}, & \{1, 3, 5, 7\}, & \{2, 6, 5, 7\}, \\
\mathcal{A}_2 : & \{4, 7, 5, 8\}, & \{3, 6, 5, 8\}, & \{3, 6, 4, 7\}.
\end{array}$$

View each base block as an ordered quadruple given above so that each block $B \in \mathcal{B}$ is ordered.

Since an $\mathrm{RHF}_g(n^3 : s)$ exists, both gn and gs are even. We separate the proof into the following two cases:

Case (1): When g is even, we will construct an $\mathrm{RHF}_g((3n)^3 : s)$ on $X = (Z_9 \times Z_2 \times Z_{gn/2}) \cup (\{\infty\} \times Z_2 \times Z_{gs/2})$ with groups $G(x, j) = \{x\} \times Z_2 \times \{j, j+n, \ldots, j+(\frac{g}{2}-1)n\}$, $x \in Z_9$, $0 \le j \le n-1$, and $G(\infty, j) = \{\infty\} \times Z_2 \times \{j, j+s, \ldots, j+(\frac{g}{2}-1)s\}$, $0 \le j \le s-1$, and three holes $F_i = \{G(i, j), G(i+3, j), G(i+6, j) : 0 \le j \le n-1\} \cup S$, $0 \le i \le 2$, which intersect on a common hole $S = \{G(\infty, j) : 0 \le j \le s-1\}$.

For each block $B \in \mathcal{B}$ containing ∞, construct an $\mathrm{RHF}_g(n^3 : s)$ on $X_B = ((B \setminus \{\infty\}) \times Z_2 \times Z_{gn/2}) \cup (\{\infty\} \times Z_2 \times Z_{gs/2})$ with group set $\{G(x, j) : x \in B \setminus \{\infty\}, 0 \le j \le n-1\} \cup S$, three holes $\{G(x, j) : 0 \le j \le n-1\} \cup S$, $x \in B \setminus \{\infty\}$ and a common hole S. Denote its block set by $\mathcal{A}_B$, which has a resolution $\{P_B(x, l) : x \in B \setminus \{\infty\}, 1 \le l \le n(n+2s-3)g^2/6\} \cup \{P_B(r', r, h) : r', r \in Z_2, 1 \le h \le (gn)^2/4\}$ such that each $P_B(x, l)$ is a partition of $(B \setminus \{\infty, x\}) \times Z_2 \times Z_{gn/2}$ and each $P_B(r', r, h)$ is a parallel class on X_B.

For each block $B = \{a, b, c, d\} \in \mathcal{B}$ and $\infty \notin B$, we shall construct a special $\mathrm{H}((gn)^4)$ on $B \times Z_2 \times Z_{gn/2}$ with groups $\{x\} \times Z_2 \times Z_{gn/2}$, $x \in B$. Denote

$$C'_B(k, i, j) = \{(a, i), (b, i+k), (c, j), (d, j+k)\} \text{ and } \mathcal{C}'_B(k) = \{C'_B(k, i, j) : i, j \in Z_2\},$$

then $\mathcal{C}'_B = \mathcal{C}'_B(0) \cup \mathcal{C}'_B(1)$ is the block set of an $\mathrm{H}(2^4)$ on $B \times Z_2$. For each $A \in \mathcal{C}'_B$, construct an $\mathrm{RH}((gn/2)^4)$ on $A \times Z_{gn/2}$ with groups $\{a\} \times Z_{gn/2}$, $a \in A$. Denote its block set by $\mathcal{B}(A)$ and the $(gn)^2/4$ parallel classes by $P(A, h)$, $1 \le h \le (gn)^2/4$. Then, $\mathcal{C}_B = \cup_{A \in \mathcal{C}'_B} \mathcal{B}(A)$ is the block set of the desired $\mathrm{H}((gn)^4)$.

Let $\mathcal{D} = (\cup_{B \in \mathcal{B}, \infty \notin B} \mathcal{C}_B) \cup (\cup_{B \in \mathcal{B}, \infty \in B} \mathcal{A}_B)$. By Theorem 2.4, $\mathcal{D}$ is the block set of an $\mathrm{HF}_g(((3n)^3 : s))$. It remains to show the resolvability. This $\mathrm{HF}_g(((3n)^3 : s))$ should be partitioned into $9g^2n^2$ parallel classes on X and $g^2n(3n+2s-3)/2$ partial parallel classes on $(Z_9 \setminus G_i) \times Z_2 \times Z_{gn/2}$ for each i, $0 \le i \le 2$.

For each i, $0 \le i \le 2$, let $P(i, x, l) = \cup_{B \in \mathcal{B}, \{x, \infty\} \subset B} P_B(x, l)$, $1 \le l \le n(n+2s-3)g^2/6$, $x \in G_i$. Then each $P(i, x, l)$ is a partition of $(Z_9 \setminus G_i) \times Z_2 \times Z_{gn/2}$. The other g^2n^2 partial parallel classes on $(Z_9 \setminus G_i) \times Z_2 \times Z_{gn/2}$ can be obtained as follows. Denote the three base blocks of $\mathcal{A}_2$ by B_0, B_1, B_2 in order. For $0 \le i \le 2$, let $\mathcal{B}_i = \{3j + B_i : 0 \le j \le 2\}$, and for $r', r \in Z_2$, let $P(i, r', r) = \{C'_B(1, r', r) : B \in \mathcal{B}_i\}$. Then $P(i, r', r)$ is a partial class on $(Z_9 \setminus G_i) \times Z_2$. Note that for $0 \le i \le 2$, $\cup_{r', r \in Z_2} P(i, r', r) = \cup_{B \in \mathcal{B}_i} \mathcal{C}'_B(1)$. Let $P(i, r', r, h) = \cup_{A \in P(i, r', r)} P(A, h)$. Then, these $P(i, r', r, h)$s with $r', r \in Z_2$ and $1 \le h \le (gn)^2/4$ are g^2n^2 partial parallel classes on $(Z_9 \setminus G_i) \times Z_2 \times Z_{gn/2}$.

Now we give the required $9g^2n^2$ parallel classes on X. Denote the three base blocks of $\mathcal{A}_1$ by A_0, A_1, A_2 in order. Let $D_0 = A_0$, $D_1 = A_1 + 3 = \{4, 6, 8, 1\}$, $D_2 = A_2 + 6 = \{8, 3, 2, 4\}$. Let $\mathcal{A}(i, 0)$ be as follows and $\mathcal{A}(i, j) = \{3j + B : B \in \mathcal{A}(i, 0)\}$ for $0 \le j \le 2$.

$$\begin{aligned}
\mathcal{A}(1, 0) &= \{\{0, 4, 8, \infty\}, A_0, A_1, A_2\}, \\
\mathcal{A}(2, 0) &= \{\{0, 1, 2, \infty\}, B_0, B_1, B_2\}, \\
\mathcal{A}(0, 0) &= \{\{0, 5, 7, \infty\}, D_0, D_1, D_2\}.
\end{aligned}$$

Let

$$\begin{aligned}
P'(1, j, r', r) &= \{C'_{A_0+3j}(0, r', r'+r), C'_{A_1+3j}(0, r'+1, r), C'_{A_2+3j}(0, r'+r+1, r+1)\}, \\
P'(2, j, r', r) &= \{C'_{B_0+3j}(0, r'+r, r'), C'_{B_1+3j}(0, r, r'+1), C'_{B_2+3j}(0, r+1, r'+r+1)\}, \\
P'(0, j, r', r) &= \{C'_{D_0+3j}(1, r', r'+r), C'_{D_1+3j}(1, r'+r+1, r'), C'_{D_2+3j}(1, r'+1, r'+r+1)\}.
\end{aligned}$$

Let $P'(i,j,r',r,h) = \cup_{A \in P'(i,j,r',r)} P(A,h)$ and $P''(i,j,r',r,h) = P_B(r',r,h) \cup P'(i,j,r',r,h)$, where $B \in \mathcal{A}(i,j)$ and $\infty \in B$. Then $P''(i,j,r',r,h)$ for $0 \le i,j \le 2$, $r',r \in Z_2$, $1 \le h \le (gn)^2/4$ are the desired $9g^2n^2$ parallel classes on X.

So $\mathcal{D}$ has the resolution $\{P(i,x,l) : 0 \le i \le 2, x \in G_i, 1 \le l \le n(n+2s-3)g^2/6\} \cup \{P(i,r',r,h) : 0 \le i \le 2, r',r \in Z_2, 1 \le h \le (gn)^2/4\} \cup \{P''(i,j,r',r,h) : 0 \le i,j \le 2, r',r \in Z_2, 1 \le h \le (gn)^2/4\}$, and the $\mathrm{HF}_g(((3n)^3 : s))$ is resolvable.

Case (2): When g is odd, both n and s must be even, we will construct an $\mathrm{RHF}_g((3n)^3 : s)$ on X with groups $G'(x,k,j) = \{x\} \times \{k\} \times \{j, j+\frac{n}{2}, \ldots, j+(g-1)\frac{n}{2}\}$, $x \in Z_9$, $k \in Z_2$, $0 \le j \le \frac{n}{2}-1$, and $G'(\infty,k,j) = \{\infty\} \times \{k\} \times \{j, j+\frac{s}{2}, \ldots, j+(g-1)\frac{s}{2}\}$, $k \in Z_2$, $0 \le j \le \frac{s}{2}-1$, and three holes $F_i' = \{G'(i,k,j), G'(i+3,k,j), G'(i+6,k,j) : k \in Z_2, 0 \le j \le \frac{n}{2}-1\} \cup S'$, $0 \le i \le 2$, which intersect on a common hole $S' = \{G'(\infty,k,j) : k \in Z_2, 0 \le j \le \frac{s}{2}-1\}$.

For each block $B \in \mathcal{B}$ containing ∞, construct an $\mathrm{RHF}_g(n^3 : s)$ on $X_B = ((B\backslash\{\infty\}) \times Z_2 \times Z_{gn/2}) \cup (\{\infty\} \times Z_2 \times Z_{gs/2})$ with group set $\{G'(x,k,j) : x \in B\backslash\{\infty\}, k \in Z_2, 0 \le j \le \frac{n}{2}-1\} \cup S'$, three holes $\{G'(x,k,j) : k \in Z_2, 0 \le j \le \frac{n}{2}-1\} \cup S'$, $x \in B\backslash\{\infty\}$ and a common hole S'. Denote its block set by $\mathcal{A}_B$, which has a resolution $\{P_B(x,l) : x \in B\backslash\{\infty\}, 1 \le l \le n(n+2s-3)g^2/6\} \cup \{P_B(r',r,h) : r',r \in Z_2, 1 \le h \le (gn)^2/4\}$ such that each $P_B(x,l)$ is a partition of $(B\backslash\{\infty,x\}) \times Z_2 \times Z_{gn/2}$ and each $P_B(r',r,h)$ is a parallel class on X_B.

The remaining proof of this case is the same as that of Case (1). □

Next, we give two tripling constructions for resolvable H-designs. They are generalizations of those for resolvable Steiner quadruple systems proposed by Hartman in [8,9], which have played an important role in the construction of $\mathrm{RSQS}(v)$. We need the following notations.

For $x \in Z_n$, we define $|x|$ by

$$|x| = \begin{cases} x, & \text{if } \ 0 \le x \le n/2, \\ -x, & \text{if } \ n/2 < x < n. \end{cases}$$

For $n \ge 2$ and $L \subseteq \{1,2,\ldots,\lfloor n/2 \rfloor\}$, define $\mathrm{G}(n,L)$ to be the regular graph with vertex set Z_n and edge set E given by $\{x,y\} \in E$ if and only if $|x-y| \in L$.

The following lemma is proved by Stern and Lenz in [15].

Lemma 2.10 *Let $L \subseteq \{1,2,\ldots,n\}$. Then $G(2n,L)$ has a one-factorization if and only if $2n/gcd(j,2n)$ is even for some $j \in L$.*

The construction given below is a variation of the construction for resolvable candelabra quadruple systems in [9].

Theorem 2.11 *Suppose that $n \ge 1$, $s \equiv 1,2 \pmod 3$ and $3s \ge 5n$. There exists an $RHF_4((3n)^3 : s)$.*

Proof Let $n \ge 1$, $s \equiv 1,2 \pmod 3$ and $3s \ge 5n$. Take $Y = \{\infty_0, \infty_1, \ldots, \infty_{4s-1}\}$ and let $X = (Z_{12n} \times Z_3) \cup Y$. We will construct an $\mathrm{RHF}_4((3n)^3 : s)(X, \mathcal{G}, \mathcal{B}, \mathcal{F})$ with groups $G(i,j) = \{(i+3kn, j) : k \in Z_4\}$, $i \in Z_{3n}$, $j \in Z_3$, and $G(\infty, j) = \{\infty_{sk+j} : k \in Z_4\}$, $0 \le j \le s-1$, and three holes $F_j = \{G(i,j) : i \in Z_{3n}\} \cup S$, $0 \le j \le 2$, which intersect on a common hole $S = \{G(\infty,j) : 0 \le j \le s-1\}$. In the sequel we shall write x_i for the ordered pair $(x,i) \in Z_{12n} \times Z_3$.

Let $h = (12n-4s)/2$. Since $3s \ge 5n$, h is even and $h \le 8n/3$. As in [9, Theorem 2.1], let

$$H_1^* = \{\{9n-i, 9n-3+i\} : 2 \le i \le 3n+1, i \not\equiv 0 \pmod 3\}, \text{ and}$$
$$H_2^* = \{\{3n-i, 3n+i\} : 1 \le i \le 3n-2, i \not\equiv 0 \pmod 3\}.$$

It is easy to check that $|H_1^*| = 2n$ and $|H_2^*| = 2n - 1$. Let H_i be any subset of H_i^* of cardinality $h/2$, $i = 1, 2$ and $H = H_1 \cup H_2$, which satisfies the following properties:

(1) $|H| = h = (12n - 4s)/2 \le 8n/3$.
(2) The pairs in H are disjoint, i.e., $|\bigcup_{\{x,y\}\in H}\{x, y\}| = 2h$.
(3) Let $LH = \{|y - x| : \{x, y\} \in H\}$, then $|LH| = h$ and $LH \cap \{3, 6, \dots, 6n\} = \emptyset$.
(4) The distances between members of H_1 are odd.
(5) $\{x, y\} \equiv \{1, 2\} \pmod 3$ for each $\{x, y\} \in H$.

Since $H_1 \subsetneq H_1^*$ and all distances between members of H_1^* are odd, the graph $G(12n, \{1, 2, \dots, 6n\}\backslash(LH \bigcup\{3n, 6n\}))$ has a one-factorization $F_1|F_2|\dots|F_{12n-2h-4}$ by Lemma 2.10. Let $F_{12n-2h-3}|F_{12n-2h-2}|F_{12n-2h-1}$ be a one-factorization of the graph $G(12n, \{3n, 6n\})$. Then it is not difficult to see that $F_1|F_2|\dots|F_{12n-2h-1}$ is a one-factorization of the graph $G(12n, \{1, 2, \dots, 6n\}\backslash LH)$. Using the above set of pairs H and the one-factorization of the graph $G(12n, \{1, 2, \dots, 6n\}\backslash LH)$, Hartman [9, Theorem 2.1] constructed a resolvable RCQS$((12n)^3 : 4s)$ on X with group set $\{Z_{12n} \times \{i\} : i \in Z_3\}$ and stem Y, as well as the block set $\mathcal{B}'$ and its resolution $\mathcal{P}$ containing the following $6n(12n - 2h - 1)$ partitions of $Z_{12n} \times \{i + 1, i + 2\}$ for each $i \in Z_3$:

$$P_{i,u,k} = \{\{x_{i+1}, y_{i+1}, z_{i+2}, t_{i+2}\} : \{x, y\} \text{ is the } m\text{th member of } F_u,\\ \{z, t\} \text{ is the } (m + k)\text{th member of } F_u, m = 1, 2, \dots, 6n\},$$

where $u = 1, 2, \dots, 12n - 2h - 1$, and $k = 0, 1, \dots, 6n - 1$.

For each $i \in Z_3$, let β_i be the union of partitions $P_{i,u,k}$ with $12n-2h-3 \le u \le 12n-2h-1$ and $0 \le k \le 6n - 1$. Then we have that $\mathcal{B} = \mathcal{B}'\backslash(\bigcup_{i\in Z_3} \beta_i)$ is the block set of the desired RHF$_4((3n)^3 : s)$ on X with group set $\mathcal{G}$ and hole set $\mathcal{F}$, where $\mathcal{B}$ has a resolution $\mathcal{P}\backslash\{P_{i,u,k} : 12n - 2h - 3 \le u \le 12n - 2h - 1, 0 \le k \le 6n - 1, i \in Z_3\}$. □

As a consequence of Theorem 2.11, we have our second tripling construction as follows.

Corollary 2.12 (Tripling Construction II) *Let* $n \equiv s \pmod 3$, $s \equiv 1$ *or* $2 \pmod 3$ *and* $14s \ge 5n$. *If there exists an IRH*$(4^n : 4^s)$, *then there exists an IRH*$(4^{3n-2s} : 4^n)$ *and an IRH*$(4^{3n-2s} : 4^s)$.

To construct resolvable H-frames with group size 6, the concept of resolvable B-pairing was introduced in [16]. To show our third tripling construction, we adapt the concept to the case of group size 4 and call it B_4-pairing as follows.

For non-negative integers n and s, a *B_4-pairing*, $B_4(n, s)$ consists of four subsets D, R_0, R_1, R_2 of $Z_{4(3n+s)}$ and three subsets PR_0, PR_1, PR_2 of $Z_{4(3n+s)} \times Z_{4(3n+s)}$ with the following properties for each $i \in \{0, 1, 2\}$:

(1) *Cardinality and symmetry conditions*

 (a) $|D| = 4s$, $|R_i| = 4n$,
 (b) $D = -D$.

(2) *Partitioning conditions*

 (a) PR_i is a partition of R_i into pairs, thus $|PR_i| = 2n$,
 (b) $Z_{4(3n+s)} = D \cup R_0 \cup R_1 \cup R_2$.

(3) *Pairing conditions*
Let $L_i = \{|x - y| : \{x, y\} \in PR_i\}$ and $N = \{3n + s, 2(3n + s)\}$,

(a) $N \cap L_i = \emptyset$,
(b) $|L_i| = 2n$,
(c) the complement G_i of the graph $G(4(3n+s), L_i \cup N)$ has a one-factorization.

Let $S_0, S_1, S_2, \overline{R}_0, \overline{R}_1, \overline{R}_2$ be subsets of $Z_{4(3n+s)}$ and PS_0, PS_1, PS_2 be subsets of $Z_{4(3n+s)} \times Z_{4(3n+s)}$. A B_4-pairing $B_4(n, s)$ with $D, R_i, PR_i, i \in \{0, 1, 2\}$, is said to be *resolvable*, denoted by $RB_4(n, s)$, if the following properties are satisfied for each $i \in \{0, 1, 2\}$:

(1) *Cardinality and symmetry conditions*

(c) $|S_i| = 4n, |\overline{R}_i| = 2n$.

(2) *Partitioning conditions*

(c) PS_i is a partition of S_i into pairs, thus $|PS_i| = 2n$,
(d) $Z_{6(n+s)} = D \cup R_i \cup S_i \cup \overline{R}_{i+1} \cup -\overline{R}_{i-1}$.

(3) *Pairing conditions*
Let $O_i = \{|x - y| : \{x, y\} \in PS_i\}$,

(d) $N \cap O_i = \emptyset$,
(e) $|O_i| = 2n$, $L_i \cap O_i = \emptyset$, and all members of O_i are odd,
(f) the complement G_i' of the graph $G(4(3n+s), L_i \cup O_i \cup N)$ has a one-factorization.

The following theorem gives the relation between B_4-pairings and H-frames with group size 4. A similar one for the case of group size 6 was proved in [16]. Hence, we omit the proof here.

Theorem 2.13 *If there exists a $B_4(n, s)$, then there exists an $HF_4((3n+s)^3 : s)$. Furthermore, if the $B_4(n, s)$ is resolvable, then the $HF_4((3n+s)^3 : s)$ is resolvable.*

Lemma 2.14 *If $D, R_i, PR_i, S_i, PS_i, \overline{R}_i, i \in \{0, 1, 2\}$ form an $RB_4(n, s)$ with the property $\{0, 3n+s, 2(3n+s), 3(3n+s)\} \subset D$, then there exists an $RHF_4((3n+s)^3 : s)$ with a sub-design $RH(4^4)$.*

Proof Using the given $RB_4(n, s)$, we construct an $\mathrm{RHF}_4((3n+s)^3 : s)$ on $X = \{a_i : a \in Z_{4(3n+s)}, i \in \{0, 1, 2\}\} \cup \{\infty_0, \infty_1, \ldots, \infty_{4s-1}\}$ with groups $G(i, j) = \{(k(3n+s)+j)_i : 0 \le k \le 3\}, i \in \{0, 1, 2\}, 0 \le j \le 3n+s-1$, $G(\infty, j) = \{\infty_{ks+j} : 0 \le k \le 3\}, 0 \le j \le s-1$, three holes $F_{1+i} = F_0 \cup \{G(i, j) : 0 \le j \le 3n+s-1\}, i \in \{0, 1, 2\}$ and a common hole $F_0 = \{G(\infty, j) : 0 \le j \le s-1\}$, as well as the block set $\mathcal{B}$ containing the following blocks (see the details in [16]):

$$\delta = \big\{\{\infty_j, (a+d)_0, (b-d)_1, (c+d)_2\} : \ a+b+c \equiv 0 \pmod{4(3n+s)},$$
$$d \text{ is the } j\text{th member of } D, 0 \le j \le 4s-1\big\}.$$

Since $k(3n+s) \in D$ for each $k, 0 \le k \le 3$, without loss of generality we may assume $k(3n+s)$ is the (ks)th element of D. Let

$$\delta_0 = \{\{\infty_{ks}, (a+d)_0, (b-d)_1, (c+d)_2\} : a+b+c \equiv 0 \pmod{4(3n+s)}, a, b, c \in$$
$$\{i(3n+s) : 0 \le i \le 3\}, d \text{ is the } (ks)\text{th member of } D \text{ and } 0 \le k \le 3\}.$$

Note that $\delta_0 \subset \delta$ and δ_0 forms the block set of an $\mathrm{RH}(4^4)$ with the group set $\{\{(k(3n+s))_i : 0 \le k \le 3\} : i \in \{0, 1, 2\}\} \cup \{\{\infty_{ks} : 0 \le k \le 3\}\}$ and parallel classes $\{\{\infty_{(i+j+k+g)s}, ((i+g)(3n+s))_0, ((j+g)(3n+s))_1, ((k+g)(3n+s))_2\} : g \in Z_4\}, i+j+k \equiv 0 \pmod 4$. Hence, the $\mathrm{RHF}_4((3n+s)^3 : s)$ contains a subdesign $\mathrm{RH}(4^4)$. □

Combining Theorem 2.13, Lemma 2.14 and the existence results of resolvable B_4-pairings established in the next subsection, we obtain the following theorem.

Theorem 2.15 *Suppose that $n \geq 0$ and $s \geq 1$. There exists an $RHF_4((3n+s)^3 : s)$. When $(n,s) \neq (1,1)$, there exists an $RHF_4((3n+s)^3 : s)$ with a sub-design $RH(4^4)$.*

As a consequence of Theorem 2.15, we have our third tripling construction as follows.

Corollary 2.16 (Tripling Construction III) *Let $n \equiv 2s \pmod 3$ and $s \geq 1$. If there exists an $IRH(4^n : 4^s)$, then there exist both an $IRH(4^{3n-2s} : 4^n)$ and an $IRH(4^{3n-2s} : 4^s)$. Furthermore, if there exists an $RH(4^n)$ or an $RH(4^s)$, then there exists an $RH(4^{3n-2s})$, as well as an $IRH(4^{3n-2s} : 4^4)$ when $(n,s) \neq (5,1)$.*

2.4 Construction of resolvable B_4-pairings

In order to construct resolvable B_4-pairings, we describe a special class of B_4-pairings with extra properties. Suppose that $D, R_i, PR_i, i \in \{0,1,2\}$ form a $B_4(n,s)$ on $Z_{4(3n+s)}$. If there exist three subsets A_0, A_1, A_2 of $Z_{4(3n+s)}$ and three subsets PA_0, PA_1, PA_2 of $Z_{4(3n+s)} \times Z_{4(3n+s)}$ satisfying the following conditions for each $i \in \{0,1,2\}$:

(1) $R_i = -R_i$, $A_i \subset R_i$, $|A_i| = 2n$,
(2) PA_i is a partition of A_i into pairs. Let $O'_i = \{|x-y| : \{x,y\} \in PA_i\}$,

 (a) $|O'_i| = n$, all O'_0, O'_1, O'_2 are disjoint and of odd members,
 (b) $(\cup_{i=0}^{2} O'_i) \bigcap (N \bigcup (\cup_{i=0}^{2} L_i)) = \emptyset$,

then let

$$\begin{aligned} S_0 &= A_1 \cup A_2, S_1 = A_0 \cup (-A_2), S_2 = (-A_0) \cup (-A_1),\\ PS_0 &= PA_1 \cup PA_2, PS_1 = PA_0 \cup (-PA_2), PS_2 = (-PA_0) \cup (-PA_1),\\ \overline{R}_0 &= -(R_0 \backslash A_0), \overline{R}_1 = R_1 \backslash A_1 \quad \text{and} \quad \overline{R}_2 = -(R_2 \backslash A_2). \end{aligned}$$

It is readily checked that $D, R_i, PR_i, S_i, PS_i, \overline{R}_i, i \in \{0,1,2\}$ form an $RB_4(n,s)$.

Now, we are in a position to construct $RB_4(n,s)$ for any $n \geq 0$ and $s \geq 1$. We list the components $D, PR_i, PA_i, i \in \{0,1,2\}$ for short or $D, PR_i, PS_i, \overline{R}_i, i \in \{0,1,2\}$ fully.

Lemma 2.17 *For each pair of integers $n \geq 0$ and $s \geq 1$, there exists an $RB_4(n,s)$.*

Proof When $n = 0$, we take $D = Z_{4(3n+s)}$ and $R_i = S_i = \overline{R}_i = \emptyset$ for each $i \in \{0,1,2\}$. When $n > 0$, $s > 0$, the desired $RB_4(n,s)$ is constructed directly as follows:

(1) For s odd and n even, let

$D = \{(3n+s)j : 0 \leq j \leq 3\} \cup \{(3n+s)i + j : 0 \leq i \leq 3, 1 \leq j \leq (s-1)/2$ or $3n + (s-1)/2 + 1 \leq j \leq 3n+s-1\}$,
$PR_0 = \{\{j,-j\} : (s-1)/2+1 \leq j \leq (s-1)/2+n$ or $(3n+s)+(s-1)/2+n+1 \leq j \leq (3n+s)+(s-1)/2+2n\}$,
$PR_1 = \{\{j,-j\} : (s-1)/2+2n+1 \leq j \leq (s-1)/2+3n$ or $(3n+s)+(s-1)/2+1 \leq j \leq (3n+s)+(s-1)/2+n\}$,
$PR_2 = \{\{j,-j\} : (s-1)/2+n+1 \leq j \leq (s-1)/2+2n$ or $(3n+s)+(s-1)/2 + 2n+1 \leq j \leq (3n+s)+(s-1)/2+3n\}$,
$PA_0 = \{\{(s-1)/2+j, 8n+3s-(s-1)/2-j\} : 1 \leq j \leq n\}$,
$PA_1 = \{\{(s-1)/2+2n+j, 4n+s+(s-1)/2-j\} : 1 \leq j \leq n-1\} \cup \{\{10n - (s-1)/2-1, 10n-(s-1)/2-2\}\}$,
$PA_2 = \{\{(s-1)/2+n+j, 6n+s+(s-1)/2+2-j\} : 2 \leq j \leq n\} \cup \{\{(s-1)/2 + n+1, 11n+4s-(s-1)/2-2\}\}$.

(2) For s even and n even, let

$D=\{(3n+s)j,(3n+s)/2+(3n+s)j:0\le j\le 3\}\cup\{(3n+s)i+j:0\le i\le 3, 1\le j\le (s-2)/2$ or $3n+s/2+1\le j\le 3n+s-1\}$,
$PR_0=\{\{j,-j\}:(s-2)/2+1\le j\le (s-2)/2+n$ or $(s-2)/2+n+1\le j\le 2n+(s-2)/2+1$ and $j\neq (3n+s)/2\}$,
$PR_1=\{\{j,-j\}:2n+(s-2)/2+2\le j\le 3n+s/2$ or $3n+s+(s-2)/2+n+1\le j\le 3n+s+(s-2)/2+2n+1$ and $j\neq 3n+s+(3n+s)/2\}$,
$PR_2=\{\{j,-j\}:3n+s+(s-2)/2+1\le j\le 3n+s+(s-2)/2+n$ or $5n+s+(s-2)/2+2\le j\le 6n+s+(s-2)/2\}$,
$PA_0=\{\{(s-2)/2+j,(s-2)/2+2n+1-j\}:1\le j\le n$ and $j\neq n/2\}\cup\{\{11n+3s+(s+2)/2,10n+(s+2)/2+3s-1\}\}$,
$PA_1=\{\{(s-2)/2+2n+1+j,8n+2s+(s+2)/2-j\}:1\le j\le n$ and $j\neq n/2\}\cup\{\{10n+3s+(s+2)/2-2,3n+s+(s-2)/2+2n+1\}\}$,
$PA_2=\{\{3n+s+(s-2)/2+j,7n+2s+(s+2)/2-1-j\}:1\le j\le n\}$.

(3) For s even and n odd,

(3.1) $n\ge 3$ odd, let
$D=\{(3n+s)j:0\le j\le 3\}\cup\{(3n+s)i+j:0\le i\le 3, 1\le j\le (s-2)/2$ or $3n+(s-2)/2+2\le j\le 3n+s-1\}\cup\{\pm((s-2)/2+1),\pm(6n+s+(s-2)/2+1)\}$,
$PR_0=\{\{j,-j\}:(s-2)/2+2\le j\le (s-2)/2+n+1$ or $(s-2)/2+2n+2\le j\le (s-2)/2+3n+1\}$,
$PR_1=\{\{j,-j\}:(s-2)/2+n+2\le j\le (s-2)/2+2n+1$ or $(5n+s)+(s-2)/2+1\le j\le (5n+s)+(s-2)/2+n\}$,
$PR_2=\{\{j,-j\}:3n+s+(s-2)/2+1\le j\le 3n+s+(s-2)/2+n$ or $3n+s+(s-2)/2+n+1\le j\le 3n+s+(s-2)/2+2n\}$,
$PA_0=\{\{(s-2)/2+2n+j,10n+4s-(s-2)/2-1-j\}:2\le j\le n\}\cup\{(s-2)/2+3,(s-2)/2+3n+1\}$,
$PA_1=\{\{(s-2)/2+n+j,6n+s+(s-2)/2+2-j\}:2\le j\le n\}\cup\{\{5n+s+(s-2)/2+1,11n+4s-(s-1)/2-2\}\}$,
$PA_2=\{\{(s-2)/2+3n+s+j,5n+s+(s-2)/2+1-j\}:1\le j\le n\}$.

(3.2) $n=1$, let
$D=\{(3+s)j:0\le j\le 3\}\cup\{(3+s)i+j:0\le i\le 3,1\le j\le (s-2)/2$ or $3+(s-2)/2+2\le j\le 3+s-1\}\cup\{\pm((s-2)/2+1),\pm((s-2)/2+2)\}$,
$PR_0=\{\{j,-j\}:(s-2)/2+3\le j\le (s-2)/2+4\}$,
$PR_1=\{\{j,-j\}:3+s+(s-2)/2+1\le j\le 3+s+(s-2)/2+2\}$,
$PR_2=\{\{j,-j\}:3+s+(s-2)/2+3\le j\le 3+s+(s-2)/2+4\}$,
$PA_0=\{\{(s-2)/2+3,(s-2)/2+4\}\}$,
$PA_1=\{\{3+s+(s-2)/2+2,8+2s+(s+2)/2\}\}$,
$PA_2=\{\{3+s+(s-2)/2+3,5+2s+(s+2)/2\}\}$.

(4) For s odd and n odd,

(4.1) $s\ge 3$ odd and $n\ge 3$ odd, let
$D=\{(3n+s)j,(3n+s)/2+(3n+s)j:0\le j\le 3\}\cup\{(3n+s)i+j:0\le i\le 3,1\le j\le (s-3)/2$ or $3n+(s-3)/2+3\le j\le 3n+s-1\}\cup\{\pm((s-3)/2+1),\pm(3n+s+3n+(s-3)/2+2)\}$,
$PR_0=\{\{j,-j\}:(s-3)/2+2\le j\le (s-3)/2+n+1$ or $(s-3)/2+n+2\le j\le 2n+(s-3)/2+2$ and $j\neq (3n+s)/2\}$,

$PR_1 = \{\{j, -j\} : 2n + (s-3)/2 + 3 \le j \le 3n + (s-3)/2 + 2$ or $3n + s + (s-3)/2 + n + 1 \le j \le 3n + s + (s-3)/2 + 2n + 1$ and $j \ne 3n + s + (3n+s)/2\}$,
$PR_2 = \{\{j, -j\} : 3n + s + (s-3)/2 + 1 \le j \le 3n + s + (s-3)/2 + n$ or $5n + s + (s-3)/2 + 2 \le j \le 6n + s + (s-3)/2 + 1\}$,
$PA_0 = \{\{(s-3)/2 + j, (s-3)/2 + 2n + 3 - j\} : 2 \le j \le n + 1$ and $j \ne n + 3 - (n+3)/2\} \cup \{\{11n + 3s + (s+3)/2 - 2, 10n + (s+3)/2 + 3s - 2\}\}$,
$PA_1 = \{\{(s-3)/2 + 2n + 2 + j, 8n + 2s + (s+3)/2 - j\} : 1 \le j \le n$ and $j \ne (n-1)/2 + 2\} \cup \{\{10n + 3s + (s+3)/2 - 4, 3n + s + (s-3)/2 + 2n + 1\}\}$,
$PA_2 = \{\{3n + s + (s-3)/2 + j, 7n + 2s + (s+3)/2 - 1 - j\} : 1 \le j \le n\}$.

(4.2) $s \ge 3$ odd and $n = 1$ odd, let
$D = \{(3+s)j, (3+s)/2 + (3+s)j : 0 \le j \le 3\} \cup \{(3+s)i + j : 0 \le i \le 3, 1 \le j \le (s-3)/2$ or $3 + (s+3)/2 \le j \le 3 + s - 1\} \cup \{\pm((s-3)/2 + 1), \pm((s-3)/2 + 2)\}$,
$PR_0 = \{\{j, -j\} : (s+3)/2 + 1 \le j \le (s+3)/2 + 2\}$,
$PR_1 = \{\{j, -j\} : 3 + s + (s-3)/2 + 1 \le j \le 3 + s + (s-3)/2 + 2\}$,
$PR_2 = \{\{j, -j\} : 3 + s + (s+3)/2 + 1 \le j \le 3 + s + (s+3)/2 + 1\}$,
$PA_0 = \{\{(s+3)/2 + 1, (s+3)/2 + 2\}\}$,
$PA_1 = \{\{3 + s + (s-3)/2 + 1, 7 + 2s + (s+3)/2\}\}$,
$PA_2 = \{\{3 + s + (s+3)/2 + 1, 7 + 2s + (s-3)/2\}\}$.

(4.3) For $s = 1$ and $n \ge 3$ odd, let
$D = \{(3n+1)i : 0 \le i \le 3\}$,
$PR_0 = \{\{j, -j\} : 1 \le j \le (n+1)/2$ or $(3n+1)/2 + n + 1 \le j \le 3n\} \cup \{\{j, -j-1\} : 3n + 1 + (n+1)/2 + 1 \le j \le 3n + 1 + (3n+1)/2 - 1\} \cup \{\{3n + 1 + (3n+1)/2, 3(3n+1) - (n+1)/2 - 1\}\}$,
$PR_1 = \{\{j, -j\} : 3n + 1 + 1 \le j \le 3n + 1 + (n+1)/2$ or $3n + 1 + (3n+1)/2 + n + 1 \le j \le 2(3n+1) - 1\} \cup \{\{j, -j-1\} : (n+1)/2 + 1 \le j \le (3n+1)/2 - 1\} \cup \{\{(3n+1)/2, 4(3n+1) - (n+1)/2 - 1\}\}$,
$PR_2 = \{\{j, -j\} : (3n+1)/2 + 1 \le j \le (3n+1)/2 + n$ or $3n + 1 + (3n+1)/2 + 1 \le j \le 3n + 1 + (3n+1)/2 + n\}$,
$PA_0 = \{\{j, -j-1\} : 3n + 1 + (n+1)/2 + 1 \le j \le 3n + 1 + (3n+1)/2 - 1\} \cup \{\{(n+1)/2 - 1, 4(3n+1) - (n+1)/2\}\}$,
$PA_1 = \{\{j, -j-1\} : (n+1)/2 + 1 \le j \le (3n+1)/2 - 1\} \cup \{\{3n+2, 3n+3\}\}$,
$PA_2 = \{\{(3n+1)/2 + 1 + j, 3(3n+1) - (3n+1)/2 - j\} : 1 \le j \le (n+1)/2\} \cup \{\{3n + 1 + (3n+1)/2 + j, 3n + 1 + (3n+1)/2 + n - j\} : 1 \le j \le (n-3)/2\} \cup V$, where $V = \{\{3n + 1 + (3n+1)/2 + n, 3(3n+1) - (3n+1)/2 - n + 1\}\}$ for $n \ge 5$ and $V = \{\{17, 22\}\}$ when $n = 3$.

(4.4) For $s = 1$ and $n = 1$, let
$D = \{0, 1, 8, 15\}$,
$PR_0 = \{\{2, 3\}, \{4, 6\}\}$, $PR_1 = \{\{5, 11\}, \{9, 14\}\}$, $PR_2 = \{\{7, 13\}, \{10, 12\}\}$,
$PS_0 = \{\{7, 10\}, \{9, 14\}\}$, $PS_1 = \{\{6, 7\}, \{10, 13\}\}$, $PS_2 = \{\{2, 3\}, \{6, 9\}\}$,
$\overline{R}_0 = \{4, 14\}$, $\overline{R}_1 = \{5, 11\}$, $\overline{R}_2 = \{3, 4\}$. □

3 Existence of RH(4^n)

In this section, we shall establish the existence of resolvable H-designs with group size 4 by using the recursive constructions developed in Sect. 2. First, we need the following initial designs.

Lemma 3.1 *There exists an RH*(4^5).

Proof Let the point set be Z_{20}, and the group set be $\{\{j, j+5, j+10, j+15\} : j = 0, 1, \ldots, 4\}$. We list the base blocks as follows, which are developed by adding 2 modulo 20:

{3, 4, 7, 11}	{6, 10, 17, 18}	{0, 2, 9, 13}	{5, 8, 19, 1}	{12, 14, 15, 16}
{8, 9, 16, 17}	{11, 19, 0, 2}	{18, 4, 12, 15}	{3, 6, 10, 14}	{1, 5, 7, 13}
{1, 7, 9, 18}	{11, 13, 14, 15}	{19, 5, 6, 12}	{8, 10, 2, 4}	{16, 17, 0, 3}
{1, 12, 18, 19}				

Each of the first three rows forms a parallel class. The last block covers the four residues modulo 4, hence gives a parallel class by adding 4 modulo 20. □

Lemma 3.2 *There exists an RH*(4^7).

Proof Let the point set be Z_{28}, and the group set be $\{\{j, j+7, j+14, j+21\} : j = 0, 1, \cdots, 6\}$. We list the base blocks as follows, each of which is developed by adding 2 modulo 28:

{3, 7, 11, 23}	{27, 9, 15, 17}	{13, 14, 19, 1}	{2, 4, 10, 20}	{18, 22, 26, 6}
{24, 0, 5, 16}	{8, 12, 21, 25}			
{0, 1, 9, 12}	{21, 25, 2, 10}	{18, 20, 5, 14}	{22, 24, 27, 4}	{13, 15, 17, 23}
{16, 19, 7, 8}	{26, 3, 6, 11}			
{3, 6, 18, 21}	{8, 9, 19, 24}	{20, 5, 7, 11}	{10, 15, 16, 0}	{4, 14, 17, 1}
{25, 2, 12, 13}	{22, 23, 26, 27}			
{2, 4, 6, 24}	{1, 5, 7, 23}	{9, 12, 20, 21}	{16, 18, 22, 0}	{3, 8, 11, 13}
{10, 15, 19, 27}	{14, 17, 25, 26}			
{3, 4, 7, 8}	{21, 26, 2, 13}	{22, 23, 24, 25}	{14, 17, 20, 11}	{18, 19, 0, 9}
{27, 1, 10, 12}	{16, 5, 6, 15}			
{2, 7, 13, 24}	{5, 7, 22, 24}	{7, 12, 13, 18}	{12, 18, 21, 27}	{13, 15, 16, 18}

The seven blocks in the ith and $(i+1)$th rows form a parallel class for each $i = 1, 3, 5, 7, 9$. Each block of the last row covers the four residues modulo 4, hence gives a parallel class by adding 4 modulo 28. □

Lemma 3.3 *There exists an RHF*$_4(3^5 : 2)$.

Proof We first construct an $\mathrm{HF}_2(3^5 : 2)$ on $Z_{30} \cup \{\infty_0, \ldots, \infty_3\}$, with groups $G'_j = \{j, j+15\}$, $j = 0, 1, \ldots, 14$, $G'_{\infty_i} = \{\infty_i, \infty_{i+2}\}$, $i = 0, 1$, five holes $F'_i = \{G'_i, G'_{i+5}, G'_{i+10}\} \cup S'$, $i = 0, 1, \ldots, 4$ and a common hole $S' = \{G'_{\infty_0}, G'_{\infty_1}\}$. We list below the set of base blocks $\mathcal{B}' = \Delta \cup \Theta$, which will be developed under the automorphism group $\langle\alpha'\rangle$, where $\alpha' = (0\ 1\ 2\ 3 \ldots 28\ 29)$.

Δ :	{0, 1, 13, 22}	{0, 3, 4, 7}	{0, 14, 16, 27}	{0, 6, 18, 19}
	{0, 3, 6, 24}	{0, 19, 21, 22}	{0, 1, 2, 8}	{0, 11, 19, 27}

	$\{0, 2, 29, \infty_0\}$	$\{0, 4, 22, \infty_0\}$	$\{0, 7, 16, \infty_0\}$	$\{0, 6, 17, \infty_0\}$
	$\{0, 3, 12, \infty_1\}$	$\{0, 2, 24, \infty_1\}$	$\{0, 16, 29, \infty_1\}$	$\{0, 4, 11, \infty_1\}$
	$\{0, 19, 28, \infty_2\}$	$\{0, 13, 27, \infty_2\}$	$\{0, 8, 26, \infty_2\}$	$\{0, 6, 7, \infty_2\}$
	$\{0, 3, 9, \infty_3\}$	$\{0, 22, 29, \infty_3\}$	$\{0, 14, 26, \infty_3\}$	$\{0, 11, 13, \infty_3\}$
Θ :	{0, 2, 18, 28}	{0, 5, 14, 18}	{0, 1, 14, 19}	{0, 2, 25, 27}
	{0, 3, 8, 25}	{0, 7, 12, 28}	{0, 7, 14, 25}	{0, 1, 6, 25}
	{0, 10, 19, 26}	{0, 9, 10, 29}	{0, 12, 20, 22}	{0, 6, 16, 22}
	{0, 3, 20, 23}	{0, 21, 25, 26}	{0, 7, 17, 24}	{0, 10, 21, 28}
	{0, 20, 24, 26}	{0, 13, 17, 21}		

For each block $B = \{a, b, c, d\} \in \mathcal{B}'$, construct an RH($2^4$) with group set $\{\{x, x'\} : x \in B\}$, where $x' = x + 30$ when $x \in Z_{30}$ or $x' = \infty_{i+4}$ when $x = \infty_i$, and block set $\mathcal{A}_B$ having a resolution $P_B(1) = \{\{a, b, c, d\}, \{a', b', c', d'\}\}$, $P_B(2) = \{\{a, b, c', d'\}, \{a', b', c, d\}\}$, $P_B(3) = \{\{a, b', c, d'\}, \{a', b, c', d\}\}$, $P_B(4) = \{\{a, b', c', d\}, \{a', b, c, d'\}\}$. Let $\mathcal{B} = \cup_{B\in\mathcal{B}'}\mathcal{A}_B$. It is clear that $\mathcal{B}$ is the set of base blocks of an HF$_4$(3^5 : 2) on $X = Z_{60} \cup \{\infty_0, \ldots, \infty_7\}$ with the group set $G_j = \{j + 15k : 0 \le k \le 3\}, j = 0, 1, \cdots, 14$, $G_{\infty_i} = \{\infty_{i+2k} : 0 \le k \le 3\}, i = 0, 1$, five holes $F_i = \{G_i, G_{i+5}, G_{i+10}\} \cup S$, $i \in Z_5$, a common hole $S = \{G_{\infty_0}, G_{\infty_1}\}$ and an automorphism group $\langle\alpha\rangle$, where $\alpha = (0\ 1\ 2\ 3 \ldots 28\ 29\ )(30\ 31\ 32\ 33 \ldots 58\ 59)$. Now, we need to give the resolution. The design should contain 16×30 parallel classes on X and 8×4 partial parallel classes on $X \backslash (\cup_{G\in F_i} G)$ for each $i \in Z_5$.

Note that each block $B \in \Delta$ covers all but one, say j, distinct residues modulo 5. Then for each $i \in \{1, 2, 3, 4\}$ and a fixed $s \in Z_5$, $P_B(i)$ gives a partial parallel class on $X \backslash (\cup_{G\in F_{j+s}} G)$ when developed by the automorphisms $\{\alpha^{5k+s} : k \in Z_6\}$. That is, $\cup_{B\in\Delta}\mathcal{A}_B$ gives 32 partial parallel classes on $X \backslash (\cup_{G\in F_i} G)$ for each $i \in Z_5$ when developed under $\langle\alpha\rangle$.

Then we shift each block $B \in \cup_{B\in\Theta}\mathcal{A}_B$ by a suitable automorphism $\alpha_B \in \langle\alpha\rangle$. The result is listed below, where the blocks in each of the four consecutive rows, namely the ith, $(i + 1)$th, $(i + 2)$th and $(i + 3)$th rows for $i \in \{4k + 1 : k = 0, 1, \ldots, 15\}$, form a parallel class.

$\{1, 37, 38, \infty_2\}$	$\{3, 44, 46, \infty_3\}$	$\{6, 40, 28, \infty_4\}$	$\{34, 7, 16, \infty_5\}$	$\{32, 8, 9, \infty_6\}$
$\{30, 11, 13, \infty_7\}$	$\{31, 36, 45, 49\}$	$\{35, 10, 19, 53\}$	$\{15, 20, 59, 33\}$	$\{57, 29, 52, 24\}$
$\{18, 51, 26, 43\}$	$\{47, 50, 25, 12\}$	$\{55, 27, 54, \infty_0\}$	$\{17, 48, 23, 42\}$	$\{4, 41, 21, 58\}$
$\{39, 22, 56, 0\}$	$\{2, 5, 14, \infty_1\}$			
$\{32, 6, 24, \infty_4\}$	$\{2, 5, 44, \infty_5\}$	$\{3, 9, 40, \infty_6\}$	$\{8, 19, 51, \infty_7\}$	$\{35, 10, 49, 23\}$
$\{11, 46, 55, 29\}$	$\{33, 38, 17, 21\}$	$\{57, 28, 41, 16\}$	$\{12, 13, 48, 37\}$	$\{59, 39, 18, 25\}$
$\{36, 43, 20, 1\}$	$\{0, 4, 22, \infty_0\}$	$\{56, 30, 7, \infty_1\}$	$\{26, 45, 54, \infty_2\}$	$\{50, 27, 34, 15\}$
$\{52, 53, 58, 47\}$	$\{31, 42, 14, \infty_3\}$			
$\{55, 26, 9, 44\}$	$\{10, 12, 5, 7\}$	$\{54, 57, 14, 17\}$	$\{13, 49, 30, \infty_0\}$	$\{4, 46, 24, 56\}$
$\{37, 43, 23, 29\}$	$\{52, 25, 31, \infty_3\}$	$\{45, 48, 27, \infty_1\}$	$\{33, 39, 50, \infty_4\}$	$\{8, 20, 28, 0\}$
$\{19, 35, 18, \infty_5\}$	$\{51, 59, 47, \infty_6\}$	$\{2, 40, 58, \infty_2\}$	$\{42, 15, 21, \infty_7\}$	$\{36, 38, 1, 3\}$
$\{34, 11, 16, 32\}$	$\{22, 53, 6, 41\}$			
$\{1, 15, 27, \infty_3\}$	$\{32, 39, 48, \infty_4\}$	$\{34, 36, 58, \infty_5\}$	$\{40, 59, 38, \infty_6\}$	$\{51, 35, 47, \infty_7\}$
$\{2, 3, 16, 21\}$	$\{10, 41, 54, 29\}$	$\{17, 49, 12, 44\}$	$\{53, 25, 18, 50\}$	$\{56, 5, 6, 55\}$
$\{7, 20, 24, 28\}$	$\{9, 52, 26, 30\}$	$\{14, 46, 43, \infty_0\}$	$\{0, 4, 11, \infty_1\}$	$\{45, 22, 57, 13\}$
$\{23, 33, 42, 19\}$	$\{31, 37, 8, \infty_2\}$			
$\{45, 28, 12, \infty_6\}$	$\{52, 58, 38, 44\}$	$\{39, 25, 8, \infty_5\}$	$\{43, 19, 29, 35\}$	$\{4, 41, 16, 32\}$
$\{21, 27, 7, 13\}$	$\{53, 0, 10, 47\}$	$\{33, 9, 50, \infty_0\}$	$\{15, 22, 59, 40\}$	$\{55, 17, 54, \infty_3\}$
$\{1, 34, 51, 24\}$	$\{23, 56, 31, 18\}$	$\{49, 5, 48, \infty_1\}$	$\{3, 46, 30, \infty_2\}$	$\{42, 14, 11, \infty_4\}$
$\{57, 6, 37, 26\}$	$\{36, 20, 2, \infty_7\}$			
$\{51, 58, 33, 49\}$	$\{50, 42, 19, \infty_3\}$	$\{8, 40, 2, \infty_5\}$	$\{20, 57, 36, \infty_0\}$	$\{48, 24, 34, 10\}$
$\{31, 11, 22, 59\}$	$\{3, 6, 53, 56\}$	$\{43, 46, 52, \infty_7\}$	$\{4, 44, 23, 30\}$	$\{0, 37, 14, 55\}$
$\{17, 25, 13, \infty_2\}$	$\{27, 29, 21, \infty_1\}$	$\{32, 9, 18, \infty_4\}$	$\{16, 54, 12, \infty_6\}$	$\{1, 38, 45, 26\}$
$\{7, 47, 28, 35\}$	$\{15, 5, 39, 41\}$			
$\{17, 20, 25, 12\}$	$\{2, 53, 27, 58\}$	$\{30, 34, 22, \infty_0\}$	$\{44, 47, 23, \infty_3\}$	$\{37, 9, 1, \infty_5\}$
$\{18, 21, 56, 43\}$	$\{3, 46, 50, 24\}$	$\{26, 59, 16, 49\}$	$\{5, 36, 41, 0\}$	$\{38, 45, 55, 32\}$
$\{48, 39, 13, 14\}$	$\{6, 19, 33, \infty_6\}$	$\{28, 31, 40, \infty_1\}$	$\{10, 29, 8, \infty_2\}$	$\{52, 54, 51, \infty_4\}$
$\{15, 57, 35, 7\}$	$\{42, 4, 11, \infty_7\}$			
$\{24, 1, 40, \infty_4\}$	$\{10, 30, 4, 36\}$	$\{33, 6, 53, 26\}$	$\{11, 0, 39, \infty_6\}$	$\{59, 32, 49, 52\}$

$\{54, 8, 50, \infty_3\}$	$\{16, 18, 41, 43\}$	$\{56, 42, 55, \infty_5\}$	$\{46, 48, 15, \infty_0\}$	$\{47, 21, 58, \infty_1\}$
$\{5, 38, 14, \infty_7\}$	$\{19, 22, 9, 12\}$	$\{7, 28, 2, 3\}$	$\{20, 27, 37, 44\}$	$\{57, 17, 51, 23\}$
$\{35, 25, 29, 31\}$	$\{45, 34, 13, \infty_2\}$			
$\{0, 7, 12, 28\}$	$\{23, 55, 47, \infty_1\}$	$\{18, 58, 39, 16\}$	$\{34, 44, 53, 30\}$	$\{20, 33, 17, \infty_6\}$
$\{41, 21, 32, 9\}$	$\{31, 11, 50, 27\}$	$\{24, 4, 45, 52\}$	$\{19, 29, 8, 15\}$	$\{22, 36, 48, \infty_3\}$
$\{40, 42, 35, 37\}$	$\{13, 26, 10, \infty_2\}$	$\{57, 3, 14, \infty_4\}$	$\{25, 59, 6, \infty_5\}$	$\{2, 54, 1, \infty_7\}$
$\{43, 46, 51, 38\}$	$\{49, 56, 5, \infty_0\}$			
$\{16, 18, 40, \infty_5\}$	$\{23, 9, 22, \infty_1\}$	$\{49, 56, 1, 17\}$	$\{41, 54, 8, \infty_2\}$	$\{31, 5, 53, \infty_0\}$
$\{10, 2, 39, \infty_7\}$	$\{55, 3, 21, \infty_6\}$	$\{52, 4, 12, 44\}$	$\{45, 57, 35, 37\}$	$\{15, 28, 32, 36\}$
$\{51, 42, 46, 47\}$	$\{14, 6, 13, \infty_3\}$	$\{27, 59, 26, \infty_4\}$	$\{48, 19, 24, 43\}$	$\{38, 20, 58, 0\}$
$\{34, 25, 29, 30\}$	$\{50, 33, 7, 11\}$			
$\{36, 25, 4, \infty_6\}$	$\{8, 11, 17, \infty_3\}$	$\{30, 50, 54, 56\}$	$\{21, 58, 38, 15\}$	$\{27, 29, 26, \infty_0\}$
$\{47, 19, 41, \infty_1\}$	$\{28, 34, 44, 20\}$	$\{32, 45, 49, 53\}$	$\{43, 35, 42, \infty_7\}$	$\{40, 23, 37, \infty_2\}$
$\{24, 31, 10, \infty_4\}$	$\{51, 22, 57, 16\}$	$\{0, 6, 46, 52\}$	$\{48, 1, 5, 39\}$	$\{33, 7, 14, \infty_5\}$
$\{59, 9, 18, 55\}$	$\{3, 12, 13, 2\}$			
$\{27, 36, 37, 26\}$	$\{23, 30, 35, 21\}$	$\{39, 47, 5, \infty_2\}$	$\{46, 32, 15, \infty_1\}$	$\{53, 54, 29, 18\}$
$\{57, 34, 41, 52\}$	$\{59, 43, 25, \infty_3\}$	$\{16, 2, 45, \infty_5\}$	$\{40, 50, 1, 8\}$	$\{19, 55, 6, \infty_4\}$
$\{7, 17, 56, 33\}$	$\{3, 9, 20, \infty_0\}$	$\{31, 44, 58, \infty_6\}$	$\{10, 13, 49, \infty_7\}$	$\{4, 24, 28, 0\}$
$\{48, 38, 12, 14\}$	$\{42, 51, 22, 11\}$			
$\{0, 34, 52, \infty_0\}$	$\{30, 3, 42, \infty_1\}$	$\{1, 7, 8, \infty_2\}$	$\{2, 13, 15, \infty_3\}$	$\{31, 35, 53, \infty_4\}$
$\{33, 36, 45, \infty_5\}$	$\{32, 38, 39, \infty_6\}$	$\{44, 55, 57, \infty_7\}$	$\{24, 26, 12, 22\}$	$\{18, 50, 6, 46\}$
$\{56, 28, 14, 54\}$	$\{41, 43, 29, 9\}$	$\{19, 21, 37, 47\}$	$\{49, 58, 59, 48\}$	$\{11, 20, 51, 40\}$
$\{25, 5, 16, 23\}$	$\{10, 17, 27, 4\}$			
$\{0, 7, 16, \infty_0\}$	$\{30, 32, 24, \infty_1\}$	$\{33, 9, 40, \infty_2\}$	$\{31, 12, 44, \infty_3\}$	$\{1, 5, 53, \infty_4\}$
$\{2, 35, 14, \infty_5\}$	$\{3, 39, 10, \infty_6\}$	$\{15, 56, 28, \infty_7\}$	$\{50, 52, 38, 48\}$	$\{55, 27, 43, 23\}$
$\{19, 51, 37, 17\}$	$\{8, 13, 22, 26\}$	$\{6, 41, 20, 54\}$	$\{58, 59, 42, 47\}$	$\{4, 11, 18, 29\}$
$\{36, 46, 57, 34\}$	$\{25, 45, 49, 21\}$			
$\{30, 7, 46, \infty_0\}$	$\{0, 34, 41, \infty_1\}$	$\{31, 20, 59, \infty_2\}$	$\{3, 36, 42, \infty_3\}$	$\{2, 8, 49, \infty_4\}$
$\{43, 47, 54, \infty_5\}$	$\{1, 9, 57, \infty_6\}$	$\{29, 13, 55, \infty_7\}$	$\{38, 39, 22, 27\}$	$\{50, 23, 28, 45\}$
$\{37, 14, 21, 32\}$	$\{24, 33, 4, 53\}$	$\{10, 11, 16, 5\}$	$\{6, 18, 56, 58\}$	$\{19, 40, 44, 15\}$
$\{26, 17, 51, 52\}$	$\{48, 25, 35, 12\}$			
$\{30, 36, 17, \infty_0\}$	$\{0, 46, 59, \infty_1\}$	$\{51, 29, 47, \infty_2\}$	$\{5, 57, 34, \infty_3\}$	$\{11, 13, 40, \infty_4\}$
$\{3, 7, 44, \infty_5\}$	$\{16, 35, 14, \infty_6\}$	$\{23, 37, 19, \infty_7\}$	$\{24, 25, 38, 43\}$	$\{1, 33, 56, 28\}$
$\{53, 26, 31, 18\}$	$\{27, 4, 39, 55\}$	$\{48, 21, 8, 41\}$	$\{50, 32, 10, 12\}$	$\{6, 42, 22, 58\}$
$\{54, 15, 49, 20\}$	$\{45, 52, 2, 9\}$			

□

As a corollary of the Tripling Construction III, we obtain

Theorem 3.4 *If there exists a constant $M \geq 6$, such that for every $n \equiv 1, 2 \pmod 3$ in the range $M \leq n < 3M$, there exists an IRH$(4^n : 4^{17})$, then for every $n \equiv 1, 2 \pmod 3$ and $n \geq M$, there exists an IRH$(4^n : 4^{17})$.*

Proof First, we claim that there exists an IRH$(4^{17} : 4^s)$ for each $s \in \{1, 2, 4, 5, 7\}$. Applying the Tripling Construction III with $(n, s) = (7, 2)$ and an RH(4^7) in Lemma 3.2, we obtain an RH(4^{17}), an IRH$(4^{17} : 4^4)$ and an IRH$(4^{17} : 4^7)$. An IRH$(4^{17} : 4^5)$ can be constructed by applying Theorem 2.5 with an RHF$_4(3^5 : 2)$ in Lemma 3.3 and an RH(4^5) in Lemma 3.1. The designs with a hole of sizes 1 or 2 are actually an RH(4^{17}).

The above statement yields that the existence of an IRH$(4^n : 4^{17})$ implies the existence of an IRH$(4^n : 4^s)$ for all $s \in \{1, 2, 4, 5, 7, 17\}$. We proceed the proof by induction.

Let $n \geq 3M$ and $n \equiv 1, 2 \pmod 3$. Assume that for each $n', M \leq n' < n, n' \equiv 1, 2 \pmod 3$, there exists an IRH$(4^{n'} : 4^{17})$. Write $n = 3m - 2s$, where $s = 7, 5, 1, 17, 4, 2$ when $n \equiv 1, 2, 4, 5, 7, 8 \pmod 9$, respectively. It is easy to check that $M \leq m < n$, $m \equiv 1, 2 \pmod 3$. Applying the Tripling Construction III, the conclusion then follows. □

Lemma 3.5 *For each integer* $n \equiv 1, 2 \pmod 3$, $n \geq 4$ *and* $n \notin \{73, 149, 181, 599\}$*, there exists an RH*$(4^n)$.

Proof Let L be the list of pairs (n, s) such that an IRH$(4^n : 4^s)$ is known. For every two pairs (n, s) and (n', s'), define $(n, s) \prec (n', s')$ if $n < n'$or, $n = n'$ and $s < s'$. We will compute the output of the Tripling Constructions I, II and III, the Doubling Construction and the Product Construction by a computer programme, which involves the following steps:

Step 1: Initialize L. Let $L = \{(4, 1), (4, 2), (5, 1), (5, 2), (7, 1), (7, 2), (13, 1), (13, 2), (13, 5), (19, 1), (19, 2), (41, 1), (41, 2)\}$. The designs with 13 groups can be constructed by applying Tripling Construction III with $(n, s) = (5, 1)$. The designs with 19 or 41 groups are constructed directly based on the corresponding block sets appeared in [2, Lemmas 5.4 and 5.2]. In order to save space, we post these two designs on the new results website for Handbook of Combinatorial Designs [18] maintained by Professor Jeff Dinitz of the University of Vermont. Sort L in ascending order. Let (n, s) be the smallest pair in L.

Step 2: Check whether (n, s) satisfies Tripling Construction III's condition, i.e., $n \equiv 2s \pmod 3$ and $(n, s) \neq (5, 1)$. If not, go to Step 3. If yes, update L by adding pairs $(3n - 2s, n)$, $(3n - 2s, 4)$ and $(3n - 2s, k)$ for all k such that $(n, k) \in L$. Sort the updated L in ascending order, then go to Step 4.

Step 3: Check whether $n - s \equiv 0 \pmod 3$. If not, go to Step 4. If yes, write $n - s = 3^x \cdot t$, such that $t > s$ and $3 \nmid t$, or $s < t < 3s$ and $3 | t$. Check whether $(t+s, s)$ satisfies Tripling Construction III's condition, i.e., $t + s \equiv 2s \pmod 3$ and $(t + s, s) \neq (5, 1)$, or Tripling Construction II's condition, i.e., $t \equiv 0 \pmod 3$ and $9s \geq 5t$. If yes, update L by adding pairs $(3n - 2s, n)$ and $(3n - 2s, k)$ for all k such that $(n, k) \in L$. Furthermore, add $(3n - 2s, 4)$ into L if $(t + s, s)$ satisfies Tripling Construction III's condition. Sort the updated L in ascending order, then go to Step 4.

Step 4: Apply the Doubling Construction and the Product Construction. Update L by adding the pair $(2n, k)$ for all k such that $(n, k) \in L$. For each m such that $(m, 1) \in L$, update L by adding pairs (mn, n), (mn, m) and (mn, k) for all k such that $(n, k) \in L$ or $(m, k) \in L$. Sort the updated L in ascending order. Let (n, s) be the next smallest pair in the updated L, then go to Step 2.

The programme was run with $n < 2000$ and $s \leq 64$, and produced two results as follows:

Result 1: For each $n \equiv 1, 2 \pmod 3$ and $4 \leq n < 1285$, there exists an RH(4^n) with four possible exceptions $\{73, 149, 181, 599\}$.

Result 2: There exists an IRH$(4^n : 4^{17})$ for all $n \equiv 1, 2 \pmod 3$ and $1285 \leq n < 3855$.

By Theorem 3.4, there exists an IRH$(4^n : 4^{17})$ for all $n \equiv 1, 2 \pmod 3$ and $n \geq 1285$. Hence there exists an RH(4^n) by Theorem 2.5. This completes the proof. □

Lemma 3.6 *There exists an RH*(4^n) *for each* $n \in \{181, 599\}$.

Proof For $n = 181$, there exists an RCQS$(1^{15} : 1)$ obtained from an RSQS(16). By Theorem 2.15, there exists an RHF$_4(4^3 : 1)$, thus an RHF$_4(12^3 : 1)$ exists by Tripling

Construction I. Applying Theorem 2.4 with an RH(48^4) and an RCQS(1^{15} : 1), we get an RHF$_4$(12^{15} : 1). Then applying Theorem 2.5 with an RH(4^{13}), we obtain an RH(4^{181}).

For $n = 599$, there exists an RCQS(1^7 : 1) obtained from an RSQS(8). By Theorem 2.15, there exists an RHF$_4$(85^3 : 4). Applying Theorem 2.4 with the RCQS(1^7 : 1), the RHF$_4$(85^3 : 4) and an RH(340^4), we get an RHF$_4$(85^7 : 4). Applying Theorem 2.5 with an IRH(4^{89} : 4^4) gives the desired RH(4^{599}). Here, the input IRH(4^{89} : 4^4) can be constructed by applying Tripling Construction III with $(n, s) = (31, 2)$ and an RH(4^{31}). □

Combining Lemmas 3.5 and 3.6, we obtain the main result in this section.

Theorem 3.7 *The necessary conditions* $n \equiv 1$ *or* $2 \pmod 3$ *and* $n \geq 4$ *for the existence of an RH*(4^n) *are sufficient except possibly for* $n \in \{73, 149\}$.

4 Conclusions

The existence problem for resolvable Steiner quadruple systems is a challenging one in combinatorial designs theory. A complete solution was obtained by a joint effort of Hartman [8,9] and Ji and Zhu [12] over twenty years long. In this section, we will provide an alternative existence proof for resolvable SQS(v)s. This new proof is beneficial not only from the tripling constructions, but also from the Group Halving Construction developed in this paper.

First, we establish the existence result of resolvable H-designs with group size 2. As a corollary of Theorem 3.7, we have the following result by the Group Halving Construction.

Lemma 4.1 *There exists an RH*(2^n) *for each* $n \equiv 2, 4 \pmod 6$ *and* $n \notin \{146, 298\}$.

Lemma 4.2 *There exists an RH*(2^{146}) *and an RH*(2^{298}).

Proof An RH(2^{146}) was constructed in [12]. For RH(2^{298}), there exists an RHF$_2$(1^3 : 1) which is actually an RH(2^4). By the Tripling Construction I, there is an RHF$_2$(9^3 : 1) and an RHF$_2$(27^3 : 1). Applying Theorem 2.4 with an RCQS(3^5 : 1) from Theorem 2.3, an RHF$_2$(9^3 : 1) and an RH(18^4), we get an RHF$_2$(27^5 : 1). Start from an URCS(1^{11} : 1) with block sizes $k \in \{4, 6\}$, which is obtained from an RG(6^2) (see [16]). Applying Theorem 2.4 again with an RHF$_2$(27^{k-1} : 1) and an RH(54^k) for $k \in \{4, 6\}$, we get an RHF$_2$(27^{11} : 1). Applying Theorem 2.5 with an RH(2^{28}), we get an RH(2^{298}). Here, the input RH(54^6) can be obtained from an RH(6^6) (see [16]) by applying the Weighting Construction with $m = 9$. □

Combining Lemmas 4.1 and 4.2, we obtain

Theorem 4.3 *The necessary conditions* $n \equiv 2$ *or* $4 \pmod 6$ *and* $n \geq 4$ *for the existence of an RH*(2^n) *are also sufficient.*

As a consequence of Theorem 4.3, we have the following corollary by the Group Halving Construction.

Corollary 4.4 *The necessary condition* $v \equiv 4$ *or* $8 \pmod{12}$ *for the existence of an RSQS*(v) *is also sufficient.*

As the other consequence of Theorem 4.3, we reestablish the existence result for resolvable H-designs with group size 6. The following construction was proved in [16], which is similar to but much stronger than the Product Construction in Theorem 2.6.

Lemma 4.5 *Suppose that there exist both an RH*(g^{2u}) *and an RH*(g^{2t})*. Then there exists an RH*(g^{2ut}).

Theorem 4.6 *There exists an RH*(6^n) *for each* $n \equiv 0 \pmod 2$ *and* $n \geq 4$.

Proof For each $n \equiv 2$ or $4 \pmod 6$ and $n \geq 4$, there exists an RH(6^n) by applying the Weighting Construction with an RH(2^n) from Theorem 4.3 and $m = 3$.

For $n = 6$, there exists an RH(6^6) [16]. For each $n = 6h$ and $h \geq 2$, the proof proceeds by induction. Assume that for each $n' \equiv 0 \pmod 6$ and $n' < n$, there exists an RH$(6^{n'})$. Thus there exists an RH(6^k) for each $k \equiv 0 \pmod 2$ and $k < n$. By Lemma 4.5, an RH(6^n) exists since there exists an RH(6^6) and an RH(6^{2h}). □

As a corollary of Theorem 4.6, we have the following result by the Group Halving Construction.

Theorem 4.7 *The necessary conditions* $n \equiv 0 \pmod 4$ *and* $n \geq 4$ *for the existence of an RH*(3^n) *are also sufficient.*

According to the necessary conditions for the existence of an RH(g^n) and by the Weighting Construction, the general existence problem of RH(g^n) depends on the solution of the following six cases, which have been listed in [16]:

(1) $g = 1$ and $n \equiv 4, 8 \pmod{12}$,
(2) $g = 2$ and $n \equiv 2, 4 \pmod 6$,
(3) $g = 3$ and $n \equiv 0 \pmod 4$,
(4) $g = 4$ and $n \equiv 1, 2 \pmod 3$,
(5) $g = 6$ and $n \equiv 0 \pmod 2$,
(6) $g = 12$ and $n \in N$.

For Case (1), an RH(1^n) is actually an RSQS(n), whose existence has been solved completely [9,12]. For Cases (2) and (4), the existence of RH(2^n) and RH(4^n) were studied in this paper. For Cases (3) and (5), the existence of RH(g^n) was established in Theorems 4.7 and 4.6, respectively. Hence, the whole problem can be reduced to the odd orders of n in Case (6) and the two remaining orders of $n = 73, 149$ in Case (4), which will be an interesting topic for further investigation. Now, Theorem 1.1 can be updated as follows.

Theorem 4.8 *The necessary conditions* $gn \equiv 0 \pmod 4$, $g(n-1)(n-2) \equiv 0 \pmod 3$ *and* $n \geq 4$ *for the existence of a resolvable H-design of type* g^n *are also sufficient for each* $g \equiv 1, 2, 3, 5, 6, 7, 9, 10, 11 \pmod{12}$*, and also sufficient for each* $g \equiv 4, 8 \pmod{12}$ *with two possible exceptions* $n = 73, 149$.

As an application of the above existence result of resolvable H-designs, we give a complete solution to the existence problem of resolvable G-designs.

A *G-design* of order v with block sizes from K, denoted by $G(t, K, v)$, is a triple $(X, \mathcal{G}, \mathcal{A})$ that satisfies the following properties:

(1) X is a set of v elements;
(3) $\mathcal{G} = \{G_1, G_2, \ldots\}$ is a set of nonempty subsets of X, which partition X;
(4) $\mathcal{A}$ is a family of subsets of X, each of cardinality from K;
(5) every t-subset T of X with $|T \cap G_i| < t$, for all i, is contained in a unique block, and no t-subset of G_i, for any i, is contained in any block.

The *type* of the G(t, K, v) is defined as the list $(|G| | G \in \mathcal{G})$. In this paper, we denote a G$(3, \{4\}, v)$ of type g^n by G(g^n) for short. Recently, Zhuralev et al. [17] investigated the existence of such designs (called *group divisible Steiner quadruple systems* as in [17]). A table was provided that includes existence results when the number of points is not more than 24. They also proved the following theorem in [17].

Theorem 4.9 *There exists a* $G(g^n)$ *if and only if* $g = 1$ *and* $n \equiv 2\,or\,4 \pmod 6$*, or* g *is even and* $g(n-1)(n-2) \equiv 0 \pmod 3$.

A $G(g^n)$ is said to be *resolvable*, denoted by $RG(g^n)$, if its block set can be partitioned into parallel classes. It is clear that the necessary conditions for the existence of an $RG(g^n)$ are $g = 1$ and $n \equiv 4$ or $8 \pmod{12}$, or g is even, $gn \equiv 0 \pmod 4$ and $g(n-1)(n-2) \equiv 0 \pmod 3$. The following lemma was proved in [16].

Lemma 4.10 [16] *If there exists an* $RH(g^{2t})$ *with g even, then there exist both an* $RG((2g)^t)$ *and an* $RG(g^{2t})$.

Lemma 4.11 *If there exists an* $RG(g^n)$*, then there exists an* $RG((2mg)^n)$ *for any positive integer* m.

Proof Let $(X, \mathcal{G}, \mathcal{B})$ be the given $RG(g^n)$ with $\mathcal{G} = \{G_1, G_2, \ldots, G_n\}$ and $\mathcal{B}$ having a resolution P_i, $1 \le i \le r$, where $r = ((gn-1)(gn-2) - (g-1)(g-2))/6$. Let $X' = X \times Z_{2m}$ and $G'_k = G_k \times Z_{2m}$, $1 \le k \le n$. We will construct an $RG((2mg)^n)$ on X' with group set $\mathcal{G}' = \{G'_k : 1 \le k \le n\}$.

For each block $B \in \mathcal{B}$, construct an $RH((2m)^4)$ on $B \times Z_{2m}$ with group set $\{\{x\} \times Z_{2m} : x \in B\}$ and block set $\mathcal{A}_B$ having resolution classes $P_B(j)$, $1 \le j \le (2m)^2$.

Let Γ be a multi-partite complete graph on the vertex set X with partite set $\mathcal{G}$. Denote its edge set by E. Then E is the block set of a $GDD(2, 2, gn)$ of type g^n on X with group set $\mathcal{G}$. Since an $RG(g^n)$ exists, gn is even. There exists a resolvable $GDD(2, 2, gn)$ of type g^n by [3], i.e., E has a resolution $\{Q_i : 1 \le i \le g(n-1)\}$ on X.

For each $x \in X$, let $\mathcal{F}^x = \{F_1^x, \ldots, F_{2m-1}^x\}$ be a one-factorization of the complete graph on $\{x\} \times Z_{2m}$. For each edge $\{x, y\} \in E$, let

$$\mathcal{E}_{\{x,y\}} = \{\{a, b, c, d\} : \{a, b\} \in F_k^x, \{c, d\} \in F_k^y, 1 \le k \le 2m-1\}.$$

Then $\mathcal{C} = (\bigcup_{B\in\mathcal{B}} \mathcal{A}_B) \bigcup (\bigcup_{\{x,y\}\in E} \mathcal{E}_{\{x,y\}})$ is the block set of the required $G((2mg)^n)$. We need to give its required resolution classes.

For each P_i, $1 \le i \le r$, $P'_{i,j} = \bigcup_{B\in P_i} P_B(j)$ is a parallel class of X', where $1 \le j \le (2m)^2$.

For each Q_i, $1 \le i \le g(n-1)$, and for each pair of k, l with $1 \le k \le 2m-1$ and $0 \le l \le m-1$,

$$Q'_{i,k,l} = \bigcup_{\{x,y\}\in Q_i} \Big\{\{a, b, c, d\} : \text{where } \{a, b\} \text{ is the } j\text{th member of } F_k^x \text{ and } \{c, d\} \text{ is the } (j+l)\text{th member of } F_k^y, 1 \le j \le m\Big\}$$

is a parallel class of X'.

Thus we obtain an $RG((2mg)^n)$. □

We close this section by the following theorem.

Theorem 4.12 *The necessary conditions* $g = 1$ *and* $n \equiv 4\,or\,8 \pmod{12}$*, or g is even,* $gn \equiv 0 \pmod 4$ *and* $g(n-1)(n-2) \equiv 0 \pmod 3$ *for the existence of an* $RG(g^n)$ *are also sufficient.*

Proof According to the necessary conditions for the existence of an $RG(g^n)$, we partition the parameters into seven classes as follows:

(1) $g = 1$ and $n \equiv 4, 8 \pmod{12}$,
(2) $g \equiv 2 \pmod{12}$ and $n \equiv 2, 4 \pmod{6}$,
(3) $g \equiv 4 \pmod{12}$ and $n \equiv 1, 2 \pmod{3}$,
(4) $g \equiv 6 \pmod{12}$ and $n \equiv 0 \pmod{2}$,
(5) $g \equiv 8 \pmod{12}$ and $n \equiv 1, 2 \pmod{3}$,
(6) $g \equiv 10 \pmod{12}$ and $n \equiv 2, 4 \pmod{6}$,
(7) $g \equiv 0 \pmod{12}$ and $n \in N$.

For Case (1), an RG(1^n) is actually an RSQS(n), whose existence has been solved completely [9,12]. For Cases (2), (4) and (6), an RG(g^n) can be obtained by applying Lemma 4.10 with an RH(g^n). For Cases (3), (5) and (7), we continue to partition them into two subcases (A) $g \equiv 4, 20, 12 \pmod{24}$ and (B) $g \equiv 16, 8, 0 \pmod{24}$. For Subcase (A), an RG$(g^n)$ can be obtained by applying Lemma 4.10 with an RH$((g/2)^{2n})$. For Subcase (B), the existence of an RG(g^n) can be obtained by applying Lemma 4.11 with an RG(4^n) or an RG(12^n). □

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References

1. Booth T.R.: A resolvable quadruple system of order 20. Ars Combin. **5**, 121–125 (1978).
2. Cao H., Ji L., Zhu L.: Construction for generalized Steiner systems. Des. Codes Cryptogr. **45** 185–197 (2007).
3. Colbourn C.J., Dinitz J.H.: CRC Handbook of Combinatorial Designs. CRC Press Inc., Boca Raton (2007).
4. Greenwell D.L., Lindner C.C.: Some remarks on resolvable quadruple systems. Ars Combin. **6**, 215–221 (1978).
5. Hanani H.: On quadruple systems. Canad. J. Math. **12**, 145–157 (1960).
6. Hartman A.: Resolvable Steiner quadruple systems. Ars Combin. **9**, 263–273 (1980).
7. Hartman A.: Kirkman's trombone player problem. Ars Combin. **10**, 19–26 (1980).
8. Hartman A.: Tripling quadruple systems. Ars Combin. **10**, 255–309 (1980).
9. Hartman A.: The existence of resolvable Steiner quadruple systems. J. Combin. Theory Ser. A **44**, 182–206 (1987).
10. Hartman A., Mills W.H., Mullin R.C.: Covering triples by quadruple: an asymptotic solution. J. Combin. Theory Ser. A **41**, 117–138 (1986).
11. Ji L.: An improvement on *H* design. J. Combin. Des. **17**, 25–35 (2009).
12. Ji L., Zhu L.: Resolvable Steiner quadruple systems for the last 23 orders. SIAM J. Discrete Math. **19**, 420–430 (2005).
13. Meng Z., Ji L., Du B.: Uniformly resolvable three-wise balanced designs with block sizes four and six. Discrete Math. (to appear). doi:10.1016/j.disc.2009.06.015.
14. Mills W.H.: On the existence of H designs. Congr. Numer. **79**, 129–141 (1990).
15. Stern G., Lenz H.: Steiner triple systems with given subspaces, another proof of the Doyen–Wilson Theorem. Boll. Unione Mat. Ital. (5) **17-A**, 109–114 (1980).
16. Zhang X., Ge G.: Maximal resolvable packings and minimal resolvable coverings of triples by quadruples. J. Combin. Des. (to appear). doi:10.1002/jcd.20234.
17. Zhuralev A.A., Keranen M.S., Kreher D.L.: Small group divisiable Steiner quadruple systems. Electron. J. Combin. **15** (2008), no. 1, Research paper 40, 14 pp.
18. http://www.emba.uvm.edu/~dinitz/hcd/Appendix_ZG.pdf.