Supplementary material: strongly nonlocal four-partite UPBs

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In this supplementary material, we show strongly nonlocal UPBs in four-partite systems.

Let

$$\begin{aligned}
\mathcal{A}_1 &:= \{|\xi_i\rangle_A|\eta_j\rangle_B|0\rangle_C|\xi_\ell\rangle_D:\ (i,j,\ell)\in\mathbb{Z}_{d_A-1}\times\mathbb{Z}_{d_B-1}\times\mathbb{Z}_{d_D-1}\setminus\{(0,0,0)\}\},\\
\mathcal{A}_2 &:= \{|\xi_i\rangle_A|d_B-1\rangle_B|\eta_k\rangle_C|\eta_\ell\rangle_D:\ (i,k,\ell)\in\mathbb{Z}_{d_A-1}\times\mathbb{Z}_{d_C-1}\times\mathbb{Z}_{d_D-1}\setminus\{(0,0,0)\}\},\\
\mathcal{A}_3 &:= \{|\xi_i\rangle_A|\xi_j\rangle_B|\xi_k\rangle_C|d_D-1\rangle_D:\ (i,j,k)\in\mathbb{Z}_{d_A-1}\times\mathbb{Z}_{d_B-1}\times\mathbb{Z}_{d_C-1}\setminus\{(0,0,0)\}\},\\
\mathcal{A}_4 &:= \{|\xi_i\rangle_A|d_B-1\rangle_B|0\rangle_C|d_D-1\rangle_D:\ i\in\mathbb{Z}_{d_A-1}\setminus\{0\}\},\\
\mathcal{A}_5 &:= \{|d_A-1\rangle_A|\eta_j\rangle_B|\xi_k\rangle_C|\eta_\ell\rangle_D:\ (j,k,\ell)\in\mathbb{Z}_{d_B-1}\times\mathbb{Z}_{d_C-1}\times\mathbb{Z}_{d_D-1}\setminus\{(0,0,0)\}\},\\
\mathcal{A}_6 &:= \{|d_A-1\rangle_A|\eta_j\rangle_B|0\rangle_C|0\rangle_D:\ j\in\mathbb{Z}_{d_B-1}\setminus\{0\}\},\\
\mathcal{A}_7 &:= \{|d_A-1\rangle_A|0\rangle_B|\xi_k\rangle_C|d_D-1\rangle_D:\ k\in\mathbb{Z}_{d_C-1}\setminus\{0\}\},\\
\mathcal{A}_8 &:= \{|d_A-1\rangle_A|d_B-1\rangle_B|d_C-1\rangle_C|\eta_\ell\rangle_D:\ \ell\in\mathbb{Z}_{d_D-1}\setminus\{0\}\},\\
\mathcal{B}_1 &:= \{|\eta_i\rangle_A|\xi_j\rangle_B|d_C-1\rangle_C|\eta_\ell\rangle_D:\ (i,j,\ell)\in\mathbb{Z}_{d_A-1}\times\mathbb{Z}_{d_B-1}\times\mathbb{Z}_{d_D-1}\setminus\{(0,0,0)\}\},\\
\mathcal{B}_2 &:= \{|\eta_i\rangle_A|0\rangle_B|\xi_k\rangle_C|\xi_\ell\rangle_D:\ (i,k,\ell)\in\mathbb{Z}_{d_A-1}\times\mathbb{Z}_{d_C-1}\times\mathbb{Z}_{d_D-1}\setminus\{(0,0,0)\}\},\\
\mathcal{B}_3 &:= \{|\eta_i\rangle_A|\eta_j\rangle_B|\eta_k\rangle_C|0\rangle_D:\ (i,j,k)\in\mathbb{Z}_{d_A-1}\times\mathbb{Z}_{d_B-1}\times\mathbb{Z}_{d_C-1}\setminus\{(0,0,0)\}\},\\
\mathcal{B}_4 &:= \{|\eta_i\rangle_A|0\rangle_B|d_C-1\rangle_C|0\rangle_D:\ i\neq 0\in\mathbb{Z}_{d_A-1}\},\\
\mathcal{B}_5 &:= \{|0\rangle_A|\xi_j\rangle_B|\eta_k\rangle_C|\xi_\ell\rangle_D:\ (j,k,\ell)\in\mathbb{Z}_{d_B-1}\times\mathbb{Z}_{d_C-1}\times\mathbb{Z}_{d_D-1}\setminus\{(0,0,0)\}\},\\
\mathcal{B}_6 &:= \{|0\rangle_A|\xi_j\rangle_B|d_C-1\rangle_C|d_D-1\rangle_D:\ j\in\mathbb{Z}_{d_B-1}\setminus\{0\}\},\\
\mathcal{B}_7 &:= \{|0\rangle_A|d_B-1\rangle_B|\eta_k\rangle_C|0\rangle_D:\ k\in\mathbb{Z}_{d_C-1}\setminus\{0\}\},\\
\mathcal{B}_8 &:= \{|0\rangle_A|0\rangle_B|0\rangle_C|\xi_\ell\rangle_D:\ \ell\in\mathbb{Z}_{d_D-1}\setminus\{0\}\},\\
\mathcal{F} &:= \{|\beta_i\rangle_A|\beta_j\rangle_B|\beta_k\rangle_C|\beta_\ell\rangle_D:\ (i,j,k,\ell)\in\mathbb{Z}_{d_A-2}\times\mathbb{Z}_{d_B-2}\times\mathbb{Z}_{d_C-2}\times\mathbb{Z}_{d_D-2}\setminus\{(0,0,0,0)\}\},\\
|S\rangle &:= \left(\sum_{i=0}^{d_A-1}|i\rangle\right)_A\left(\sum_{j=0}^{d_B-1}|j\rangle\right)_B\left(\sum_{k=0}^{d_C-1}|k\rangle\right)_C\left(\sum_{\ell=0}^{d_D-1}|\ell\rangle\right)_D,
\end{aligned}\tag{1}$$

where $|\eta_s\rangle_X=\sum_{t=0}^{d_X-2}w_{d_X-1}^{st}|t\rangle_X$, and $|\xi_s\rangle_X=\sum_{t=0}^{d_X-2}w_{d_X-1}^{st}|t+1\rangle_X$ for $s\in\mathbb{Z}_{d_X-1}$, and $X\in\{A,B,C,D\}$, $|\beta_s\rangle_X=\sum_{t=0}^{d_X-3}w_{d_X-2}^{st}|t+1\rangle_X$ for $s\in\mathbb{Z}_{d_X-2}$, and $X\in\{A,B,C,D\}$.

Note that $\{|\eta_s\rangle_X\}_{s\in\mathbb{Z}_{d_X-1}}$, $\{|\xi_s\rangle_X\}_{s\in\mathbb{Z}_{d_X-1}}$, and $\{|\beta_s\rangle_X\}_{s\in\mathbb{Z}_{d_X-2}}$ are three orthogonal sets, $X\in\{A,B,C,D\}$, which are spanned by $\{|t\rangle_X\}_{t=0}^{d_X-2}$, $\{|t\rangle_X\}_{t=1}^{d_X-1}$, and $\{|t\rangle_X\}_{t=1}^{d_X-2}$, respectively. The 17 subsets $\mathcal{A}_i,\mathcal{B}_i$ $(i=1,\cdots,8)$, $\mathcal{F}$ in $A|BCD$ bipartition correspond to the 17 blocks of $d_A\times d_Bd_Cd_D$ grid in Fig. 1.

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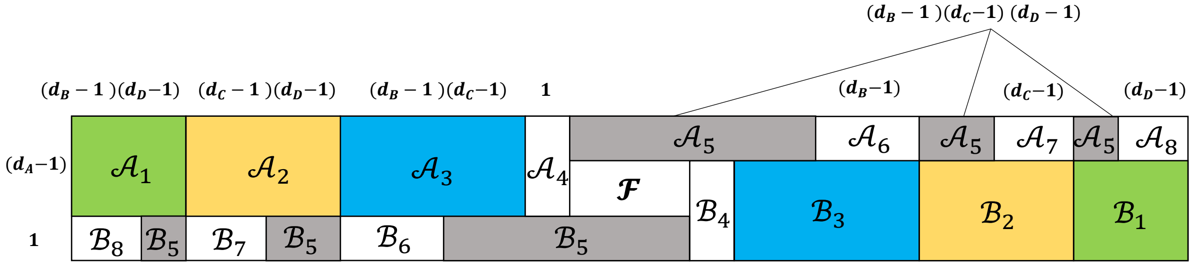

FIG. 1: The corresponding $d_A \times d_B d_C d_D$ grid of $\mathcal{A}_i, \mathcal{B}_i$ $(i = 1, \cdots, 8)$, $\mathcal{F}$ (Eq. (1)) in $A|BCD$ bipartition. Moreover, $\mathcal{A}_i$ is symmetrical to $\mathcal{B}_i$ for $1 \leq i \leq 8$.

Now, we show that the union of the states from the 17 subsets and $\{|S\rangle\}$ is a UPB.

Proposition 1 *In $d_A \otimes d_B \otimes d_C \otimes d_D$, $3 \leq d_A \leq d_B \leq d_C \leq d_D$, the set of states $\cup_{i=1}^{8}(\mathcal{A}_i \cup \mathcal{B}_i) \cup \mathcal{F} \cup \{|S\rangle\}$ given by Eq. (1) is a UPB of size $d_A d_B d_C d_D - 16$.*

Proof. By Fig. 1, we know that the set of states $\cup_{i=1}^{8}(\mathcal{A}_i \cup \mathcal{B}_i) \cup \mathcal{F} \cup \{|S\rangle\}$ has a similar structure for any $3 \leq d_A \leq d_B \leq d_C \leq d_D$. Without loss of generality, we only consider the case for $d_A = d_B = d_C = d_D = 3$. Note that in this case, $\mathcal{F} = \emptyset$. For the same discussion as Example 3 in the main text, we proved it by contradiction. Assume that there exists a product state $|\psi\rangle$ in the complementary space of the space spanned by the states in $\cup_{i=1}^{8}(\mathcal{A}_i \cup \mathcal{B}_i)$. Then we consider the matrix form of $|\psi\rangle$ in $A|BCD$, $AB|CD$ and $ABC|D$ bipartitions. First, we consider $A|BCD$ bipartition. We have

$$M = \begin{bmatrix} a_0 & a_0 & a_0 & a_0 & b_0 & b_0 & b_0 & b_0 & c_0 & c_0 & c_0 & c_0 & d_0 & e_0 & e_0 & e_0 & e_0 & f_0 & f_0 & e_0 & e_0 & g_0 & g_0 & e_0 & e_0 & h_0 & h_0 \\ a_0 & a_0 & a_0 & a_0 & b_0 & b_0 & b_0 & b_0 & c_0 & c_0 & c_0 & c_0 & d_0 & s & d_1 & c_1 & c_1 & c_1 & c_1 & b_1 & b_1 & b_1 & b_1 & a_1 & a_1 & a_1 & a_1 \\ h_1 & h_1 & e_1 & e_1 & g_1 & g_1 & e_1 & e_1 & f_1 & f_1 & e_1 & e_1 & e_1 & e_1 & d_1 & c_1 & c_1 & c_1 & c_1 & b_1 & b_1 & b_1 & b_1 & a_1 & a_1 & a_1 & a_1 \end{bmatrix} \tag{2}$$

It satisfies

$$\operatorname{rank}(M) = 1, \quad \operatorname{sum}(M) = 0. \tag{3}$$

Then we consider $AB|CD$ bipartition. We can rearrange the first row of M to the 3×9 matrix M_2 through (AB, CD) coordinates of M, and rearrange the last row of M to the 3×9 matrix M_0 through (AB, CD) coordinates of M, where

$$M_2 = \begin{bmatrix} b_0 & b_0 & d_0 & b_0 & b_0 & h_0 & h_0 & c_0 & c_0 \\ f_0 & a_0 & a_0 & e_0 & e_0 & e_0 & e_0 & c_0 & c_0 \\ f_0 & a_0 & a_0 & e_0 & e_0 & e_0 & e_0 & g_0 & g_0 \end{bmatrix}, \quad \operatorname{rank}(M_2) = 0 \text{ or } 1, \tag{4}$$

$$M_0 = \begin{bmatrix} g_1 & g_1 & e_1 & e_1 & e_1 & e_1 & a_1 & a_1 & f_1 \\ c_1 & c_1 & e_1 & e_1 & e_1 & e_1 & a_1 & a_1 & f_1 \\ c_1 & c_1 & h_1 & h_1 & b_1 & b_1 & d_1 & b_1 & b_1 \end{bmatrix}, \quad \operatorname{rank}(M_0) = 0 \text{ or } 1, \tag{5}$$

Next, we consider $ABC|D$ bipartition. We can rearrange the first row of M to the 9×3 matrix N_2 through (ABC, D) coordinates of M, and rearrange the last row of M to the 9×3 matrix N_0 through (ABC, D) coordinates of M, where

$$N_2 = \begin{bmatrix} h_0 & b_0 & b_0 & e_0 & e_0 & e_0 & e_0 & f_0 & f_0 \\ h_0 & b_0 & b_0 & e_0 & e_0 & e_0 & e_0 & a_0 & a_0 \\ c_0 & c_0 & d_0 & c_0 & c_0 & g_0 & g_0 & a_0 & a_0 \end{bmatrix}^T, \quad \operatorname{rank}(N_2) = 0 \text{ or } 1, \tag{6}$$

$$N_0 = \begin{bmatrix} a_1 & a_1 & g_1 & g_1 & c_1 & c_1 & d_1 & c_1 & c_1 \\ a_1 & a_1 & e_1 & e_1 & e_1 & e_1 & b_1 & b_1 & h_1 \\ f_1 & f_1 & e_1 & e_1 & e_1 & e_1 & b_1 & b_1 & h_1 \end{bmatrix}^T, \quad \operatorname{rank}(N_0) = 0 \text{ or } 1. \tag{7}$$

Assume $a_0 \neq 0$. Since $\operatorname{rank}(M) = 1$, we have $h_1 = e_1$ and $e_0 = s = d_1 = c_1 = f_0 = b_1 = g_0 = a_1 = h_0$. By Eq. (6), we obtain $e_0 = c_0 = b_0 = d_0$.

(i) If $e_0 \neq 0$, then $e_0 = a_0$ by Eq. (4), and $e_0 = g_1 = e_1 = f_1$ by Eq. (5). It is impossible for sum(M)=0.

(ii) If $e_0 = 0$, then $g_1 = e_1 = f_1 = 0$ since rank$(M) = 1$. It is impossible for sum(M)=0.

Thus we have $a_0 = 0$. By the symmetry of M, we must have $a_1 = 0$.

Assume $b_0 \neq 0$. Since rank$(M) = 1$, we have $h_1 = e_1 = g_1 = 0$, and $e_0 = s = d_1 = c_1 = f_0 = b_1 = g_0 = a_1 = h_0 = 0$. By Eq. (6), we obtain $c_0 = d_0 = 0$. Further, since rank$(M) = 1$, we have $f_1 = 0$. It is impossible for sum(M)=0. We must have $b_0 = b_1 = 0$.

Assume $c_0 \neq 0$. Since rank$(M) = 1$, we have $h_1 = e_1 = g_1 = f_1 = 0$, and $e_0 = s = d_1 = c_1 = f_0 = b_1 = g_0 = a_1 = h_0 = 0$. By Eq. (4), we obtain $d_0 = 0$. It is impossible for sum(M)=0. We must have $c_0 = c_1 = 0$.

Assume $d_0 \neq 0$. Since rank$(M) = 1$, we have $h_1 = e_1 = g_1 = f_1 = 0$, and $e_0 = s = d_1 = c_1 = f_0 = b_1 = g_0 = a_1 = h_0 = 0$. It is impossible for sum(M)=0. We must have $d_0 = d_1 = 0$.

Assume $h_1 \neq 0$. Since rank$(M) = 1$, we have $e_0 = s = f_0 = g_0 = h_0 = 0$. By Eq. (5), we obtain $e_1 = g_1 = f_1 = 0$. It is impossible for sum(M)=0. We must have $h_0 = h_1 = 0$.

Assume $e_1 \neq 0$. Since rank$(M) = 1$, we have $e_0 = s = f_0 = g_0 = 0$. By Eq. (7), we obtain $f_1 = g_1 = 0$. It is impossible for sum(M)=0. We must have $e_0 = e_1 = 0$.

Assume $g_1 \neq 0$. Since rank$(M) = 1$, we have $s = f_0 = g_0 = 0$. By Eq. (5), we obtain $f_1 = 0$. It is impossible for sum(M)=0. We must have $g_0 = g_1 = 0$.

Assume $f_1 \neq 0$. Since rank$(M) = 1$, we have $s = f_0 = 0$. It is impossible for sum(M)=0. We must have $f_0 = f_1 = 0$.

Since sum$(M) = 0$, WE MUST HAVE $s = 0$. It is impossible for rank$(M) = 1$.

Thus $|\psi\rangle$ must be an entangled state, and the set of states $\cup_{i=1}^{8}(\mathcal{A}_i \cup \mathcal{B}_i) \cup \{|S\rangle\}$ is a UPB. The proof for the case $d_A, d_B, d_C, d_D \geq 3$ is similar to above proof. □

Now, we show the strong quantum nonlocality for the above UPBs.

Proposition 2 *In $d_A \otimes d_B \otimes d_C \otimes d_D$, $3 \leq d_A \leq d_B \leq d_C \leq d_D$, the UPB $\cup_{i=1}^{8}(\mathcal{A}_i \cup \mathcal{B}_i) \cup \mathcal{F} \cup \{|S\rangle\}$ given by Eq. (1) is strongly nonlocal.*

Proof. Denote $\mathcal{U} := \cup_{i=1}^{8}(\mathcal{A}_i \cup \mathcal{B}_i) \cup \mathcal{F} \cup \{|S\rangle\}$. Let B, C and D come together to perform a joint orthogonality-preserving POVM $\{E = M^\dagger M\}$, where $E = (a_{ijk,\ell mn})_{i,\ell \in \mathbb{Z}_{d_B}, j,m \in \mathbb{Z}_{d_C}, k,n \in \mathbb{Z}_{d_D}}$. Then the postmeasurement states $\{\mathbb{I}_A \otimes M|\psi\rangle : \ |\psi\rangle \in \mathcal{U}\}$ should be mutually orthogonal.

Step 1 Since $\langle \xi_1|\eta_1\rangle_A \neq 0$, applying Block Zeros Lemma to any two elements of $\cup_{i=1}^{4}\{\mathcal{A}_i(|\xi_1\rangle_A), \mathcal{B}_i(|\eta_1\rangle_A)\}$, we obtain

$$
{}_{\mathcal{A}_i^{(A)}}E_{\mathcal{A}_j^{(A)}} = \mathbf{0}, \quad {}_{\mathcal{A}_i^{(A)}}E_{\mathcal{B}_k^{(A)}} = \mathbf{0}, \quad {}_{\mathcal{B}_k^{(A)}}E_{\mathcal{B}_\ell^{(A)}} = \mathbf{0}, \quad {}_{\mathcal{B}_k^{(A)}}E_{\mathcal{A}_i^{(A)}} = \mathbf{0}, \tag{8}
$$

for $1 \leq i \neq j \leq 4$, $1 \leq k \neq \ell \leq 4$. Next, we consider $\mathcal{A}_5(|d_A - 1\rangle_A)$ and $\mathcal{A}_1(|\xi_1\rangle_A)$. Then for $(j, k, \ell) \in \mathbb{Z}_{d_B-1} \times \mathbb{Z}_{d_C-1} \times \mathbb{Z}_{d_D-1} \setminus \{(0,0,0)\}$, and $(m, n) \in \mathbb{Z}_{d_B-1} \times \mathbb{Z}_{d_D-1}$, we have

$$
\begin{aligned}
&{}_B\langle \eta_j|_C\langle \xi_k|_D\langle \eta_\ell|E|\eta_m\rangle_B|0\rangle_C|\xi_n\rangle_D = \\
&{}_B\left(\sum_{t_1=0}^{d_B-2} w_{d_B-1}^{-jt_1}\langle t_1|\right)_C\left(\sum_{t_2=0}^{d_C-2} w_{d_C-1}^{-kt_2}\langle t_2+1|\right)_D\left(\sum_{t_3=0}^{d_D-2} w_{d_D-1}^{-\ell t_3}\langle t_3|\right)E \\
&\left(\sum_{t_4=0}^{d_B-2} w_{d_B-1}^{mt_4}|t_4\rangle\right)_B|0\rangle_C\left(\sum_{t_5=0}^{d_D-2} w_{d_D-1}^{nt_5}|t_5+1\rangle\right)_D = 0.
\end{aligned} \tag{9}
$$

We have shown that ${}_{\mathcal{B}_i^{(A)}}E_{\mathcal{A}_1^{(A)}} = \mathbf{0}$ for $1 \leq i \leq 4$ by Eq. (8). It means that ${}_{\mathcal{B}_i^{(A)} \cap \mathcal{A}_5^{(A)}}E_{\mathcal{A}_1^{(A)}} = \mathbf{0}$ for $1 \leq i \leq 4$. Then Eq. (9) can be expressed by

$$
\begin{aligned}
&{}_B\left(\sum_{t_1=1}^{d_B-2} w_{d_B-1}^{-jt_1}\langle t_1|\right)_C\left(\sum_{t_2=0}^{d_C-3} w_{d_C-1}^{-kt_2}\langle t_2+1|\right)_D\left(\sum_{t_3=1}^{d_D-2} w_{d_D-1}^{-\ell t_3}\langle t_3|\right)E \\
&\left(\sum_{t_4=0}^{d_B-2} w_{d_B-1}^{mt_4}|t_4\rangle\right)_B|0\rangle_C\left(\sum_{t_5=0}^{d_D-2} w_{d_D-1}^{nt_5}|t_5+1\rangle\right)_D = 0,
\end{aligned} \tag{10}
$$

i.e.

$$
\sum_{t_1=1}^{d_B-2}\sum_{t_2=0}^{d_C-3}\sum_{t_3=1}^{d_D-2}\sum_{t_4=0}^{d_B-2}\sum_{t_5=0}^{d_D-2} w_{d_B-1}^{-jt_1} w_{d_C-1}^{-kt_2} w_{d_D-1}^{-\ell t_3} w_{d_B-1}^{mt_4} w_{d_D-1}^{nt_5}\, {}_B\langle t_1|_C\langle t_2+1|_D\langle t_3|E|t_4\rangle_B|0\rangle_C|t_5+1\rangle_D = 0, \tag{11}
$$

for any $1 \leq j \leq d_B - 2$, $0 \leq k \leq d_C - 3$, $1 \leq \ell \leq d_D - 2$, $0 \leq m \leq d_B - 2$, $0 \leq n \leq d_D - 2$. It means that

$$[H_1^\dagger \otimes H_2^\dagger \otimes H_3^\dagger \otimes H_4 \otimes H_5]X = \mathbf{0}, \tag{12}$$

where

$$H_1 = \begin{pmatrix} w_{d_B-1} & w_{d_B-1}^2 & \cdots & w_{d_B-1}^{(d_B-2)} \\ w_{d_B-1}^2 & w_{d_B-1}^4 & \cdots & w_{d_B-1}^{2(d_B-2)} \\ \vdots & \vdots & \ddots & \vdots \\ w_{d_B-1}^{(d_B-2)} & w_{d_B-1}^{2(d_B-2)} & \cdots & w_{d_B-1}^{(d_B-2)^2} \end{pmatrix}, \quad H_2 = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & w_{d_C-1} & \cdots & w_{d_C-1}^{(d_C-3)} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & w_{d_C-1}^{(d_C-3)} & \cdots & w_{d_C-1}^{(d_C-3)^2} \end{pmatrix} \tag{13}$$

$$H_3 = \begin{pmatrix} w_{d_D-1} & w_{d_D-1}^2 & \cdots & w_{d_D-1}^{(d_D-2)} \\ w_{d_D-1}^2 & w_{d_D-1}^4 & \cdots & w_{d_D-1}^{2(d_D-2)} \\ \vdots & \vdots & \ddots & \vdots \\ w_{d_D-1}^{(d_D-2)} & w_{d_D-1}^{2(d_D-2)} & \cdots & w_{d_D-1}^{(d_D-2)^2} \end{pmatrix}, \tag{14}$$

$H_4 = (w_{d_B-1}^{ij})_{i,j\in\mathbb{Z}_{d_B-1}}$, $H_5 = (w_{d_D-1}^{ij})_{i,j\in\mathbb{Z}_{d_D-1}}$ and X is a column vector,

$$X = ({}_B\langle t_1|_C\langle t_2+1|_D\langle t_3|E|t_4\rangle_B|0\rangle_C|t_5+1\rangle_D)_{\{1\leq t_1\leq d_B-2,\ 0\leq t_2\leq d_C-3,\ 1\leq t_3\leq d_D-2\ 0\leq t_4\leq d_B-2,\ 0\leq t_5\leq d_D-2\}}. \tag{15}$$

Since H_1, H_2, H_3, H_4, H_5 are all full-rank matrices, it implies that $H_1^\dagger \otimes H_2^\dagger \otimes H_3^\dagger \otimes H_4 \otimes H_5$ is a full-rank matrix. Then $X = \mathbf{0}$, i.e.

$${}_B\langle t_1|_C\langle t_2+1|_D\langle t_3|E|t_4\rangle_B|0\rangle_C(|t_5+1\rangle_D = 0, \tag{16}$$

for $1 \leq t_1 \leq d_B-2, 0 \leq t_2 \leq d_C-3, 1 \leq t_3 \leq d_D-2, 0 \leq t_4 \leq d_B-2, 0 \leq t_5 \leq d_D-2$. It also means that

$${}_{\mathcal{F}^{(A)}}E_{\mathcal{A}_1^{(A)}} = \mathbf{0}. \tag{17}$$

By using $\mathcal{A}_5(|d_A-1\rangle_A)$ and $\mathcal{A}_i(|\xi_1\rangle_A)$ for $1 \leq i \leq 4$, we can also show that

$${}_{\mathcal{F}^{(A)}}E_{\mathcal{A}_i^{(A)}} = \mathbf{0}, \tag{18}$$

for $1 \leq i \leq 4$ by the same discussion as above. Further, by the symmetry of Fig. 1, we can also obtain that

$${}_{\mathcal{F}^{(A)}}E_{\mathcal{B}_j^{(A)}} = \mathbf{0}, \tag{19}$$

for $1 \leq j \leq 4$. Thus, by Eqs. (8), (18) and (19), E is a block diagonal matrix. It can be expressed by

$$E = E_{\mathcal{A}_1^{(A)}} \oplus E_{\mathcal{A}_2^{(A)}} \oplus E_{\mathcal{A}_3^{(A)}} \oplus E_{\mathcal{A}_4^{(A)}} \oplus E_{\mathcal{F}^{(A)}} \oplus E_{\mathcal{B}_4^{(A)}} \oplus E_{\mathcal{B}_3^{(A)}} \oplus E_{\mathcal{B}_2^{(A)}} \oplus E_{\mathcal{B}_1^{(A)}}. \tag{20}$$

Step 2 Considering $|S\rangle$ and $\{|\beta_0\rangle_A|\beta_j\rangle_B|\beta_k\rangle_C|\beta_\ell\rangle_D\}_{(j,k,\ell)\in\mathbb{Z}_{d_B-2}\times\mathbb{Z}_{d_C-2}\times\mathbb{Z}_{d_D-2}\setminus\{(0,0,0)\}} \subset \mathcal{F}$, where $d_D \geq 4$. Then we have

$$\begin{aligned} &{}_B\left(\sum_{i_1=0}^{d_B-1}\langle i_1|\right){}_C\left(\sum_{i_2=0}^{d_C-1}\langle i_2|\right){}_D\left(\sum_{i_3=0}^{d_D-1}\langle i_3|\right)E|\beta_j\rangle_B|\beta_k\rangle_C|\beta_\ell\rangle_D = \\ &{}_B\left(\sum_{i_1=1}^{d_B-2}\langle i_1|\right){}_C\left(\sum_{i_2=1}^{d_C-2}\langle i_2|\right){}_D\left(\sum_{i_3=1}^{d_D-2}\langle i_3|\right)E|\beta_j\rangle_B|\beta_k\rangle_C|\beta_\ell\rangle_D = 0, \end{aligned} \tag{21}$$

for $(j,k,\ell) \in \mathbb{Z}_{d_B-1} \times \mathbb{Z}_{d_C-1} \times \mathbb{Z}_{d_D-1} \setminus \{(0,0,0)\}$. Moreover, we have

$$\left(\sum_{i_1=1}^{d_B-2}|i_1\rangle\right)_B\left(\sum_{i_2=1}^{d_C-2}|i_2\rangle\right)_C\left(\sum_{i_3=1}^{d_D-2}|i_3\rangle\right)_D = |\beta_0\rangle_B|\beta_0\rangle_C|\beta_0\rangle_D. \tag{22}$$

Therefore, by using the states in $\{|S\rangle\} \cup \{|\beta_0\rangle_A|\beta_j\rangle_B|\beta_k\rangle_C|\beta_\ell\rangle_D\}_{(j,k,\ell)\in\mathbb{Z}_{d_B-2}\times\mathbb{Z}_{d_C-2}\times\mathbb{Z}_{d_D-2}\setminus\{(0,0,0)\}}$, we obtain

$$ {}_B\langle\beta_i|_C\langle\beta_j|_D\langle\beta_k|E|\beta_\ell\rangle_B|\beta_m\rangle_C|\beta_n\rangle_D = 0, \quad \text{for } (i,j,k)\neq(\ell,m,n)\in\mathbb{Z}_{d_B-2}\times\mathbb{Z}_{d_C-2}\times\mathbb{Z}_{d_D-2}. \tag{23}$$

Then there exists a real number $e_{r,s,t}$ for $(r,s,t)\in\mathbb{Z}_{d_B-2}\times\mathbb{Z}_{d_C-2}\times\mathbb{Z}_{d_D-2}$, such that

$$ E_{\mathcal{F}^{(A)}} = \sum_{r=0}^{d_B-3}\sum_{s=0}^{d_C-3}\sum_{t=0}^{d_D-3} e_{r,s,t}|\beta_r\rangle_B\langle\beta_r|\otimes|\beta_s\rangle_C\langle\beta_s|\otimes|\beta_t\rangle_D\langle\beta_t|. \tag{24}$$

Note that Eq. (24) also holds for $d_D=3$. Next, by using the states in $\mathcal{A}_1(|\xi_1\rangle_A)$, we have

$$ {}_B\langle\eta_i|_C\langle 0|_D\langle\xi_j|E|\eta_k\rangle_B|0\rangle_C|\xi_\ell\rangle_D = 0, \quad \text{for } (i,j)\neq(k,\ell)\in\mathbb{Z}_{d_B-1}\times\mathbb{Z}_{d_D-1}. \tag{25}$$

Then there exists a real number $a_{s,t}$ for $(s,t)\in\mathbb{Z}_{d_B-1}\times\mathbb{Z}_{d_D-1}$ such that

$$ E_{\mathcal{A}_1^{(A)}} = \sum_{s=0}^{d_B-2}\sum_{t=0}^{d_D-2} a_{s,t}|\eta_s\rangle_B\langle\eta_s|\otimes|0\rangle_C\langle 0|\otimes|\xi_t\rangle_D\langle\xi_t|. \tag{26}$$

In the same way, there exist real numbers $a_{s,t}, b_{s,t}, c_{s,t}, p, e_{r.s,t}, q, g_{s,t}, h_{s,t}, i_{s,t}$ such that the operator

$$\begin{aligned} E = &\sum_{s=0}^{d_B-2}\sum_{t=0}^{d_D-2} a_{s,t}|\eta_s\rangle_B\langle\eta_s|\otimes|0\rangle_C\langle 0|\otimes|\xi_t\rangle_D\langle\xi_t| + \sum_{s=0}^{d_C-2}\sum_{t=0}^{d_D-2} b_{s,t}|d_B-1\rangle_B\langle d_B-1|\otimes|\eta_s\rangle_C\langle\eta_s|\otimes|\eta_t\rangle_D\langle\eta_t| \\ &+\sum_{s=0}^{d_B-2}\sum_{t=0}^{d_C-2} c_{s,t}|\xi_s\rangle_B\langle\xi_s|\otimes|\xi_t\rangle_C\langle\xi_t|\otimes|d_D-1\rangle_D\langle d_D-1| + p|d_B-1\rangle_B\langle d_B-1|\otimes|0\rangle_C\langle 0|\otimes|d_D-1\rangle_D\langle d_D-1| \\ &+\sum_{r=0}^{d_B-3}\sum_{s=0}^{d_C-3}\sum_{t=0}^{d_D-3} e_{r,s,t}|\beta_r\rangle_B\langle\beta_r|\otimes|\beta_s\rangle_C\langle\beta_s|\otimes|\beta_t\rangle_D\langle\beta_t| + q|0\rangle_B\langle 0|\otimes|d_C-1\rangle_C\langle d_C-1|\otimes|0\rangle_D\langle 0| \\ &+\sum_{s=0}^{d_B-2}\sum_{t=0}^{d_C-2} g_{s,t}|\eta_s\rangle_B\langle\eta_s|\otimes|\eta_t\rangle_C\langle\eta_t|\otimes|0\rangle_D\langle 0| + \sum_{s=0}^{d_C-2}\sum_{t=0}^{d_D-2} h_{s,t}|0\rangle_B\langle 0|\otimes|\xi_s\rangle_C\langle\xi_s|\otimes|\xi_t\rangle_D\langle\xi_t| \\ &+\sum_{s=0}^{d_B-2}\sum_{t=0}^{d_D-2} i_{s,t}|\xi_s\rangle_B\langle\xi_s|\otimes|d_C-1\rangle_C\langle d_C-1|\otimes|\eta_t\rangle_D\langle\eta_t|. \end{aligned} \tag{27}$$

By using those states $\{|0\rangle_A|\xi_i\rangle_B|\eta_j\rangle_C|\xi_k\rangle_D\}_{(i,j,k)\ \in\mathbb{Z}_{d_B-1}\times\mathbb{Z}_{d_C-1}\times\mathbb{Z}_{d_D-1}\setminus\{(0,0,0)\}} = \mathcal{B}_5$, we have

$$ {}_B\langle\xi_i|_C\langle\eta_j|_D\langle\xi_k|E|\xi_\ell\rangle_B|\eta_m\rangle_C|\xi_n\rangle_D = 0, \quad \text{for } (i,j,k)\neq(\ell,m,n)\in\mathbb{Z}_{d_B-1}\times\mathbb{Z}_{d_C-1}\times\mathbb{Z}_{d_D-1}\setminus\{(0,0,0)\}. \tag{28}$$

There are three cases.

(i) Assume $i\neq 0, j=0, k=0, \ell=0, m\neq 0, n=0$. By Eq. (27), we have

$$\begin{aligned} 0 =& {}_B\langle\xi_i|_C\langle\eta_0|_D\langle\xi_0|E|\xi_0\rangle_B|\eta_m\rangle_C|\xi_0\rangle_D \\ =&(d_D-1)^2\sum_{s=0}^{d_B-2} a_{s,0}\langle\xi_i|\eta_s\rangle_B\langle\eta_s|\xi_0\rangle_B + w^i_{d_B-1}p + w^i_{d_B-1}(d_B-2)(d_C-2)(d_D-2)^2 e_{0,0,0} \\ =&(d_D-1)^2 w^i_{d_B-1}\left(\sum_{s=0}^{d_B-2} a_{s,0} - (d_B-1)(a_{0,0}+a_{i,0})\right) + w^i_{d_B-1}p + w^i_{d_B-1}(d_B-2)(d_C-2)(d_D-2)^2 e_{0,0,0}. \end{aligned} \tag{29}$$

That is,

$$ (d_D-1)^2\left(\sum_{s=0}^{d_B-2} a_{s,0} - (d_B-1)(a_{0,0}+a_{i,0})\right) + p + (d_B-2)(d_C-2)(d_D-2)^2 e_{0,0,0} = 0. \tag{30}$$

Since $i\in\mathbb{Z}_{d_B-1}\setminus\{0\}$, we obtain $a_{1,0}=a_{2,0}=\cdots=a_{d_B-2,0}$. Then Eq. (30) can be expressed by

$$ -(d_D-1)^2((d_B-2)a_{0,0}+a_{1,0}) + p + (d_B-2)(d_C-2)(d_D-2)^2 e_{0,0,0} = 0. \tag{31}$$

Next, by using the states $|S\rangle$ and $|0\rangle_A|\xi_1\rangle_B|\eta_1\rangle_C|\xi_0\rangle_D \in \mathcal{B}_5$, we have

$$\begin{aligned} 0 &= {}_B\left(\sum_{i_1=0}^{d_B-1}\langle i_1|\right){}_C\left(\sum_{i_2=0}^{d_C-1}\langle i_2|\right){}_D\left(\sum_{i_3=0}^{d_D-1}\langle i_3|\right)E|\xi_1\rangle_B|\eta_1\rangle_C|\xi_0\rangle_D \\ &= -w_{d_B-1}^{d_B-2}(d_B-1)(d_D-1)^2a_{0,0}+w_{d_B-1}^{d_B-2}p+w_{d_B-1}^{d_B-2}(d_B-2)(d_C-2)(d_D-2)^2e_{0,0,0}. \end{aligned} \tag{32}$$

That is

$$-(d_B-1)(d_D-1)^2a_{0,0}+p+(d_B-2)(d_C-2)(d_D-2)^2e_{0,0,0}=0. \tag{33}$$

Considering Eqs. (31) and (33), it implies $a_{0,0}=a_{1,0}$. It means that $a_{0,0}=a_{1,0}=\cdots=a_{d_B-2,0}=a$. Next, we consider $\{|0\rangle_A|0\rangle_B|0\rangle_C|\xi_i\rangle_D\}_{i\in\mathbb{Z}_{d_D-1}\setminus\{0\}}=\mathcal{B}_8$ and $\{|0\rangle_A|\xi_j\rangle_B|\eta_k\rangle_C|\xi_\ell\rangle_D\}_{(j,k,\ell)\in\mathbb{Z}_{d_B-1}\times\mathbb{Z}_{d_C-1}\times\mathbb{Z}_{d_D-1}\setminus\{(0,0,0)\}}$ $=\mathcal{B}_5$. Then by the orthogonality-preserving POVM, we have

$${}_B\langle 0|_C\langle 0|_D\langle \xi_i|E|\xi_j\rangle_B|\eta_k\rangle_C|\xi_\ell\rangle_D=0, \tag{34}$$

for $i\in\mathbb{Z}_{d_D-1}\setminus\{0\}$ and $(j,k,\ell)\in\mathbb{Z}_{d_B-1}\times\mathbb{Z}_{d_C-1}\times\mathbb{Z}_{d_D-1}\setminus\{(0,0,0)\}$. By Eq. (27), we obtain that

$${}_B\langle 0|_C\langle 0|_D\langle \xi_i|E|\xi_0\rangle_B|\eta_0\rangle_C|\xi_0\rangle_D=0, \tag{35}$$

for $i\in\mathbb{Z}_{d_D-1}\setminus\{0\}$ and $(j,k,\ell)=(0,0,0)$, and

$${}_B\langle 0|_C\langle 0|_D\langle \xi_0|E|\xi_j\rangle_B|\eta_k\rangle_C|\xi_\ell\rangle_D=0, \tag{36}$$

for $(j,k)\in\mathbb{Z}_{d_B-1}\times\mathbb{Z}_{d_C-1}$ and $\ell\in\mathbb{Z}_{d_D-1}\setminus\{0\}$. Further, by Eq. (27), we have

$$\begin{aligned} {}_B\langle 0|_C\langle 0|_D\langle \xi_0|E|\xi_j\rangle_B|\eta_k\rangle_C|\xi_0\rangle_D &= (d_D-1)^2\sum_{s=0}^{d_B-2}a_{s,0}\langle \eta_s|\xi_j\rangle_B = a(d_D-1)^2\sum_{s=0}^{d_B-2}\langle\eta_s|\xi_j\rangle_B \\ &= a(d_D-1)^2\sum_{s=0}^{d_B-2}\sum_{n=1}^{d_B-2}w_{d_B-1}^{j(n-1)-ns}=a(d_D-1)^2w_{d_B-1}^{-j}\sum_{s=0}^{d_B-2}\sum_{n=1}^{d_B-2}w_{d_B-1}^{n(j-s)}=0, \end{aligned} \tag{37}$$

for $(j,k)\in\mathbb{Z}_{d_B-1}\times\mathbb{Z}_{d_C-1}$. Thus by Eqs. (34), (35) (36) and (37), we have

$${}_B\langle 0|_C\langle 0|_D\langle \xi_i|E|\xi_j\rangle_B|\eta_k\rangle_C|\xi_\ell\rangle_D=0, \tag{38}$$

for $i\in\mathbb{Z}_{d_D-1}$ and $(j,k,\ell)\in\mathbb{Z}_{d_B-1}\times\mathbb{Z}_{d_C-1}\times\mathbb{Z}_{d_D-1}$. Thus, applying Block Zeros Lemma to $\{|0\rangle_B|0\rangle_C|\xi_i\rangle_D\}_{i\in\mathbb{Z}_{d_D-1}}$ and $\{|\xi_j\rangle_B|\eta_k\rangle_C|\xi_\ell\rangle_D\}_{(j,k,\ell)\in\mathbb{Z}_{d_B-1}\times\mathbb{Z}_{d_C-1}\times\mathbb{Z}_{d_D-1}}$, we obtain that

$${}_{\mathcal{B}_8^{(A)}}E_{\mathcal{B}_5^{(A)}}=\mathbf{0}. \tag{39}$$

(ii) Assume $i\neq 0, j=0, k=0, \ell=0, m=0, n\neq 0$. By Eq. (27), we have

$$\begin{aligned} 0 ={}& {}_B\langle\xi_i|_C\langle\eta_0|_D\langle\xi_0|E|\xi_0\rangle_B|\eta_0\rangle_C|\xi_n\rangle_D = w_{d_B-1}^{i}(d_C-1)^2w_{d_D-1}^{-n}\left(\sum_{t=0}^{d_D-2}b_{0,t}-(d_D-1)(b_{0,0}+b_{0,n})\right) \\ &+w_{d_B-1}^{i}w_{d_D-1}^{-n}p+w_{d_B-1}^{i}w_{d_D-1}^{-n}(d_B-2)(d_C-2)^2(d_D-2)e_{0,0,0}. \end{aligned} \tag{40}$$

That is,

$$(d_C-1)^2\left(\sum_{t=0}^{d_D-2}b_{0,t}-(d_D-1)(b_{0,0}+b_{0,n})\right)+p+(d_B-2)(d_C-2)^2(d_D-2)e_{0,0,0}=0. \tag{41}$$

Since $n\in\mathbb{Z}_{d_D-1}\setminus\{0\}$, we obtain $b_{0,1}=b_{0,2}=\cdots=b_{0,d_D-2}$. Then Eq. (41) can be expressed by

$$-(d_C-1)^2((d_D-2)b_{0,0}+b_{0,1})+p+(d_B-2)(d_C-2)^2(d_D-2)e_{0,0,0}=0. \tag{42}$$

Next, by using the states $|S\rangle$ and $|0\rangle_A|\xi_1\rangle_B|\eta_0\rangle_C|\xi_1\rangle_D \in \mathcal{B}_5$, we have

$$
\begin{aligned}
0 &= {}_B\left(\sum_{i_1=0}^{d_B-1}\langle i_1|\right){}_C\left(\sum_{i_2=0}^{d_C-1}\langle i_2|\right){}_D\left(\sum_{i_3=0}^{d_D-1}\langle i_3|\right)E|\xi_1\rangle_B|\eta_0\rangle_C|\xi_1\rangle_D \\
&= -w_{d_B-1}^{d_B-2}w_{d_D-1}^{d_D-2}(d_C-1)^2(d_D-1)b_{0,0} + w_{d_B-1}^{d_B-2}w_{d_D-1}^{d_D-2}p + w_{d_B-1}^{d_B-2}w_{d_D-1}^{d_D-2}(d_B-2)(d_C-2)^2(d_D-2)e_{0,0,0}.
\end{aligned} \tag{43}
$$

That is

$$
-(d_C-1)^2(d_D-1)b_{0,0} + p + (d_B-2)(d_C-2)^2(d_D-2)e_{0,0,0} = 0. \tag{44}
$$

Considering Eqs. (42) and (44), it implies $b_{0,0} = b_{0,1}$. It means that $b_{0,0} = b_{0,1} = \cdots = b_{0,d_D-2} = b$.
Next, we consider $\{|0\rangle_A|d_B-1\rangle_B|\eta_i\rangle_C|0\rangle_D\}_{i\in\mathbb{Z}_{d_C-1}\setminus\{0\}} = \mathcal{B}_7$ and $\{|0\rangle_A|\xi_j\rangle_B|\eta_k\rangle_C|\xi_\ell\rangle_D\}_{(j,k,\ell)\in\mathbb{Z}_{d_B-1}\times\mathbb{Z}_{d_C-1}\times\mathbb{Z}_{d_D-1}\setminus\{(0,0,0)\}} = \mathcal{B}_5$. For the same discussion as Eqs. (34), (35) (36) and (37), we obtain that

$$
{}_B\langle d_B-1|_C\langle\eta_i|_D\langle 0|E|\xi_j\rangle_B|\eta_k\rangle_C|\xi_\ell\rangle_D = 0, \tag{45}
$$

for $i\in\mathbb{Z}_{d_C-1}$ and $(j,k,\ell)\in\mathbb{Z}_{d_B-1}\times\mathbb{Z}_{d_C-1}\times\mathbb{Z}_{d_D-1}$. Applying Block Zeros Lemma to $\{|d_B-1\rangle_B|\eta_i\rangle_C|0\rangle_D\}_{i\in\mathbb{Z}_{d_C-1}}$ and $\{|\xi_j\rangle_B|\eta_k\rangle_C|\xi_\ell\rangle_D\}_{(j,k,\ell)\in\mathbb{Z}_{d_B-1}\times\mathbb{Z}_{d_C-1}\times\mathbb{Z}_{d_D-1}}$, we obtain that

$$
{}_{\mathcal{B}_7^{(A)}}E_{\mathcal{B}_5^{(A)}} = \mathbf{0}. \tag{46}
$$

(iii) Assume $i=0, j=0, k\neq 0, \ell=0, m\neq 0, n=0$. By Eq. (27), we have

$$
\begin{aligned}
0 ={}& {}_B\langle\xi_0|_C\langle\eta_0|_D\langle\xi_k|E|\xi_0\rangle_B|\eta_m\rangle_C|\xi_0\rangle_D = w_{d_D-1}^k(d_B-1)^2\left(\sum_{t=0}^{d_C-2}c_{0,t} - (d_C-1)(c_{0,0}+c_{0,m})\right) \\
&+ w_{d_D-1}^k p + w_{d_D-1}^k(d_B-2)^2(d_C-2)(d_D-2)e_{0,0,0}.
\end{aligned} \tag{47}
$$

That is,

$$
(d_B-1)^2\left(\sum_{t=0}^{d_C-2}c_{0,t} - (d_C-1)(c_{0,0}+c_{0,m})\right) + p + (d_B-2)^2(d_C-2)(d_D-2)e_{0,0,0} = 0. \tag{48}
$$

Since $m\in\mathbb{Z}_{d_C-1}\setminus\{0\}$, we obtain $c_{0,1} = c_{0,2} = \cdots = c_{0,d_C-2}$. Then Eq. (48) can be expressed as

$$
-(d_B-1)^2((d_C-2)c_{0,0}+c_{0,1}) + p + (d_B-2)^2(d_C-2)(d_D-2)e_{0,0,0} = 0. \tag{49}
$$

Next, by using the states $|S\rangle$ and $|0\rangle_A|\xi_0\rangle_B|\eta_1\rangle_C|\xi_1\rangle_D \in \mathcal{B}_5$, we have

$$
\begin{aligned}
0 &= {}_B\left(\sum_{i_1=0}^{d_B-1}\langle i_1|\right){}_C\left(\sum_{i_2=0}^{d_C-1}\langle i_2|\right){}_D\left(\sum_{i_3=0}^{d_D-1}\langle i_3|\right)E|\xi_0\rangle_B|\eta_1\rangle_C|\xi_1\rangle_D \\
&= -w_{d_D-1}^{d_D-2}(d_B-1)^2(d_C-1)c_{0,0} + w_{d_D-1}^{d_D-2}p + w_{d_D-1}^{d_D-2}(d_B-2)^2(d_C-2)(d_D-2)e_{0,0,0}.
\end{aligned} \tag{50}
$$

That is

$$
-(d_B-1)^2(d_C-1)c_{0,0} + p + (d_B-2)^2(d_C-2)(d_D-2)e_{0,0,0} = 0. \tag{51}
$$

Considering Eqs. (49) and (51), it implies $c_{0,0} = c_{0,1}$. It means that $c_{0,0} = c_{0,1} = \cdots = c_{0,d_D-2} = c$.
Next, we consider $\{|0\rangle_A|\xi_i\rangle_B|d_C-1\rangle_C|d_D-1\rangle_D\}_{i\in\mathbb{Z}_{d_B-1}\setminus\{0\}} = \mathcal{B}_6$ and $\{|0\rangle_A|\xi_j\rangle_B|\eta_k\rangle_C|\xi_\ell\rangle_D\}_{(j,k,\ell)\in\mathbb{Z}_{d_B-1}\times\mathbb{Z}_{d_C-1}\times\mathbb{Z}_{d_D-1}\setminus\{(0,0,0)\}} = \mathcal{B}_5$. For the same discussion as Eqs. (34), (35) (36) and (37), we obtain that

$$
{}_B\langle\xi_i|_C\langle d_C-1|_D\langle d_D-1|E|\xi_j\rangle_B|\eta_k\rangle_C|\xi_\ell\rangle_D = 0, \tag{52}
$$

for $i\in\mathbb{Z}_{d_B-1}$ and $(j,k,\ell)\in\mathbb{Z}_{d_B-1}\times\mathbb{Z}_{d_C-1}\times\mathbb{Z}_{d_D-1}$. Applying Block Zeros Lemma to $\{|\xi_i\rangle_B|d_C-1\rangle_C|d_D-1\rangle_D\}_{i\in\mathbb{Z}_{d_B-1}}$ and $\{|\xi_j\rangle_B|\eta_k\rangle_C|\xi_\ell\rangle_D\}_{(j,k,\ell)\in\mathbb{Z}_{d_B-1}\times\mathbb{Z}_{d_C-1}\times\mathbb{Z}_{d_D-1}}$, then we obtain that

$$
{}_{\mathcal{B}_6^{(A)}}E_{\mathcal{B}_5^{(A)}} = \mathbf{0}. \tag{53}
$$

Step 3 Considering $|S\rangle$ and $\{|0\rangle_A|\xi_j\rangle_B|\eta_k\rangle_C|\xi_\ell\rangle_D\}_{(j,k,\ell)\in\mathbb{Z}_{d_B-1}\times\mathbb{Z}_{d_C-1}\times\mathbb{Z}_{d_D-1}\setminus\{(0,0,0)\}}=\mathcal{B}_5$. By using Eqs. (20), (39), (46) and (53), we have the following equality

$$\begin{aligned}&{}_B\left(\sum_{i_1=0}^{d_B-1}\langle i_1|\right){}_C\left(\sum_{i_2=0}^{d_C-1}\langle i_2|\right){}_D\left(\sum_{i_3=0}^{d_D-1}\langle i_3|\right)E|\xi_j\rangle_B|\eta_k\rangle_C|\xi_\ell\rangle_D\\ &={}_B\left(\sum_{i_1=1}^{d_B-1}\langle i_1|\right){}_C\left(\sum_{i_2=0}^{d_C-2}\langle i_2|\right){}_D\left(\sum_{i_3=1}^{d_D-1}\langle i_3|\right)E|\xi_j\rangle_B|\eta_k\rangle_C|\xi_\ell\rangle_D=0.\end{aligned}\tag{54}$$

Moreover, we have

$$\left(\sum_{i_1=1}^{d_B-1}|i_1\rangle\right)_B\left(\sum_{i_2=0}^{d_C-2}|i_2\rangle\right)_C\left(\sum_{i_3=1}^{d_D-1}|i_3\rangle\right)_D=|\xi_0\rangle_B|\eta_0\rangle_C|\xi_0\rangle_D.\tag{55}$$

Therefore, by using the states $\{|S\rangle\}\cup\{|0\rangle_A|\xi_j\rangle_B|\eta_k\rangle_C|\xi_\ell\rangle_D\}_{(j,k,\ell)\in\mathbb{Z}_{d_B-1}\times\mathbb{Z}_{d_C-1}\times\mathbb{Z}_{d_D-1}\setminus\{(0,0,0)\}}$, we have

$${}_B\langle\xi_i|_C\langle\eta_j|_D\langle\xi_k|E|\xi_\ell\rangle_B|\eta_m\rangle_C|\xi_n\rangle_D=0,\quad\text{for }(i,j,k)\neq(\ell,m,n)\in\mathbb{Z}_{d_B-1}\times\mathbb{Z}_{d_C-1}\times\mathbb{Z}_{d_D-1}.\tag{56}$$

By Eq. (20), we know that ${}_{\mathcal{A}_4^{(A)}}E_{\mathcal{B}_5^{(A)}\setminus\mathcal{A}_4^{(A)}}=\mathbf{0}$, where $\mathcal{A}_4^{(A)}=\{|d_B-1\rangle_B|0\rangle_C|d_D-1\rangle_D\}$, $\mathcal{A}_4^{(A)}\subset\mathcal{B}_5^{(A)}$. Applying Block Trivial Lemma to $\{|\xi_j\rangle_B|\eta_k\rangle_C|\xi_\ell\rangle_D\}_{(j,k,\ell)\in\mathbb{Z}_{d_B-1}\times\mathbb{Z}_{d_C-1}\times\mathbb{Z}_{d_D-1}}$, we have

$$E_{\mathcal{B}_5^{(A)}}=k\mathbb{I}_{\mathcal{B}_5^{(A)}}.\tag{57}$$

Next, for any $|j\rangle_B|k\rangle_C|\ell\rangle_D\in\mathcal{A}_i^{(A)}\cap\mathcal{B}_5^{(A)}$ for $i=1,2,3$, we have ${}_{\{|j\rangle_B|k\rangle_C|\ell\rangle_D\}}E_{\mathcal{A}_i^{(A)}\setminus\{|j\rangle_B|k\rangle_C|\ell\rangle_D\}}=\mathbf{0}$ for $i=1,2,3$ by Eqs. (39), (46), (53) and (57). Applying Block Trivial Lemma to $\mathcal{A}_i(|\xi_1\rangle)$ for $i=1,2,3$, we have

$$E_{\mathcal{A}_i^{(A)}}=k_i\mathbb{I}_{\mathcal{A}_i^{(A)}}.\tag{58}$$

Since $\mathcal{A}_i^{(A)}\cap\mathcal{B}_5^{(A)}\neq\emptyset$ for $i=1,2,3$, it implies that $k_i=k$ for $i=1,2,3$. Thus by Eqs. (57) and (58), we have

$$E_{\{\cup_{i=1}^3\mathcal{A}_i^{(A)}\}\cup\mathcal{B}_5^{(A)}}=k\mathbb{I}_{\{\cup_{i=1}^3\mathcal{A}_i^{(A)}\}\cup\mathcal{B}_5^{(A)}}.\tag{59}$$

By the he symmetry of Fig. 1, we can obtain that $E=k\mathbb{I}$. Thus E is trivial. Further, since the 17 subsets $\cup_{i=1}^8\{\mathcal{A}_i,\mathcal{B}_i\}\cup\{\mathcal{F}\}$ in any bipartition of $\{A|BCD,D|ABC,C|DAB,B|CDA\}$ correspond to a similar grid as Fig. 1, it implies that any of the party $\{BCD,ABC,DAB,CDA\}$ can only perform a trivial orthogonality-preserving POVM. This completes the proof. □

Note that the states in $\mathcal{A}_i$ or $\mathcal{B}_i$ $(i=1,\ldots,8)$ in Eq. (1) are defined by the outermost layer of a $d_A\times d_B\times d_C\times d_D$ hypercube, and the states in $\mathcal{F}$ are just defined by all inner cells. By observing this, we can construct more strongly nonlocal UPBs in $d_A\otimes d_B\otimes d_C\otimes d_D$ by continuing decompose $\mathcal{F}$ in Fig. 1 by the similar tiling method. Suppose we are on the n-th layer from outside to inside, $0\leq n\leq\lfloor\frac{d_A-3}{2}\rfloor$. Let $X_n:=d_X-2n$ for $X\in\{A,B,C\}$. Then we can define the following states,

$$
\begin{aligned}
\mathcal{A}_1^{(n)} :=&\{|\xi_i^{(n)}\rangle_A|\eta_j^{(n)}\rangle_B|n\rangle_C|\xi_\ell^{(n)}\rangle_D:\ (i,j,\ell)\in\mathbb{Z}_{A_n-1}\times\mathbb{Z}_{B_n-1}\times\mathbb{Z}_{D_n-1}\setminus\{(0,0,0)\}\},\\
\mathcal{A}_2^{(n)} :=&\{|\xi_i^{(n)}\rangle_A|d_B-1-n\rangle_B|\eta_k^{(n)}\rangle_C|\eta_\ell^{(n)}\rangle_D:\\
&\quad(i,k,\ell)\in\mathbb{Z}_{A_n-1}\times\mathbb{Z}_{C_n-1}\times\mathbb{Z}_{D_n-1}\setminus\{(0,0,0)\}\},\\
\mathcal{A}_3^{(n)} :=&\{|\xi_i^{(n)}\rangle_A|\xi_j^{(n)}\rangle_B|\xi_k^{(n)}\rangle_C|d_C-1-n\rangle_D:\\
&\quad(i,j,k)\in\mathbb{Z}_{A_n-1}\times\mathbb{Z}_{B_n-1}\times\mathbb{Z}_{C_n-1}\setminus\{(0,0,0)\}\},\\
\mathcal{A}_4^{(n)} :=&\{|\xi_i^{(n)}\rangle_A|d_B-1-n\rangle_B|n\rangle_C|d_D-1-n\rangle_D:\ i\in\mathbb{Z}_{A_n-1}\setminus\{0\}\},\\
\mathcal{A}_5^{(n)} :=&\{|d_A-1-n\rangle_A|\eta_j^{(n)}\rangle_B|\xi_k^{(n)}\rangle_C|\eta_\ell^{(n)}\rangle_D:\\
&\quad(j,k,\ell)\in\mathbb{Z}_{B_n-1}\times\mathbb{Z}_{C_n-1}\times\mathbb{Z}_{D_n-1}\setminus\{(0,0,0)\}\},\\
\mathcal{A}_6^{(n)} :=&\{|d_A-1-n\rangle_A|\eta_j^{(n)}\rangle_B|n\rangle_C|n\rangle_D:\ j\in\mathbb{Z}_{B_n-1}\setminus\{0\}\},\\
\mathcal{A}_7^{(n)} :=&\{|d_A-1-n\rangle_A|n\rangle_B|\xi_k^{(n)}\rangle_C|d_D-1-n\rangle_D:\ k\in\mathbb{Z}_{C_n-1}\setminus\{0\}\},\\
\mathcal{A}_8^{(n)} :=&\{|d_A-1-n\rangle_A|d_B-1-n\rangle_B|d_C-1-n\rangle_C|\eta_\ell^{(n)}\rangle_D:\ \ell\in\mathbb{Z}_{D_n-1}\setminus\{0\}\},\\
\mathcal{B}_1^{(n)} :=&\{|\eta_i^{(n)}\rangle_A|\xi_j^{(n)}\rangle_B|d_C-1-n\rangle_C|\eta_\ell^{(n)}\rangle_D:\\
&\quad(i,j,\ell)\in\mathbb{Z}_{A_n-1}\times\mathbb{Z}_{B_n-1}\times\mathbb{Z}_{D_n-1}\setminus\{(0,0,0)\}\},\\
\mathcal{B}_2^{(n)} :=&\{|\eta_i^{(n)}\rangle_A|n\rangle_B|\xi_k^{(n)}\rangle_C|\xi_\ell^{(n)}\rangle_D:\ (i,k,\ell)\in\mathbb{Z}_{A_n-1}\times\mathbb{Z}_{C_n-1}\times\mathbb{Z}_{D_n-1}\setminus\{(0,0,0)\}\},\\
\mathcal{B}_3^{(n)} :=&\{|\eta_i^{(n)}\rangle_A|\eta_j^{(n)}\rangle_B|\eta_k^{(n)}\rangle_C|0\rangle_D:\ (i,j,k)\in\mathbb{Z}_{A_n-1}\times\mathbb{Z}_{B_n-1}\times\mathbb{Z}_{C_n-1}\setminus\{(0,0,0)\}\},\\
\mathcal{B}_4^{(n)} :=&\{|\eta_i^{(n)}\rangle_A|n\rangle_B|d_C-1-n\rangle_C|n\rangle_D:\ i\in\mathbb{Z}_{A_n-1}\setminus\{0\}\},\\
\mathcal{B}_5^{(n)} :=&\{|n\rangle_A|\xi_j^{(n)}\rangle_B|\eta_k^{(n)}\rangle_C|\xi_\ell^{(n)}\rangle_D:\ (j,k,\ell)\in\mathbb{Z}_{B_n-1}\times\mathbb{Z}_{C_n-1}\times\mathbb{Z}_{D_n-1}\setminus\{(0,0,0)\}\},\\
\mathcal{B}_6^{(n)} :=&\{|n\rangle_A|\xi_j^{(n)}\rangle_B|d_C-1-n\rangle_C|d_D-1-n\rangle_D:\ j\in\mathbb{Z}_{B_n-1}\setminus\{0\}\},\\
\mathcal{B}_7^{(n)} :=&\{|n\rangle_A|d_B-1-n\rangle_B|\eta_k^{(n)}\rangle_C|n\rangle_D:\ k\in\mathbb{Z}_{C_n-1}\setminus\{0\}\},\\
\mathcal{B}_8^{(n)} :=&\{|n\rangle_A|n\rangle_B|n\rangle_C|\xi_\ell^{(n)}\rangle_D:\ \ell\in\mathbb{Z}_{D_n-1}\setminus\{(0,0,0)\}\},
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{F}^{(n)} :=&\{|\beta_i^{(n)}\rangle_A|\beta_j^{(n)}\rangle_B|\beta_k^{(n)}\rangle_C|\beta_\ell^{(n)}\rangle_D:\\
&\quad(i,j,k,\ell)\in\mathbb{Z}_{A_n-2}\times\mathbb{Z}_{B_n-2}\times\mathbb{Z}_{C_n-2}\times\mathbb{Z}_{D_n-2}\setminus\{(0,0,0,0)\}\},\\
|S\rangle :=&\left(\sum_{i=0}^{d_A-1}|i\rangle\right)_A\left(\sum_{j=0}^{d_B-1}|j\rangle\right)_B\left(\sum_{k=0}^{d_C-1}|k\rangle\right)_C\left(\sum_{\ell=0}^{d_D-1}|\ell\rangle\right)_D,
\end{aligned}
\tag{60}
$$

where $|\eta_s^{(n)}\rangle_X=\sum_{t=n}^{X_n+n-2}w_{X_n-1}^{s(t-n)}|t\rangle_X$, and $|\xi_s^{(n)}\rangle_X=\sum_{t=n}^{X_n+n-2}w_{X_n-1}^{s(t-n)}|t+1\rangle_X$, for $s\in\mathbb{Z}_{X_n-1}$, and $X\in\{A,B,C,D\}$, $|\beta_s^{(n)}\rangle_X=\sum_{t=n}^{X_n+n-3}w_{X_n-2}^{s(t-n)}|t+1\rangle_X$ for $s\in\mathbb{Z}_{X_n-2}$, and $X\in\{A,B,C,D\}$.

Note that $\{|\eta_s^{(n)}\rangle_X\}_{s\in\mathbb{Z}_{X_n-1}}$, $\{|\xi_s^{(n)}\rangle_X\}_{s\in\mathbb{Z}_{X_n-1}}$, and $\{|\beta_s^{(n)}\rangle_X\}_{s\in\mathbb{Z}_{X_n-2}}$ are three orthogonal sets, $X\in\{A,B,C\}$, which are spanned by $\{|t\rangle_X\}_{t=n}^{X_n+n-2}$, $\{|t\rangle_X\}_{t=n+1}^{X_n+n-1}$, and $\{|t\rangle_X\}_{t=n+1}^{X_n+n-2}$, respectively. This extends the definition of states for Eq. (1) from $n=0$ to general n. Now, we have the following theorem.

Theorem 3 *In $d_A\otimes d_B\otimes d_C\otimes d_D$, $3\leq d_A\leq d_B\leq d_C\leq d_D$, for any $0\leq n\leq\lfloor\frac{d_A-3}{2}\rfloor$,*

$$
\mathcal{U}_n:=\cup_{t=0}^{n}(\cup_{i=1}^{8}(\mathcal{A}_i^{(t)}\cup\mathcal{B}_i^{(t)}))\cup\mathcal{F}^{(n)}\cup\{|S\rangle\}
\tag{61}
$$

given by Eq. (60) is a strongly nonlocal UPB of size $d_Ad_Bd_C-16(n+1)$.

Theorem 3 can be easily obtained by induction on t along with Proposition 1 and Proposition 2.