# Supplementary material: The strongest nonlocal sets in $d_1 \otimes d_2 \otimes d_3$ and $d \otimes d \otimes d \otimes d$

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In this supplementary material, we show our constructions in $d_1\otimes d_2\otimes d_3 (d_3\geq d_2\geq d_1\geq 2)$ and $d\otimes d\otimes d\otimes d (d\geq 2)$ are strongest nonlocal.

In $d_1\otimes d_2\otimes d_3$, where $2\leq d_1\leq d_2\leq d_3$. let

$$\begin{aligned}
\mathcal{A}_0 :=&\{|000\rangle\},\\
\mathcal{A}_1 :=&\{|i00\rangle : 1\leq i\leq d_1-1\}\bigcup\{|0i0\rangle : 1\leq i\leq d_2-1\}\bigcup\{|00i\rangle : 1\leq i\leq d_3-1\},\\
\mathcal{A}_2 :=&\left\{\frac{1}{\sqrt{3}}(|0ij\rangle+|j0i\rangle+|ij0\rangle) : 1\leq i,j\leq d_1-1\right\}\bigcup\left\{\frac{1}{\sqrt{2}}(|0ij\rangle+|j0i\rangle) : d_1\leq i\leq d_2-1,\ 1\leq j\leq d_1-1\right\}\\
&\bigcup\left\{\frac{1}{\sqrt{2}}(|0ij\rangle+|ij0\rangle) : 1\leq i\leq d_1-1,\ d_1\leq j\leq d_2-1\right\}\bigcup\{|0ij\rangle : d_1\leq i,j\leq d_2-1\}\\
&\bigcup\left\{\frac{1}{\sqrt{2}}(|0ij\rangle+|i0j\rangle) : 1\leq i\leq d_1-1,\ d_2\leq j\leq d_3-1\right\}\bigcup\{|0ij\rangle : d_1\leq i\leq d_2-1,\ d_2\leq j\leq d_3-1\},
\end{aligned}$$

where $|\mathcal{A}_0|=1$, $|\mathcal{A}_1|=d_1+d_2+d_3-3$, $|\mathcal{A}_2|=(d_2-1)(d_3-1)$, and $|\mathcal{A}_0|+|\mathcal{A}_1|+|\mathcal{A}_2|=d_2d_3+d_1-1=S(d_1,d_2,d_3)$. We denote $|\alpha_0\rangle=|000\rangle$ and $\{|\alpha_i\rangle\}_{i=1}^{S(d_1,d_2,d_3)-1}=\mathcal{A}_1\bigcup\mathcal{A}_2$. Then we can construct a set of orthogonal states $\{|\psi_i\rangle\}_{i\in\mathbb{Z}_{S(d_1,d_2,d_3)}}$, where

$$\begin{aligned}
|\psi_0\rangle &= |\alpha_0\rangle,\\
|\psi_{i+1}\rangle &= \sum_{j\in\mathbb{Z}_{S(d_1,d_2,d_3)-1}} w_{S(d_1,d_2,d_3)-1}^{ij}|\alpha_{j+1}\rangle,\quad i\in\mathbb{Z}_{S(d_1,d_2,d_3)-1}.
\end{aligned}\tag{1}$$

**Proposition 1** *In $d_1\otimes d_2\otimes d_3$, $2\leq d_1\leq d_2\leq d_3$, the set of orthogonal states $\{|\psi_i\rangle\}_{i\in\mathbb{Z}_{S(d_1,d_2,d_3)}}$ given by Eq. (1) is strongest nonlocal.*

**Proof.** Since $\{|\psi_i\rangle\}_{i\in\mathbb{Z}_{S(d_1,d_2,d_3)}}$ has a similar structure under the bipartitions $\{A_1|A_2A_3,\ A_2|A_1A_3,\ A_3|A_2A_1\}$, we only need to show that any OPLM $\{E\}$ performed on $A_2A_3$ is trivial. We assume that $A_2A_3$ performs an OPLM $\{E\}$, then we have $\langle\psi_i|\mathbb{I}_{A_1}\otimes E|\psi_j\rangle=0$ for $i\neq j\in\mathbb{Z}_{S(d_1,d_2,d_3)}$.

Note that $\{|\psi_0\rangle\}$ is spanned by $\{|\alpha_0\rangle\}$, and $\{|\psi_i\rangle\}_{i=1}^{S(d_1,d_2,d_3)-1}$ is spanned by $\{|\alpha_i\rangle\}_{i=1}^{S(d_1,d_2,d_3)-1}$. Since $\langle\psi_0|\mathbb{I}_{A_1}\otimes E|\psi_i\rangle=0$ for $1\leq i\leq S(d_1,d_2,d_3)-1$, we have $\langle\alpha_0|\mathbb{I}_{A_1}\otimes E|\alpha_i\rangle=0$ for $1\leq i\leq S(d_1,d_2,d_3)-1$ by Block Zeros Lemma. It implies that

$$\langle 00|E|ij\rangle=\langle ij|E|00\rangle=0,\quad (i,j)\in\mathbb{Z}_{d_2}\times\mathbb{Z}_{d_3}\setminus\{(0,0)\}.\tag{2}$$

Without loss of generality, we assume $|\alpha_1\rangle=|100\rangle$. Then we have

$$\begin{aligned}
&\langle\alpha_1|\mathbb{I}_{A_1}\otimes E|\alpha_i\rangle=0,\quad 2\leq i\leq S(d_1,d_2,d_3)-1;\\
&\langle\alpha_1|\psi_i\rangle\neq 0,\quad 1\leq i\leq S(d_1,d_2,d_3)-1.
\end{aligned}$$

Applying Block Trivial Lemma to $\{|\psi_i\rangle\}_{i=1}^{S(d_1,d_2,d_3)-1}$, we obtain

$$\begin{aligned}
&\langle\alpha_i|\mathbb{I}_{A_1}\otimes E|\alpha_j\rangle=0,\quad 1\leq i\neq j\leq S(d_1,d_2,d_3)-1;\\
&\langle\alpha_i|\mathbb{I}_{A_1}\otimes E|\alpha_i\rangle=\langle\alpha_j|\mathbb{I}_{A_1}\otimes E|\alpha_j\rangle,\quad 1\leq i\neq j\leq S(d_1,d_2,d_3)-1.
\end{aligned}$$

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Now we prove that the off-diagonal elements of $E$ are zeros, which is more involved. We can summarize it as follows:

Assume $x_1 = |i_1 j_1\rangle,\ x_2 = |i_2 j_2\rangle \in \mathbb{Z}_{d_2}\times\mathbb{Z}_{d_3}$, then $|0x_1\rangle, |0x_2\rangle \in \mathbb{Z}_{d_1}\times\mathbb{Z}_{d_2}\times\mathbb{Z}_{d_3}$, correspond to two states in $\mathcal{A}_{wt(x_1)}$ and $\mathcal{A}_{wt(x_2)}$. Denote the two states by $\alpha_1, \alpha_2$, respectively. When $x_1 \neq x_2$, one can check $0 = \langle\alpha_1|E|\alpha_2\rangle = \langle 0x_1|\mathbb{I}\otimes E|0x_2\rangle = \langle i_1 j_1|E|i_2 j_2\rangle$(Here we ignore the coefficients, which have no effect on the result). The detailed process is as follows:

If $|\alpha_i\rangle, |\alpha_j\rangle \in \mathcal{A}_1$, then we have

$$\langle i_1 j_1|E|i_2 j_2\rangle = 0, \quad wt(i_1,j_1) = wt(i_2,j_2) = 1,\ (i_1,j_1)\neq(i_2,j_2)\in\mathbb{Z}_{d_2}\times\mathbb{Z}_{d_3}. \tag{3}$$

If $|\alpha_i\rangle \in \mathcal{A}_1, |\alpha_j\rangle \in \mathcal{A}_2$, then we have

$$\begin{aligned}
0 =& \langle 00i_1|\mathbb{I}_{A_1}\otimes E\frac{1}{\sqrt{3}}(|0i_2j_2\rangle + |j_20i_2\rangle + |i_2j_20\rangle) = \frac{1}{\sqrt{3}}(\langle 0i_2j_2| + \langle j_20i_2| + \langle i_2j_20|)\mathbb{I}_{A_1}\otimes E|00i_1\rangle \\
=& \langle 0i_1|E|i_2j_2\rangle = \langle i_2j_2|E|0i_1\rangle, \quad 1\le i_1\le d_3-1,\ 1\le i_2,j_2\le d_1-1, \\
0 =& \langle 0i_10|\mathbb{I}_{A_1}\otimes E\frac{1}{\sqrt{3}}(|0i_2j_2\rangle + |j_20i_2\rangle + |i_2j_20\rangle) = \frac{1}{\sqrt{3}}(\langle 0i_2j_2| + \langle j_20i_2| + \langle i_2j_20|)\mathbb{I}_{A_1}\otimes E|0i_10\rangle \\
=& \langle i_10|E|i_2j_2\rangle = \langle i_2j_2|E|i_10\rangle, \quad 1\le i_1\le d_2-1,\ 1\le i_2,j_2\le d_1-1, \\
0 =& \langle 00i_1|\mathbb{I}_{A_1}\otimes E\frac{1}{\sqrt{2}}(|0i_2j_2\rangle + |j_20i_2\rangle) = \frac{1}{\sqrt{2}}(\langle 0i_2j_2| + \langle j_20i_2|)\mathbb{I}_{A_1}\otimes E|00i_1\rangle \\
=& \frac{1}{\sqrt{2}}\langle 0i_1|E|i_2j_2\rangle = \frac{1}{\sqrt{2}}\langle i_2j_2|E|0i_1\rangle, \quad 1\le i_1\le d_3-1,\ d_1\le i_2\le d_2-1,\ 1\le j_2\le d_1-1, \\
0 =& \langle 0i_10|\mathbb{I}_{A_1}\otimes E\frac{1}{\sqrt{2}}(|0i_2j_2\rangle + |j_20i_2\rangle) = \frac{1}{\sqrt{2}}(\langle 0i_2j_2| + \langle j_20i_2|)\mathbb{I}_{A_1}\otimes E|0i_10\rangle \\
=& \frac{1}{\sqrt{2}}\langle i_10|E|i_2j_2\rangle = \frac{1}{\sqrt{2}}\langle i_2j_2|E|i_10\rangle, \quad 1\le i_1\le d_2-1,\ d_1\le i_2\le d_2-1,\ 1\le j_2\le d_1-1, \\
0 =& \langle 00i_1|\mathbb{I}_{A_1}\otimes E\frac{1}{\sqrt{2}}(|0i_2j_2\rangle + |i_2j_20\rangle) = \frac{1}{\sqrt{2}}(\langle 0i_2j_2| + \langle i_2j_20|)\mathbb{I}_{A_1}\otimes E|00i_1\rangle \\
=& \langle 0i_1|\otimes E\frac{1}{\sqrt{2}}|i_2j_2\rangle = \frac{1}{\sqrt{2}}\langle i_2j_2|\otimes E|0i_1\rangle, \quad 1\le i_1\le d_3-1,\ 1\le i_2\le d_1-1,\ d_1\le j_2\le d_2-1, \\
0 =& \langle 0i_10|\mathbb{I}_{A_1}\otimes E\frac{1}{\sqrt{2}}(|0i_2j_2\rangle + |i_2j_20\rangle) = \frac{1}{\sqrt{2}}(\langle 0i_2j_2| + \langle i_2j_20|)\mathbb{I}_{A_1}\otimes E|0i_10\rangle \\
=& \langle i_10|\otimes E\frac{1}{\sqrt{2}}|i_2j_2\rangle = \frac{1}{\sqrt{2}}\langle i_2j_2|\otimes E|i_10\rangle, \quad 1\le i_1\le d_2-1,\ 1\le i_2\le d_1-1,\ d_1\le j_2\le d_2-1, \\
0 =& \langle 00i_1|\mathbb{I}_{A_1}\otimes E|0i_2j_2\rangle = \langle 0i_2j_2|\mathbb{I}_{A_1}\otimes E|00i_1\rangle \\
=& \langle 0i_1|\otimes E|i_2j_2\rangle = \langle i_2j_2|\otimes E|0i_1\rangle, \quad 1\le i_1\le d_3-1,\ d_1\le i_2,j_2\le d_2-1, \\
0 =& \langle 0i_10|\mathbb{I}_{A_1}\otimes E|0i_2j_2\rangle = \langle 0i_2j_2|\mathbb{I}_{A_1}\otimes E|0i_10\rangle \\
=& \langle i_10|\otimes E|i_2j_2\rangle = \langle i_2j_2|\otimes E|i_10\rangle, \quad 1\le i_1\le d_2-1,\ d_1\le i_2,j_2\le d_2-1, \\
0 =& \langle 00i_1|\mathbb{I}_{A_1}\otimes E\frac{1}{\sqrt{2}}(|0i_2j_2\rangle + |i_20j_2\rangle) = \frac{1}{\sqrt{2}}(\langle 0i_2j_2| + \langle i_20j_2|)\mathbb{I}_{A_1}\otimes E|00i_1\rangle \\
=& \langle 0i_1|E|i_2j_2\rangle = \langle i_2j_2|E|0i_1\rangle, \quad 1\le i_1\le d_3-1,\ 1\le i_2\le d_1-1,\ d_2\le j_2\le d_3-1, \\
0 =& \langle 0i_10|\mathbb{I}_{A_1}\otimes E\frac{1}{\sqrt{2}}(|0i_2j_2\rangle + |i_20j_2\rangle) = \frac{1}{\sqrt{2}}(\langle 0i_2j_2| + \langle i_20j_2|)\mathbb{I}_{A_1}\otimes E|0i_10\rangle \\
=& \langle i_10|E|i_2j_2\rangle = \langle i_2j_2|E|i_10\rangle, \quad 1\le i_1\le d_2-1,\ 1\le i_2\le d_1-1,\ d_2\le j_2\le d_3-1, \\
0 =& \langle 00i_1|\mathbb{I}_{A_1}\otimes E|0i_2j_2\rangle = \langle 0i_2j_2|\mathbb{I}_{A_1}\otimes E|00i_1\rangle \\
=& \langle 0i_1|\otimes E|i_2j_2\rangle = \langle i_2j_2|\otimes E|0i_1\rangle, \quad 1\le i_1\le d_3-1,\ d_1\le i_2\le d_2-1,\ d_2\le j_2\le d_3-1, \\
0 =& \langle 0i_10|\mathbb{I}_{A_1}\otimes E|0i_2j_2\rangle = \langle 0i_2j_2|\mathbb{I}_{A_1}\otimes E|0i_10\rangle \\
=& \langle i_10|\otimes E|i_2j_2\rangle = \langle i_2j_2|\otimes E|i_10\rangle, \quad 1\le i_1\le d_2-1,\ d_1\le i_2\le d_2-1,\ d_2\le j_2\le d_3-1.
\end{aligned}$$

It means that

$$\langle i_1j_1|E|i_2j_2\rangle = \langle i_2j_2|E|i_1j_1\rangle = 0, \quad wt(i_1,j_1) = 1, wt(i_2,j_2) = 2,\ (i_1,j_1),(i_2,j_2)\in\mathbb{Z}_{d_2}\times\mathbb{Z}_{d_3}. \tag{4}$$

If $|\alpha_i\rangle \in \mathcal{A}_2, |\alpha_j\rangle \in \mathcal{A}_2$, then we have

$$\begin{aligned}
0 =&\frac{1}{\sqrt{3}}(\langle 0i_1j_1| + \langle j_10i_1| + \langle i_1j_10|)\mathbb{I}_{A_1} \otimes E\frac{1}{\sqrt{3}}(|0i_2j_2\rangle + |j_20i_2\rangle + |i_2j_20\rangle)\\
=&\frac{1}{3}(\langle i_1j_1|E|i_2j_2\rangle + \langle j_1|j_2\rangle\langle 0i_1|E|0i_2\rangle + \langle j_1|i_2\rangle\langle 0i_1|E|j_20\rangle + \langle i_1|j_2\rangle\langle j_10|E|0i_2\rangle + \langle i_1|i_2\rangle\langle j_10|E|j_20\rangle)\\
=&\frac{1}{3}(\langle i_1j_1|E|i_2j_2\rangle + \langle j_1|j_2\rangle\langle 0i_1|E|0i_2\rangle + \langle i_1|i_2\rangle\langle j_10|E|j_20\rangle)\\
=&\frac{1}{3}(\langle i_1j_1|E|i_2j_2\rangle, \quad 1 \le i_1, j_1, i_2, j_2 \le d_1 - 1,\ (i_1, j_1) \ne (i_2, j_2),\\
0 =&\frac{1}{\sqrt{3}}(\langle 0i_1j_1| + \langle j_10i_1| + \langle i_1j_10|)\mathbb{I}_{A_1} \otimes E\frac{1}{\sqrt{2}}(|0i_2j_2\rangle + |j_20i_2\rangle)\\
=&\frac{1}{\sqrt{2}}(\langle 0i_2j_2| + \langle j_20i_2|)\mathbb{I}_{A_1} \otimes E\frac{1}{\sqrt{3}}(|0i_1j_1\rangle + |j_10i_1\rangle + |i_1j_10\rangle)\\
=&\frac{1}{\sqrt{6}}(\langle i_1j_1|E|i_2j_2\rangle + \langle j_1|j_2\rangle\langle 0i_1|E|0i_2\rangle + \langle i_1|j_2\rangle\langle j_10|E|0i_2\rangle)\\
=&\frac{1}{\sqrt{6}}(\langle i_2j_2|E|i_1j_1\rangle + \langle j_2|j_1\rangle\langle 0i_2|E|0i_1\rangle + \langle j_2|i_1\rangle\langle 0i_2|E|j_10\rangle)\\
=&\frac{1}{\sqrt{6}}\langle i_1j_1|E|i_2j_2\rangle = \frac{1}{\sqrt{6}}\langle i_2j_2|E|i_1j_1\rangle, \quad 1 \le i_1, j_1 \le d_1 - 1,\ d_1 \le i_2 \le d_2 - 1,\ 1 \le j_2 \le d_1 - 1,\\
0 =&\frac{1}{\sqrt{3}}(\langle 0i_1j_1| + \langle j_10i_1| + \langle i_1j_10|)\mathbb{I}_{A_1} \otimes E\frac{1}{\sqrt{2}}(|0i_2j_2\rangle + |i_2j_20\rangle)\\
=&\frac{1}{\sqrt{3}}(\langle 0i_2j_2| + \langle i_2j_20|)\mathbb{I}_{A_1} \otimes E\frac{1}{\sqrt{2}}(|0i_1j_1\rangle + |j_10i_1\rangle + |i_1j_10\rangle)\\
=&\frac{1}{\sqrt{6}}(\langle i_1j_1|E|i_2j_2\rangle + \langle j_1|i_2\rangle\langle 0i_1|E|j_20\rangle + \langle i_1|i_2\rangle\langle j_10|E|j_20\rangle)\\
=&\frac{1}{\sqrt{6}}(\langle i_2j_2|E|i_1j_1\rangle + \langle i_2|j_1\rangle\langle 0j_2|E|0i_1\rangle + \langle i_2|i_1\rangle\langle j_20|E|j_10\rangle)\\
=&\frac{1}{\sqrt{6}}\langle i_1j_1|E|i_2j_2\rangle = \frac{1}{\sqrt{6}}\langle i_2j_2|E|i_1j_1\rangle, \quad 1 \le i_1, j_1 \le d_1 - 1,\ 1 \le i_2 \le d_1 - 1,\ d_1 - 1 \le j_2 \le d_2 - 1,\\
0 =&\frac{1}{\sqrt{3}}(\langle 0i_1j_1| + \langle j_10i_1| + \langle i_1j_10|)\mathbb{I}_{A_1} \otimes E|0i_2j_2\rangle\\
=&\langle 0i_2j_2| \otimes E\frac{1}{\sqrt{3}}(|0i_1j_1\rangle + |j_10i_1\rangle + |i_1j_10\rangle)\\
=&\frac{1}{\sqrt{3}}\langle i_1j_1|E|i_2j_2\rangle = \frac{1}{\sqrt{3}}\langle i_2j_2|E|i_1j_1\rangle, \quad 1 \le i_1, j_1 \le d_1 - 1,\ d_1 \le i_2, j_2 \le d_2 - 1,\\
0 =&\frac{1}{\sqrt{3}}(\langle 0i_1j_1| + \langle j_10i_1| + \langle i_1j_10|)\mathbb{I}_{A_1} \otimes E\frac{1}{\sqrt{2}}(|0i_2j_2\rangle + |i_20j_2\rangle)\\
=&\frac{1}{\sqrt{2}}(\langle 0i_2j_2| + \langle i_20j_2|)\mathbb{I}_{A_1} \otimes E\frac{1}{\sqrt{3}}(|0i_1j_1\rangle + |j_10i_1\rangle + |i_1j_10\rangle)\\
=&\frac{1}{\sqrt{6}}(\langle i_1j_1|E|i_2j_2\rangle + \langle j_1|i_2\rangle\langle 0i_1|E|0j_2\rangle + \langle i_1|i_2\rangle\langle j_10|E|0j_2\rangle)\\
=&\frac{1}{\sqrt{6}}(\langle i_2j_2|E|i_1j_1\rangle + \langle i_2|j_1\rangle\langle 0j_2|E|0i_1\rangle + \langle i_2|i_1\rangle\langle 0j_2|E|j_10\rangle)\\
=&\frac{1}{\sqrt{6}}\langle i_1j_1|E|i_2j_2\rangle = \frac{1}{\sqrt{6}}\langle i_2j_2|E|i_1j_1\rangle, \quad 1 \le i_1, j_1 \le d_1 - 1,\ 1 \le i_2 \le d_1 - 1,\ d_2 \le j_2 \le d_3 - 1,\\
0 =&\frac{1}{\sqrt{3}}(\langle 0i_1j_1| + \langle j_10i_1| + \langle i_1j_10|)\mathbb{I}_{A_1} \otimes E|0i_2j_2\rangle\\
=&\langle 0i_2j_2|\mathbb{I}_{A_1} \otimes E\frac{1}{\sqrt{3}}(|0i_1j_1\rangle + |j_10i_1\rangle + |i_1j_10\rangle)\\
=&\frac{1}{\sqrt{3}}\langle i_1j_1|E|i_2j_2\rangle = \frac{1}{\sqrt{3}}\langle i_2j_2|E|i_1j_1\rangle, \quad 1 \le i_1, j_1 \le d_1 - 1,\ d_1 \le i_2 \le d_2 - 1,\ d_2 \le j_2 \le d_3 - 1,\\
0 =&\frac{1}{\sqrt{2}}(\langle 0i_1j_1| + \langle j_10i_1|)\mathbb{I}_{A_1} \otimes E\frac{1}{\sqrt{2}}(|0i_2j_2\rangle + |j_20i_2\rangle)\\
=&\frac{1}{2}(\langle i_1j_1|E|i_2j_2\rangle + \langle j_1|j_2\rangle\langle 0i_1|E|0i_2\rangle)\\
=&\frac{1}{2}\langle i_1j_1|E|i_2j_2\rangle, \quad d_1 \le i_1, i_2 \le d_2 - 1,\ 1 \le j_1, j_2 \le d_1 - 1,\ (i_1, j_1) \ne (i_2, j_2),
\end{aligned}$$

$$
\begin{aligned}
0 =&\frac{1}{\sqrt{2}}(\langle 0i_1j_1| + \langle j_10i_1|)\mathbb{I}_{A_1}\otimes E\frac{1}{\sqrt{2}}(|0i_2j_2\rangle + |i_2j_20\rangle) = \frac{1}{\sqrt{2}}(\langle 0i_2j_2| + \langle i_2j_20|)\mathbb{I}_{A_1}\otimes E\frac{1}{\sqrt{2}}(|0i_1j_1\rangle + |j_10i_1\rangle)\\
=&\frac{1}{2}(\langle i_1j_1|E|i_2j_2\rangle + \langle j_1|i_2\rangle\langle 0i_1|E|j_20\rangle) = \frac{1}{2}(\langle i_2j_2|E|i_1j_1\rangle + \langle i_2|j_1\rangle\langle j_20|E|0i_1\rangle)\\
=&\frac{1}{2}\langle i_1j_1|E|i_2j_2\rangle = \frac{1}{2}\langle i_2j_2|E|i_1j_1\rangle,\quad d_1\le i_1\le d_2-1,\ 1\le j_1\le d_1-1,\ 1\le i_2\le d_1-1,\ d_1\le j_2\le d_2-1,\\
0 =&\frac{1}{\sqrt{2}}(\langle 0i_1j_1| + \langle j_10i_1|)\mathbb{I}_{A_1}\otimes E|0i_2j_2\rangle = \langle 0i_2j_2|\mathbb{I}_{A_1}\otimes E\frac{1}{\sqrt{2}}(|0i_1j_1\rangle + |j_10i_1\rangle)\\
=&\frac{1}{\sqrt{2}}\langle i_1j_1|E|i_2j_2\rangle = \frac{1}{\sqrt{2}}\langle i_2j_2|E|i_1j_1\rangle,\quad d_1\le i_1\le d_2-1,\ 1\le j_1\le d_1-1,\ d_1\le i_2,j_2\le d_2-1,\\
0 =&\frac{1}{\sqrt{2}}(\langle 0i_1j_1| + \langle j_10i_1|)\mathbb{I}_{A_1}\otimes E\frac{1}{\sqrt{2}}(|0i_2j_2\rangle + |i_20j_2\rangle) = \frac{1}{\sqrt{2}}(\langle 0i_2j_2| + \langle i_20j_2|)\mathbb{I}_{A_1}\otimes E\frac{1}{\sqrt{2}}(|0i_1j_1\rangle + |j_10i_1\rangle)\\
=&\frac{1}{2}(\langle i_1j_1|E|i_2j_2\rangle + \langle j_1|i_2\rangle\langle 0i_1|E|0j_2\rangle) = \frac{1}{2}(\langle i_2j_2|E|i_1j_1\rangle + \langle i_2|j_1\rangle\langle 0j_2|E|0i_1\rangle)\\
=&\frac{1}{2}\langle i_1j_1|E|i_2j_2\rangle = \frac{1}{2}\langle i_2j_2|E|i_1j_1\rangle,\quad d_1\le i_1\le d_2-1,\ 1\le j_1\le d_1-1,\ 1\le i_2\le d_1-1,\ d_2\le j_2\le d_3-1,\\
0 =&\frac{1}{\sqrt{2}}(\langle 0i_1j_1| + \langle j_10i_1|)\mathbb{I}_{A_1}\otimes E|0i_2j_2\rangle = \langle 0i_2j_2|\mathbb{I}_{A_1}\otimes E\frac{1}{\sqrt{2}}(|0i_1j_1\rangle + |j_10i_1\rangle)\\
=&\frac{1}{\sqrt{2}}\langle i_1j_1|E|i_2j_2\rangle = \frac{1}{\sqrt{2}}\langle i_2j_2|E|i_1j_1\rangle,\quad d_1\le i_1\le d_2-1,\ 1\le j_1\le d_1-1,\ d_1\le i_2\le d_2-1,\ d_2\le j_2\le d_3-1,\\
0 =&\frac{1}{\sqrt{2}}(\langle 0i_1j_1| + \langle i_1j_10|)\mathbb{I}_{A_1}\otimes E\frac{1}{\sqrt{2}}(|0i_2j_2\rangle + |i_2j_20\rangle) = \frac{1}{2}(\langle i_1j_1|E|i_2j_2\rangle + \langle i_1|i_2\rangle\langle j_10|E|j_20\rangle)\\
=&\frac{1}{2}\langle i_1j_1|E|i_2j_2\rangle,\quad 1\le i_1,i_2\le d_1-1,\ d_1\le j_1,j_2\le d_2-1,\ (i_1,j_1)\ne(i_2,j_2),\\
0 =&\frac{1}{\sqrt{2}}(\langle 0i_1j_1| + \langle i_1j_10|)\mathbb{I}_{A_1}\otimes E|0i_2j_2\rangle = \langle 0i_2j_2|\mathbb{I}_{A_1}\otimes E\frac{1}{\sqrt{2}}(|0i_1j_1\rangle + |i_1j_10\rangle)\\
=&\frac{1}{\sqrt{2}}\langle i_1j_1|E|i_2j_2\rangle = \frac{1}{\sqrt{2}}\langle i_2j_2|E|i_1j_1\rangle,\quad 1\le i_1\le d_1-1, d_1\le j_1\le d_2-1,\ d_1\le i_2,j_2\le d_2-1,\\
0 =&\frac{1}{\sqrt{2}}(\langle 0i_1j_1| + \langle i_1j_10|)\mathbb{I}_{A_1}\otimes E\frac{1}{\sqrt{2}}(|0i_2j_2\rangle + |i_20j_2\rangle) = \frac{1}{\sqrt{2}}(\langle 0i_2j_2| + \langle i_20j_2|)\mathbb{I}_{A_1}\otimes E\frac{1}{\sqrt{2}}(|0i_1j_1\rangle + |i_1j_10\rangle)\\
=&\frac{1}{2}(\langle i_1j_1|E|i_2j_2\rangle + \langle i_1|i_2\rangle\langle j_10|E|0j_2\rangle) = \frac{1}{2}(\langle i_2j_2|E|i_1j_1\rangle + \langle i_2|i_1\rangle\langle 0j_2|E|j_10\rangle)\\
=&\frac{1}{2}\langle i_1j_1|E|i_2j_2\rangle = \frac{1}{2}\langle i_2j_2|E|i_1j_1\rangle,\quad 1\le i_1\le d_1-1,\ d_1\le j_1\le d_2-1,\ 1\le i_2\le d_1-1,\ d_2\le j_2\le d_3-1,\\
0 =&\frac{1}{\sqrt{2}}(\langle 0i_1j_1| + \langle i_1j_10|)\mathbb{I}_{A_1}\otimes E|0i_2j_2\rangle = \langle 0i_2j_2|\mathbb{I}_{A_1}\otimes E\frac{1}{\sqrt{2}}(|0i_1j_1\rangle + |i_1j_10\rangle)\\
=&\frac{1}{\sqrt{2}}\langle i_1j_1|E|i_2j_2\rangle = \frac{1}{\sqrt{2}}\langle i_2j_2|E|i_1j_1\rangle,\quad 1\le i_1\le d_1-1,\ d_1\le j_1\le d_2-1,\ d_1\le i_2\le d_2-1,\ d_2\le j_2\le d_3-1,\\
0 =&\langle 0i_1j_1|\mathbb{I}_{A_1}\otimes E|0i_2j_2\rangle = \langle i_1j_1|E|i_2j_2\rangle,\quad d_1\le i_1,i_2,j_1,j_2\le d_2-1,\ (i_1,j_1)\ne(i_2,j_2),\\
0 =&\langle 0i_1j_1|\mathbb{I}_{A_1}\otimes E\frac{1}{\sqrt{2}}(|0i_2j_2\rangle + |i_20j_2\rangle) = \frac{1}{\sqrt{2}}(\langle 0i_2j_2| + \langle i_20j_2|)\mathbb{I}_{A_1}\otimes E|0i_1j_1\rangle\\
=&\frac{1}{\sqrt{2}}\langle i_1j_1|E|i_2j_2\rangle = \frac{1}{\sqrt{2}}\langle i_2j_2|E|i_1j_1\rangle,\quad d_1\le i_1,j_1\le d_2-1,\ 1\le i_2\le d_1-1,\ d_2\le j_2\le d_3-1,\\
0 =&\langle 0i_1j_1|\mathbb{I}_{A_1}\otimes E|0i_2j_2\rangle = \langle i_1j_1|E|i_2j_2\rangle,\quad d_1\le i_1,j_1\le d_2-1,\ d_1\le i_2\le d_2-1,\ d_2\le j_2\le d_3-1,\\
0 =&\frac{1}{\sqrt{2}}(\langle 0i_1j_1| + \langle i_10j_1|)\mathbb{I}_{A_1}\otimes E\frac{1}{\sqrt{2}}(|0i_2j_2\rangle + |i_20j_2\rangle)\\
=&\frac{1}{\sqrt{2}}(\langle i_1j_1|E|i_2j_2\rangle + \langle i_1|i_2\rangle\langle 0j_1|E|0j_2\rangle)\\
=&\frac{1}{\sqrt{2}}\langle i_1j_1|E|i_2j_2\rangle,\quad 1\le i_1,i_2\le d_1-1, d_2\le i_2,j_2\le d_3-1,\ (i_1,j_1)\ne(i_2,j_2),\\
0 =&\frac{1}{\sqrt{2}}(\langle 0i_1j_1| + \langle i_10j_1|)\mathbb{I}_{A_1}\otimes E|0i_2j_2\rangle = \langle 0i_2j_2|\mathbb{I}_{A_1}\otimes E\frac{1}{\sqrt{2}}(|0i_1j_1\rangle + |i_10j_1\rangle)\\
=&\frac{1}{\sqrt{2}}\langle i_1j_1|E|i_2j_2\rangle = \frac{1}{\sqrt{2}}\langle i_2j_2|E|i_1j_1\rangle,\quad 1\le i_1\le d_1-1,\ d_2\le j_1\le d_3-1,\ d_1\le i_2\le d_2-1,\ d_2\le j_2\le d_3-1,\\
0 =&\langle 0i_1j_1|\mathbb{I}_{A_1}\otimes E|0i_2j_2\rangle = \langle 0i_2j_2|\mathbb{I}_{A_1}\otimes E|0i_1j_1\rangle\\
=&\langle i_1j_1|E|i_2j_2\rangle,\quad d_1\le i_1,i_2\le d_2-1,\ d_2\le j_1,j_2\le d_3-1,\ (i_1,j_1)\ne(i_2,j_2).
\end{aligned}
$$

It means that

$$\langle i_1j_1|E|i_2j_2\rangle = 0, \quad wt(i_1,j_1)=wt(i_2,j_2)=2,\ (i_1,j_1)\neq(i_2,j_2)\in\mathbb{Z}_{d_2}\times\mathbb{Z}_{d_3}. \tag{5}$$

By Eqs. (2), (3), (4) and (5), we obtain that the off-diagonal elements of $E$ are all zeros.

Next, we consider the diagonal elements of $E$. If $|\alpha_i\rangle\in\mathcal{A}_1$, then we have

$$\langle 00|E|00\rangle = \langle\alpha_1|\mathbb{I}_{A_1}\otimes E|\alpha_1\rangle = \langle\alpha_i|\mathbb{I}_{A_1}\otimes E|\alpha_i\rangle = \langle i_1j_1|E|i_1j_1\rangle, \quad wt(i_1,j_1)=1,\ (i_1,j_1)\in\mathbb{Z}_{d_2}\times\mathbb{Z}_{d_3}. \tag{6}$$

If $|\alpha_i\rangle\in\mathcal{A}_2$, then we have $\langle\alpha_1|\mathbb{I}_{A_1}\otimes E|\alpha_1\rangle = \langle\alpha_i|\mathbb{I}_{A_1}\otimes E|\alpha_i\rangle$, i.e.,

$$\begin{aligned}
\langle 00|E|00\rangle =&\langle\alpha_i|\mathbb{I}_{A_1}\otimes E|\alpha_i\rangle = \frac{1}{\sqrt{3}}(\langle 0i_1j_1|+\langle j_10i_1|+\langle i_1j_10|)\mathbb{I}_{A_1}\otimes E\frac{1}{\sqrt{3}}(|0i_1j_1\rangle+|j_10i_1\rangle+|i_1j_10\rangle)\\
=&\frac{1}{3}(\langle i_1j_1|E|i_1j_1\rangle+\langle 0i_1|E|0i_1\rangle+\langle j_10|E|j_10\rangle), \quad 1\le i,j\le d_1-1,\\
\langle 00|E|00\rangle =&\frac{1}{\sqrt{2}}(\langle 0i_1j_1|+\langle j_10i_1|)\mathbb{I}\otimes E\frac{1}{\sqrt{2}}(|0i_1j_1\rangle+|j_10i_1\rangle)\\
=&\frac{1}{2}(\langle i_1j_1|E|i_1j_1\rangle+\langle 0i_1|E|0i_1\rangle), \quad d_1\le i_1\le d_2-1,\ 1\le j_1\le d_1-1,\\
\langle 00|E|00\rangle =&\frac{1}{\sqrt{2}}(\langle 0i_1j_1|+\langle i_1j_10|)\mathbb{I}\otimes\frac{1}{\sqrt{2}}(|0i_1j_1\rangle+|i_1j_10\rangle)\\
=&\frac{1}{2}(\langle i_1j_1|E|i_1j_1\rangle+\langle j_10|E|j_10\rangle), \quad 1\le i\le d_1-1,\ d_1\le j\le d_2-1,\\
\langle 00|E|00\rangle =&\langle 0i_1j_1|\mathbb{I}\otimes E|0i_1j_1\rangle = \langle i_1j_1|E|i_1j_1\rangle, \quad d_1\le i,j\le d_2-1,\\
\langle 00|E|00\rangle =&\frac{1}{\sqrt{2}}(\langle 0i_1j_1|+\langle i_10j_1|)\mathbb{I}\otimes E(\frac{1}{\sqrt{2}}|0i_1j_1\rangle+|i_10j_1\rangle)\\
=&\frac{1}{2}(\langle i_1j_1|E|i_1j_1\rangle+\langle 0j_1|E|0j_1\rangle), \quad 1\le i_1\le d_1-1,\ d_2\le j_1\le d_3-1,\\
\langle 00|E|00\rangle =&\langle 0i_1j_1|\mathbb{I}\otimes|0i_1j_1\rangle = \langle i_1j_1|E|i_1j_1\rangle, \quad d_1\le i_1\le d_2-1,\ d_2\le j_1\le d_3-1.
\end{aligned}$$

That is

$$\langle 00|E|00\rangle = \langle i_1j_1|E|i_1j_1\rangle, \quad wt(i_1,j_1)=2,\ (i_1,j_1)\in\mathbb{Z}_{d_2}\times\mathbb{Z}_{d_3}. \tag{7}$$

By Eqs. (6), (7), we obtain that the diagonal elements of $E$ are all equal. Thus $E\propto\mathbb{I}$. This completes the proof. $\square$

In $d\otimes d\otimes d\otimes d$. Let

$$\begin{aligned}
\mathcal{A}_0 :=&\{|0000\rangle\},\\
\mathcal{A}_1 :=&\{|i000\rangle,|0i00\rangle,|00i0\rangle,|000i\rangle\}_{i=1}^{d-1},\\
\mathcal{A}_2 :=&\left\{\frac{1}{\sqrt{2}}(|00ij\rangle+|ij00\rangle):1\le i,j\le d-1\right\}\bigcup\left\{\frac{1}{\sqrt{2}}(|0ij0\rangle+|j00i\rangle):1\le i,j\le d-1\right\}\\
&\bigcup\left\{\frac{1}{\sqrt{2}}(|0i0j\rangle+|i0j0\rangle):1\le i,j\le d-1\right\},\\
\mathcal{A}_3 :=&\left\{\frac{1}{2}(|0ijk\rangle+|k0ij\rangle+|jk0i\rangle+|ijk0\rangle):1\le i,j,k\le d-1\right\},
\end{aligned}$$

where $|\mathcal{A}_0|=1$, $|\mathcal{A}_1|=4(d-1)$, $|\mathcal{A}_2|=3(d-1)^2$, and $|\mathcal{A}_3|=(d-1)^3$, and $|\mathcal{A}_0|+|\mathcal{A}_1|+|\mathcal{A}_2|+|\mathcal{A}_3|=d^3+d-1=S(d,d,d,d)$.

We denote $|\alpha_0\rangle=|0000\rangle$ and $\{|\alpha_i\rangle\}_{i=1}^{S(d,d,d,d)-1}=\mathcal{A}_1\bigcup\mathcal{A}_2\bigcup\mathcal{A}_3$. Then we can construct a set of orthogonal states $\{|\psi_i\rangle\}_{i\in\mathbb{Z}_{S(d,d,d,d)}}$, where

$$\begin{aligned}
|\psi_0\rangle &= |\alpha_0\rangle\\
|\psi_{i+1}\rangle &= \sum_{j\in\mathbb{Z}_{S(d,d,d,d)-1}} w_{S(d,d,d,d)-1}^{ij}|\alpha_{j+1}\rangle, \quad i\in\mathbb{Z}_{S(d,d,d,d)-1}.
\end{aligned} \tag{8}$$

**Proposition 2** *In $d\otimes d\otimes d\otimes d$, $d\geq 2$, the set of orthogonal states $\{|\psi_i\rangle\}_{i\in\mathbb{Z}_{S(d,d,d,d)}}$ given by Eq. (8) is strongest nonlocal.*

**Proof.** Since $\{|\psi_i\rangle\}_{i\in\mathbb{Z}_{S(d,d,d,d)}}$ has a similar structure under the cyclic permutation of the parties $\{A_1,A_2,A_3,A_4\}$. We only need to show any OPLM performed on $A_2A_3A_4$ is trivial. We assume that $A_2A_3A_4$ performs an OPLM $\{E\}$, then we have $\langle\psi_i|\mathbb{I}_{A_1}\otimes E|\psi_j\rangle=0$, $i\neq j\in\mathbb{Z}_{S(d,d,d,d)}$.

Note that $\{|\psi_0\rangle\}$ is spanned by $\{|\alpha_0\rangle\}$, and $\{|\psi_i\rangle\}_{i=1}^{S(d,d,d,d)-1}$ is spanned by $\{|\alpha_i\rangle\}_{i=1}^{S(d,d,d,d)-1}$. Since $\langle\psi_0|\mathbb{I}_{A_1}\otimes E|\psi_i\rangle=0$ for $1\leq i\leq S(d,d,d,d)-1$, we have $\langle\alpha_0|\mathbb{I}_{A_1}\otimes E|\alpha_i\rangle=0$ for $1\leq i\leq S(d,d,d,d)-1$ by Block Zeros Lemma. It implies that

$$\langle 000|E|ijk\rangle=\langle ijk|E|000\rangle=0,\quad (i,j,k)\in\mathbb{Z}_d\times\mathbb{Z}_d\times\mathbb{Z}_d\setminus\{(0,0,0)\}. \tag{9}$$

Without loss of generality, we assume $|\alpha_1\rangle=|1000\rangle$. Then we have

$$\begin{aligned}&\langle\alpha_1|\mathbb{I}_{A_1}\otimes E|\alpha_i\rangle=0,\quad 2\leq i\leq S(d,d,d,d)-1;\\ &\langle\alpha_1|\psi_i\rangle\neq 0,\quad 1\leq i\leq S(d,d,d,d)-1.\end{aligned}$$

Applying Block Trivial Lemma to $\{|\psi_i\rangle\}_{i=1}^{S(d,d,d,d)-1}$, we obtain

$$\begin{aligned}&\langle\alpha_i|\mathbb{I}_{A_1}\otimes E|\alpha_j\rangle=0,\quad 1\leq i\neq j\leq S(d,d,d,d)-1;\\ &\langle\alpha_i|\mathbb{I}_{A_1}\otimes E|\alpha_i\rangle=\langle\alpha_j|\mathbb{I}_{A_1}\otimes E|\alpha_j\rangle,\quad 1\leq i\neq j\leq S(d,d,d,d)-1.\end{aligned}$$

If $|\alpha_i\rangle,|\alpha_j\rangle\in\mathcal{A}_1$, then we have

$$\langle i_1j_1k_1|E|i_2j_2k_2\rangle=0,\quad wt(i_1,j_1,k_1)=wt(i_2,j_2,k_2)=1,\ (i_1,j_1,k_1)\neq(i_2,j_2,k_2)\in\mathbb{Z}_d\times\mathbb{Z}_d\times\mathbb{Z}_d. \tag{10}$$

If $|\alpha_i\rangle\in\mathcal{A}_1$, and $|\alpha_j\rangle\in\mathcal{A}_2$

$$\begin{aligned}
0=&\langle 000i_1|\mathbb{I}_{A_1}\otimes E\frac{1}{\sqrt{2}}(|00i_2j_2\rangle+|i_2j_200\rangle)=\frac{1}{\sqrt{2}}(\langle 00i_2j_2|+\langle i_2j_200|)\mathbb{I}_{A_1}\otimes E|000i_1\rangle\\
=&\frac{1}{\sqrt{2}}\langle 00i_1|E|0i_2j_2\rangle=\frac{1}{\sqrt{2}}\langle 0i_2j_2|E|00i_1\rangle,\quad i_1,j_2,j_2\in\mathbb{Z}_d\setminus\{0\},\\
0=&\langle 000i_1|\mathbb{I}_{A_1}\otimes E\frac{1}{\sqrt{2}}(|0i_2j_20\rangle+|j_200i_2\rangle)=\frac{1}{\sqrt{2}}(\langle 0i_2j_20|+\langle j_200i_2|)\mathbb{I}_{A_1}\otimes E|000i_1\rangle\\
=&\frac{1}{\sqrt{2}}\langle 00i_1|E|0i_2j_2\rangle=\frac{1}{\sqrt{2}}\langle 0i_2j_2|E|00i_1\rangle,\quad i_1,j_2,j_2\in\mathbb{Z}_d\setminus\{0\},\\
0=&\langle 000i_1|\mathbb{I}_{A_1}\otimes E\frac{1}{\sqrt{2}}(|0i_20j_2\rangle+|i_20j_20\rangle)=\frac{1}{\sqrt{2}}(\langle 0i_20j_2|+\langle i_20j_20|)\mathbb{I}_{A_1}\otimes E|000i_1\rangle\\
=&\frac{1}{\sqrt{2}}\langle 00i_1|E|i_20j_2\rangle=\frac{1}{\sqrt{2}}\langle i_20j_2|E|00i_1\rangle,\quad i_1,j_2,j_2\in\mathbb{Z}_d\setminus\{0\},\\
0=&\langle 00i_10|\mathbb{I}_{A_1}\otimes E\frac{1}{\sqrt{2}}(|00i_2j_2\rangle+|i_2j_200\rangle)=\frac{1}{\sqrt{2}}(\langle 00i_2j_2|+\langle i_2j_200|)\mathbb{I}_{A_1}\otimes E|00i_10\rangle\\
=&\frac{1}{\sqrt{2}}\langle 0i_10|E|0i_2j_2\rangle=\frac{1}{\sqrt{2}}\langle 0i_2j_2|E|0i_10\rangle,\quad i_1,j_2,j_2\in\mathbb{Z}_d\setminus\{0\},\\
0=&\langle 00i_10|\mathbb{I}_{A_1}\otimes E\frac{1}{\sqrt{2}}(|0i_2j_20\rangle+|j_200i_2\rangle)=\frac{1}{\sqrt{2}}(\langle 0i_2j_20|+\langle j_200i_2|)\mathbb{I}_{A_1}\otimes E|00i_10\rangle\\
=&\frac{1}{\sqrt{2}}\langle 00i_1|E|0i_2j_2\rangle=\frac{1}{\sqrt{2}}\langle 0i_2j_2|E|0i_10\rangle,\quad i_1,j_2,j_2\in\mathbb{Z}_d\setminus\{0\},\\
0=&\langle 00i_10|\mathbb{I}_{A_1}\otimes E\frac{1}{\sqrt{2}}(|0i_20j_2\rangle+|i_20j_20\rangle)=\frac{1}{\sqrt{2}}(\langle 0i_20j_2|+\langle i_20j_20|)\mathbb{I}_{A_1}\otimes E|00i_10\rangle\\
=&\frac{1}{\sqrt{2}}\langle 0i_10|E|i_20j_2\rangle=\frac{1}{\sqrt{2}}\langle i_20j_2|E|0i_10\rangle,\quad i_1,j_2,j_2\in\mathbb{Z}_d\setminus\{0\},\\
0=&\langle 0i_100|\mathbb{I}_{A_1}\otimes E\frac{1}{\sqrt{2}}(|00i_2j_2\rangle+|i_2j_200\rangle)=\frac{1}{\sqrt{2}}(\langle 00i_2j_2|+\langle i_2j_200|)\mathbb{I}_{A_1}\otimes E|0i_100\rangle\\
=&\frac{1}{\sqrt{2}}\langle i_100|E|0i_2j_2\rangle=\frac{1}{\sqrt{2}}\langle 0i_2j_2|E|i_100\rangle,\quad i_1,j_2,j_2\in\mathbb{Z}_d\setminus\{0\},
\end{aligned}$$

$$
\begin{aligned}
0 =&\langle 0i_100|\mathbb{I}_{A_1}\otimes E\frac{1}{\sqrt{2}}(|0i_2j_20\rangle+|j_200i_2\rangle)=\frac{1}{\sqrt{2}}(\langle 0i_2j_20|+\langle j_200i_2|)\mathbb{I}_{A_1}\otimes E|0i_100\rangle\\
=&\frac{1}{\sqrt{2}}\langle 00i_1|E|0i_2j_2\rangle=\frac{1}{\sqrt{2}}\langle 0i_2j_2|E|i_100\rangle,\quad i_1,j_2,j_2\in\mathbb{Z}_d\setminus\{0\},\\
0 =&\langle 0i_100|\mathbb{I}_{A_1}\otimes E\frac{1}{\sqrt{2}}(|0i_20j_2\rangle+|i_20j_20\rangle)=\frac{1}{\sqrt{2}}(\langle 0i_20j_2|+\langle i_20j_20|)\mathbb{I}_{A_1}\otimes E|0i_100\rangle\\
=&\frac{1}{\sqrt{2}}\langle i_100|E|i_20j_2\rangle=\frac{1}{\sqrt{2}}\langle i_20j_2|E|i_100\rangle,\quad i_1,j_2,j_2\in\mathbb{Z}_d\setminus\{0\}.
\end{aligned}
$$

It means that

$$
\langle i_1j_1k_1|E|i_2j_2k_2\rangle=\langle i_2j_2k_2|E|i_1j_1k_1\rangle=0,\ wt(i_1,j_1,k_1)=1,wt(i_2,j_2,k_2)=2,\ (i_1,j_1,k_1)\neq(i_2,j_2,k_2)\in\mathbb{Z}_d\times\mathbb{Z}_d\times\mathbb{Z}_d. \tag{11}
$$

If $|\alpha_i\rangle\in\mathcal{A}_1$, and $|\alpha_j\rangle\in\mathcal{A}_3$, we have

$$
\begin{aligned}
0 =&\langle 000i_1|\mathbb{I}_{A_1}\otimes E\frac{1}{2}(|0i_2j_2k_2\rangle+|k_20i_2j_2\rangle+|j_2k_20i_2\rangle+|i_2j_2k_20\rangle)\\
=&\frac{1}{2}(\langle 0i_2j_2k_2|+\langle k_20i_2j_2|+\langle j_2k_20i_2|+\langle i_2j_2k_20|)\mathbb{I}_{A_1}\otimes E|000i_1\rangle\\
=&\langle 00i_1|E\frac{1}{2}|i_2j_2k_2\rangle=\frac{1}{2}\langle i_2j_2k_2|E|00i_1\rangle,\quad i_1,i_2,j_2,k_2\in\mathbb{Z}_d\setminus\{0\},\\
0 =&\langle 00i_10|\mathbb{I}_{A_1}\otimes E\frac{1}{2}(|0i_2j_2k_2\rangle+|k_20i_2j_2\rangle+|j_2k_20i_2\rangle+|i_2j_2k_20\rangle)\\
=&\frac{1}{2}(\langle 0i_2j_2k_2|+\langle k_20i_2j_2|+\langle j_2k_20i_2|+\langle i_2j_2k_20|)\mathbb{I}_{A_1}\otimes E|00i_10\rangle\\
=&\frac{1}{2}\langle 0i_10|E|i_2j_2k_2\rangle=\frac{1}{2}\langle i_2j_2k_2|E|0i_10\rangle,\quad i_1,i_2,j_2,k_2\in\mathbb{Z}_d\setminus\{0\},\\
0 =&\langle 0i_100|\mathbb{I}_{A_1}\otimes E\frac{1}{2}(|0i_2j_2k_2\rangle+|k_20i_2j_2\rangle+|j_2k_20i_2\rangle+|i_2j_2k_20\rangle)\\
=&\frac{1}{2}(\langle 0i_2j_2k_2|+\langle k_20i_2j_2|+\langle j_2k_20i_2|+\langle i_2j_2k_20|)\mathbb{I}_{A_1}\otimes E|0i_100\rangle\\
=&\frac{1}{2}\langle i_100|E\frac{1}{2}|i_2j_2k_2\rangle=\langle i_2j_2k_2|E|i_100\rangle,\quad i_1,i_2,j_2,k_2\in\mathbb{Z}_d\setminus\{0\}.
\end{aligned}
$$

Then we have

$$
\langle i_1j_1k_1|E|i_2j_2k_2\rangle=\langle i_2j_2k_2|E|i_1j_1k_1\rangle=0,\ \ wt(i_1,j_1,k_1)=1,wt(i_2,j_2,k_2)=3,\ (i_1,j_1,k_1)\neq(i_2,j_2,k_2)\in\mathbb{Z}_d\times\mathbb{Z}_d\times\mathbb{Z}_d. \tag{12}
$$

If $|\alpha_i\rangle,|\alpha_j\rangle\in\mathcal{A}_2$, then we have

$$
\begin{aligned}
0 =&\frac{1}{\sqrt{2}}(\langle 00i_1j_1|+\langle i_1j_100|)\mathbb{I}_{A_1}\otimes E\frac{1}{\sqrt{2}}(|00i_2j_2\rangle+|i_2j_200\rangle)=\frac{1}{2}(\langle 0i_1j_1|E|0i_2j_2\rangle+\langle i_1|i_2\rangle\langle j_100|E|j_200\rangle)\\
=&\frac{1}{2}\langle 0i_1j_1|E|0i_2j_2\rangle,\quad i_1,j_1,i_2,j_2\in\mathbb{Z}_d\setminus\{0\},\ (i_1,j_1)\neq(i_2,j_2),\\
0 =&\frac{1}{\sqrt{2}}(\langle 00i_1j_1|+\langle i_1j_100|)\mathbb{I}_{A_1}\otimes E\frac{1}{\sqrt{2}}(|0i_2j_20\rangle+|j_200i_2\rangle)=\frac{1}{\sqrt{2}}(\langle 0i_2j_20|+\langle j_200i_2|)\mathbb{I}_{A_1}\otimes E\frac{1}{\sqrt{2}}(|00i_1j_1\rangle+|i_1j_100\rangle)\\
=&\frac{1}{2}(\langle i_2j_20|E|0i_1j_1\rangle+\langle j_2|i_1\rangle\langle 00i_2|E|j_100\rangle)=\frac{1}{2}(\langle 0i_1j_1|E|i_2j_20\rangle+\langle i_1|j_2\rangle\langle j_100|E|00i_2\rangle)\\
=&\frac{1}{2}\langle 0i_1j_1|E|i_2j_20\rangle=\frac{1}{2}\langle i_2j_20|E|0i_1j_1\rangle,\quad i_1,j_1,i_2,j_2\in\mathbb{Z}_d\setminus\{0\},\\
0 =&\frac{1}{\sqrt{2}}(\langle 00i_1j_1|+\langle i_1j_100|)\mathbb{I}_{A_1}\otimes E\frac{1}{\sqrt{2}}(|0i_20j_2\rangle+|i_20j_20\rangle)=\frac{1}{\sqrt{2}}(\langle 0i_20j_2|+\langle i_20j_20|)\mathbb{I}_{A_1}\otimes E\frac{1}{\sqrt{2}}(|00i_1j_1\rangle+|i_1j_100\rangle)\\
=&\frac{1}{2}(\langle 0i_1j_1|E|i_20j_2\rangle+\langle i_1|i_2\rangle\langle j_100|E|0j_20\rangle)=\frac{1}{2}(\langle i_20j_2|E|0i_1j_1\rangle+\langle i_2|i_1\rangle\langle j_100|E|0j_20\rangle\\
=&\langle 0i_1j_1|E|i_20j_2\rangle=\langle i_20j_2|E|0i_1j_1\rangle,\quad i_1,j_1,i_2,j_2\in\mathbb{Z}_d\setminus\{0\},
\end{aligned}
$$

$$
\begin{aligned}
0=&\frac{1}{\sqrt{2}}(\langle 0i_1j_10|+\langle j_100i_1|)\mathbb{I}_{A_1}\otimes E\frac{1}{\sqrt{2}}(|0i_2j_20\rangle+|j_200i_2\rangle)=\frac{1}{2}(\langle i_1j_10|E|i_2j_20\rangle+\langle j_1|j_2\rangle\langle 00i_1|E|00i_2\rangle)\\
=&\frac{1}{2}\langle i_1j_10|E|i_2j_20\rangle,\quad i_1,j_1,i_2,j_2\in\mathbb{Z}_d\setminus\{0\},\ (i_1,j_1)\neq(i_2,j_2),\\
0=&\frac{1}{\sqrt{2}}(\langle 0i_1j_10|+\langle j_100i_1|)\mathbb{I}_{A_1}\otimes E\frac{1}{\sqrt{2}}(|0i_20j_2\rangle+|i_20j_20\rangle)=\frac{1}{\sqrt{2}}(\langle 0i_20j_2|+\langle i_20j_20|)\mathbb{I}_{A_1}\otimes E\frac{1}{\sqrt{2}}(|0i_1j_10\rangle+|j_100i_1\rangle)\\
=&\frac{1}{2}(\langle i_1j_10|E|i_20j_2\rangle+\langle j_1|i_2\rangle\langle 00i_1|E|0j_20\rangle)=\frac{1}{2}(\langle i_20j_2|E|i_1j_10\rangle+\langle i_2|j_1\rangle\langle 0j_20|E|00i_1\rangle)\\
=&\frac{1}{2}\langle 0i_1j_1|E|i_20j_2\rangle=\frac{1}{2}\langle i_20j_2|E|0i_1j_1\rangle,\quad i_1,j_1,i_2,j_2\in\mathbb{Z}_d\setminus\{0\},\\
0=&\frac{1}{\sqrt{2}}(\langle 0i_10j_1|+\langle i_10j_10|)\mathbb{I}_{A_1}\otimes E\frac{1}{\sqrt{2}}(|0i_20j_2\rangle+|i_20j_20\rangle)=\frac{1}{2}(\langle i_10j_1|E|i_20j_2\rangle+\langle i_1|i_2\rangle\langle 0j_10|E|0j_20\rangle)\\
=&\frac{1}{2}\langle i_10j_1|E|i_20j_2\rangle,\quad i_1,j_1,i_2,j_2\in\mathbb{Z}_d\setminus\{0\},\ (i_1,j_1)\neq(i_2,j_2).
\end{aligned}
$$

Then we have

$$
\langle i_1j_1k_1|E|i_2j_2k_2\rangle=0,\quad wt(i_1,j_1,k_1)=wt(i_2,j_2,k_2)=2,\ (i_1,j_1,k_1)\neq(i_2,j_2,k_2)\in\mathbb{Z}_d\times\mathbb{Z}_d\times\mathbb{Z}_d. \tag{13}
$$

If $|\alpha_i\rangle\in\mathcal{A}_2$, and $|\alpha_j\rangle\in\mathcal{A}_3$, then we have

$$
\begin{aligned}
0=&\frac{1}{\sqrt{2}}(\langle 00i_1j_1|+\langle i_1j_100|)\mathbb{I}_{A_1}\otimes E\frac{1}{2}(|0i_2j_2k_2\rangle+|k_20i_2j_2\rangle+|j_2k_20i_2\rangle+|i_2j_2k_20\rangle)\\
=&\frac{1}{\sqrt{2}}(\langle 0i_2j_2k_2|+\langle k_20i_2j_2|+\langle j_2k_20i_2|+\langle i_2j_2k_20|)\mathbb{I}_{A_1}\otimes E\frac{1}{2}(|00i_1j_1\rangle+|i_1j_100\rangle)\\
=&\frac{1}{2\sqrt{2}}(\langle 0i_1j_1|E|i_2j_2k_2\rangle+\langle i_1|k_2\rangle\langle j_100|E|0i_2j_2\rangle+\langle i_1|j_2\rangle\langle j_100|E|k_20i_2\rangle+\langle i_1|i_2\rangle\langle j_100|E|j_2k_20\rangle)\\
=&\frac{1}{2\sqrt{2}}(\langle i_2j_2k_2|E|0i_1j_1\rangle+\langle k_2|i_1\rangle\langle 0i_2j_2|E|j_100\rangle+\langle j_2|i_1\rangle\langle k_20i_2|E|j_100\rangle+\langle j_2|i_1\rangle\langle j_2k_20|E|j_100\rangle)\\
=&\frac{1}{2\sqrt{2}}\langle 0i_1j_1|E|i_2j_2k_2\rangle=\frac{1}{2\sqrt{2}}\langle i_2j_2k_2|E|0i_1j_1\rangle,\quad i_1,j_1,i_2,j_2,k_2\in\mathbb{Z}_d\setminus\{0\},\\
0=&\frac{1}{\sqrt{2}}(\langle 0i_1j_10|+\langle j_100i_1|)\mathbb{I}_{A_1}\otimes E\frac{1}{2}(|0i_2j_2k_2\rangle+|k_20i_2j_2\rangle+|j_2k_20i_2\rangle+|i_2j_2k_20\rangle)\\
=&\frac{1}{\sqrt{2}}(\langle 0i_2j_2k_2|+\langle k_20i_2j_2|+\langle j_2k_20i_2|+\langle i_2j_2k_20|)\mathbb{I}_{A_1}\otimes E\frac{1}{2}(|0i_1j_10\rangle+|j_100i_1\rangle)\\
=&\frac{1}{2\sqrt{2}}(\langle i_1j_10|E|i_2j_2k_2\rangle+\langle j_1|k_2\rangle\langle 00i_1|E|0i_2j_2\rangle+\langle j_1|j_2\rangle\langle 00i_1|E|k_20i_2\rangle+\langle j_1|i_2\rangle\langle 00i_1|E|j_2k_20\rangle)\\
=&\frac{1}{2\sqrt{2}}(\langle i_2j_2k_2|E|0i_1j_1\rangle+\langle k_2|j_1\rangle\langle 0i_2j_2|E|00i_1\rangle+\langle j_2|j_1\rangle\langle k_20i_2|E|00i_1\rangle+\langle j_2|j_1\rangle\langle j_2k_20|E|00i_1\rangle)\\
=&\frac{1}{2\sqrt{2}}\langle i_1j_10|E|i_2j_2k_2\rangle=\frac{1}{2\sqrt{2}}\langle i_2j_2k_2|E|i_1j_10\rangle,\quad i_1,j_1,i_2,j_2,k_2\in\mathbb{Z}_d\setminus\{0\},\\
0=&\frac{1}{\sqrt{2}}(\langle 0i_10j_1|+\langle i_10j_10|)\mathbb{I}_{A_1}\otimes E\frac{1}{2}(|0i_2j_2k_2\rangle+|k_20i_2j_2\rangle+|j_2k_20i_2\rangle+|i_2j_2k_20\rangle)\\
=&\frac{1}{\sqrt{2}}(\langle 0i_2j_2k_2|+\langle k_20i_2j_2|+\langle j_2k_20i_2|+\langle i_2j_2k_20|)\mathbb{I}_{A_1}\otimes E\frac{1}{2}(|0i_10j_1\rangle+|i_10j_10\rangle)\\
=&\frac{1}{2\sqrt{2}}(\langle i_10j_1|E|i_2j_2k_2\rangle+\langle i_1|k_2\rangle\langle 0j_10|E|0i_2j_2\rangle+\langle i_1|j_2\rangle\langle 0j_10|E|k_20i_2\rangle+\langle i_1|i_2\rangle\langle 0j_10|E|j_2k_20\rangle)\\
=&\frac{1}{2\sqrt{2}}(\langle i_2j_2k_2|E|0i_1j_1\rangle+\langle k_2|i_1\rangle\langle 0i_2j_2|E|0j_10\rangle+\langle j_2|i_1\rangle\langle k_20i_2|E|0j_10\rangle+\langle j_2|i_1\rangle\langle j_2k_20|E|0j_10\rangle)\\
=&\frac{1}{2\sqrt{2}}\langle i_1j_10|E|i_2j_2k_2\rangle=\frac{1}{2\sqrt{2}}\langle i_2j_2k_2|E|i_1j_10\rangle,\quad i_1,j_1,i_2,j_2,k_2\in\mathbb{Z}_d\setminus\{0\}.
\end{aligned}
$$

That is

$$
\langle i_1j_1k_1|E|i_2j_2k_2\rangle=\langle i_2j_2k_2|E|i_1j_1k_1\rangle=0,\quad wt(i_1,j_1,k_1)=2,wt(i_2,j_2,k_2)=3,\ (i_1,j_1,k_1),(i_2,j_2,k_2)\in\mathbb{Z}_d\times\mathbb{Z}_d\times\mathbb{Z}_d. \tag{14}
$$

If $|\alpha_i\rangle, |\alpha_j\rangle \in \mathcal{A}_3$, then we have

$$\begin{aligned}
0 =&\frac{1}{2}(\langle 0i_1j_1k_1| + \langle k_10i_1j_1| + \langle j_1k_10i_1|) + \langle i_1j_1k_10|)\mathbb{I}_{A_1} \otimes E\frac{1}{2}(|0i_2j_2k_2\rangle + |k_20i_2j_2\rangle + |j_2k_20i_2\rangle + |i_2j_2k_20\rangle)) \\
=&\frac{1}{4}(\langle i_1j_1k_1|E|i_2j_2k_2\rangle + \langle k_1|k_2\rangle\langle 0i_1j_1|E|0i_2j_2\rangle + \langle k_1|j_2\rangle\langle 0i_1j_1|E|k_20i_2\rangle + \langle k_1|i_2\rangle\langle 0i_1j_1|E|j_2k_20\rangle \\
&+ \langle j_1|k_2\rangle\langle k_10i_1|E|0i_2j_2\rangle + \langle j_1|j_2\rangle\langle k_10i_1|E|k_20i_2\rangle + \langle j_1|k_2\rangle\langle k_10i_1|E|j_2k_20\rangle + \langle i_1|k_2\rangle\langle j_1k_10|E|0i_2j_2\rangle \\
&+ \langle i_1|j_2\rangle\langle j_1k_10|E|k_20i_2\rangle + \langle i_1|i_2\rangle\langle j_1k_10|E|j_2k_20\rangle) \\
=&\frac{1}{4}\langle i_1j_1k_1|E|i_2j_2k_2\rangle, \quad i_1, j_1, k_1, i_2, j_2, k_2 \in \mathbb{Z}_d \setminus \{0\},\ (i_1, j_1, k_1) \neq (i_2, j_2, k_2).
\end{aligned}$$

That is

$$\langle i_1j_1k_1|E|i_2j_2k_2\rangle = 0, \quad wt(i_1, j_1, k_1) = wt(i_2, j_2, k_2) = 3,\ (i_1, j_1, k_1) \neq (i_2, j_2, k_2) \in \mathbb{Z}_d \times \mathbb{Z}_d \times \mathbb{Z}_d. \tag{15}$$

By Eqs. (10), (11), (12), (13), (14), (15), we obtain that the off-diagonal elements of E are all zeros.

Next, we consider the diagonal elements of $E$. If $|\alpha_i\rangle \in \mathcal{A}_1$, then we have

$$\langle 000|E|000\rangle = \langle \alpha_1|\mathbb{I}_{A_1} \otimes E|\alpha_1\rangle = \langle \alpha_i|\mathbb{I}_{A_1} \otimes E|\alpha_i\rangle = \langle i_1j_1k_1|E|i_1j_1k_1\rangle, \quad wt(i_1, j_1, k_1) = 1,\ (i_1, j_1, k_1) \in \mathbb{Z}_d \times \mathbb{Z}_d \times \mathbb{Z}_d. \tag{16}$$

If $|\alpha_i\rangle \in \mathcal{A}_2$, then we have

$$\begin{aligned}
\langle 000|E|000\rangle =&\langle \alpha_1|\mathbb{I}_{A_1} \otimes E|\alpha_1\rangle = \langle \alpha_i|\mathbb{I}_{A_1} \otimes E|\alpha_i\rangle = \frac{1}{2}(\langle 0i_1j_1|E|0i_1j_1\rangle + \langle j_100|E|j_100\rangle \\
=&\frac{1}{2}(\langle i_1j_10|E|i_1j_10\rangle + \langle 00i_1|E|00i_1\rangle) = \frac{1}{2}(\langle i_10j_1|E|i_10j_1\rangle + \langle 0j_10|E|0j_10\rangle), \\
&wt(i_1, j_1, k_1) = 2,\ (i_1, j_1, k_1) \in \mathbb{Z}_d \times \mathbb{Z}_d \times \mathbb{Z}_d.
\end{aligned}$$

That is

$$\langle 000|E|000\rangle = \langle i_1j_1k_1|E|i_1j_1k_1\rangle, \quad wt(i_1, j_1, k_1) = 2,\ (i_1, j_1, k_1) \in \mathbb{Z}_d \times \mathbb{Z}_d \times \mathbb{Z}_d \setminus \{0, 0, 0, 0\}. \tag{17}$$

If $|\alpha_i\rangle \in \mathcal{A}_3$, then we have

$$\begin{aligned}
\langle 000|E|000\rangle =&\langle \alpha_1|\mathbb{I}_{A_1} \otimes E|\alpha_1\rangle = \langle \alpha_i|\mathbb{I}_{A_1} \otimes E|\alpha_i\rangle \\
=&\frac{1}{4}(\langle i_1j_1k_1|E|i_1j_1k_1\rangle + \langle 0i_1j_1|E|0i_2j_2\rangle + \langle k_10i_1|E|k_10i_1\rangle + \langle j_1k_10|E|j_1k_10\rangle), \\
&wt(i_1, j_1, k_1) = 3,\ (i_1, j_1, k_1) \in \mathbb{Z}_d \times \mathbb{Z}_d \times \mathbb{Z}_d.
\end{aligned}$$

That is

$$\langle 000|E|000\rangle = \langle i_1j_1k_1|E|i_1j_1k_1\rangle, \quad wt(i_1, j_1, k_1) = 3,\ (i_1, j_1, k_1) \in \mathbb{Z}_d \times \mathbb{Z}_d \times \mathbb{Z}_d \setminus \{0, 0, 0, 0\}. \tag{18}$$

By Eqs. (16), (17), (18), we obtain that the diagonal elements of $E$ are all equal. Thus $E \propto \mathbb{I}$. This completes the proof. $\square$