Combinatorics, 2017 Fall, USTC
Week 13 Note 2

2017.12.15, Friday

Probabilistic Methods

- **Ideas:**

  ① Image we need to find some combinatorial object satisfying certain property, call them “good” objects. We consider a random object. If the probability that the random object is ”good” is positive, then there must exist ”good” objects.

  ② To compute the probability of being ”good”, we often compute the probability of being ”bad” and aim to prove the probability is strictly less than 1.

Definition 1. *A finite probability space is* $(\Omega, \Pr)$*, where* Ω *is a finite set and* $\Pr: \ \Omega \to [0,1]$ *such that* $\sum\limits_{x\in\Omega} \Pr[x] = 1$*.*

- $A \subseteq \Omega$ *is called an event,* $\Pr[A] = \sum\limits_{x\in A} \Pr[x]$*.*
- $\Pr[\emptyset] = 0$*,* $\Pr[\Omega] = 1$*.*
- $\Pr[A \cup B] = \Pr[A] + \Pr[B] - \Pr[A \cap B] \leq \Pr[A] + \Pr[B]$*(union bound).* *”=” holds if* $A \cap B = \emptyset$*.*
- *Two events* A *and* B *are* ***independent*** *if* $\Pr[A \cap B] = \Pr[A] \cdot \Pr[B]$*.*
- *A random variable is a function* $X: \ \Omega \to \mathbb{R}$*.*
- $\forall A \subseteq \Omega$*, its indicator random variable* X_A *is defined as*

$$X_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases} .$$

- ***Expectation****:* $\mathrm{E}[X] = \sum\limits_{\omega\in\Omega} \Pr[\omega] X(\omega)$*.*
- *linearity of expectation:* $\mathrm{E}[aX + bY] = a\mathrm{E}[X] + b\mathrm{E}[Y]$*,* $a, b \in \mathbb{R}$*.*

Theorem 1. *Let* $n, s \in \mathbb{Z}$ *satisfying* $\binom{n}{s} 2^{1-\binom{s}{2}} < 1$*, then* $R(s,s) > n$*.*

Proof. *We need to find a 2-edge-coloring of* K_n *such that it has NO monochromatic clique* K_s*(i.e. a clique with all edges of the same color). Consider a* *random 2-edge-coloring of K_n: each edge is colored blue or red, each with probability $\frac{1}{2}$, independent of other edges.*

Let A be the event that the defined K_n has a monochromatic K_s. For each $B \in \binom{[n]}{s}$, let A_B be the event that K_n has a monochromatic K_s with vertex set B. Then $A = \bigcup_{B \in \binom{[n]}{s}} A_B$.

$\Pr[A] = \Pr[\bigcup_{B \in \binom{[n]}{s}} A_B] \leq \sum_{B \in \binom{[n]}{s}} \Pr[A_B] = \binom{n}{s} 2^{1-\binom{s}{2}} < 1$. *Thus* $\Pr[\bar{A}] = 1 - \Pr[A] > 0$, *i.e. the probability that K_n has NO monochromatic K_s is positive. So there must exist a 2-edge-coloring of K_n such that it has NO monochromatic K_s.* □

Corollary 2. $R(k,k) \geq \frac{1}{\mathrm{e}\sqrt{2}} k 2^{\frac{k}{2}}$.

Proof. *Let* $n = \frac{1}{\mathrm{e}\sqrt{2}} k 2^{\frac{k}{2}} \left(\frac{\mathrm{e}}{2}\right)^{\frac{1}{k}}$. *Recall* $\binom{n}{k} < \frac{n^k}{k!}$ *and* $k! > \mathrm{e}\left(\frac{k}{\mathrm{e}}\right)^k$. *We have*

$$\binom{n}{k} 2^{1-\binom{k}{2}} < \frac{n^k}{\mathrm{e}\left(\frac{k}{\mathrm{e}}\right)^k} 2^{1-\binom{k}{2}} = \left(\frac{\mathrm{e}n}{k}\right)^k \left(\frac{2}{\mathrm{e}}\right) 2^{-\binom{k}{2}}.$$

Since $n = \frac{1}{\mathrm{e}\sqrt{2}} k 2^{\frac{k}{2}} \left(\frac{\mathrm{e}}{2}\right)^{\frac{1}{k}}$, *we have* $\binom{n}{k} 2^{1-\binom{k}{2}} < 1$. *Then* $R(k,k) > n \geq \frac{1}{\mathrm{e}\sqrt{2}} k 2^{\frac{k}{2}}$. □

Definition 2. *A* ***tournament*** *of n vertices is an orientation of K_n. Say vertex x beats vertex y if $(x, y) \in E$. Say a tournament T has the property P_k, if for any k-subset $A \subset V$, there is a vertex who beats all vertices in A.*

Question: For $\forall k \geq 2$, does there exists a T with property P_k?

Theorem 3. $\forall k \geq 2$, *if* $\binom{n}{k}\left(1 - \frac{1}{2^k}\right)^{n-k} < 1$, *then there exists a tournament with n players with property P_k.*

Proof. *Consider a random tournament of n players. For any $i < j$, the arc $i \to j$ occurs with probability $\frac{1}{2}$, independent of other choices. Let B be the event that T doesn't satisfy P_k. For $A \in \binom{[n]}{k}$, let B_A be the event that T satisfies that all vertices in $[n] \setminus A$ can not beat every vertices in A. Then $B = \bigcup_{A \in \binom{[n]}{k}} B_A$.*

For $x \in [n] \setminus A$, let $B_{A,x}$ be the event that T satisfies that x can not beat every vertices in A. Then $B_A = \bigcap_{x \in [n] \setminus A} B_{A,x}$. Clearly, $\Pr[B_{A,x}] = 1 - \left(\frac{1}{2}\right)^k$.

Note that only arcs between x and A will affect the event $B_{A,x}$, and these arcs for distinct x's are disjoint. Thus all events $B_{A,x}$'s are independent, $x \in [n] \setminus A$. So $\Pr[B_A] = \prod_{x \in [n] \setminus A} \left(1 - \frac{1}{2^k}\right) = \left(1 - \frac{1}{2^k}\right)^{n-k}$.

By union bound,

$$\Pr[B] \leq \sum_{A \in \binom{[n]}{k}} \Pr[B_A] = \binom{n}{k} \left(1 - \frac{1}{2^k}\right)^{n-k} < 1.$$

Thus $\Pr[\bar{B}] > 0$, *i.e.* $\exists$ *a tournament T with property P_k.* □

Corollary 4. *If $n \geq k^2 2^{k+1}$, then $\exists$ a tournament T with n players with property P_k.*

Proof. *Exercise.* $\square$

Definition 3. *$\mathcal{F} \subseteq 2^{[n]}$, say $\mathcal{F}$ is 2-colorable if $\exists$ a function $f: [n] \to \{$blue, red$\}$, s.t. every set $A \in \mathcal{F}$ is not monochromatic.*

Note: $\mathcal{F} \subseteq \binom{[n]}{2}$, $\mathcal{F}$ is 2-colorable iff $G = ([n], \mathcal{F})$ is bipartite.

Theorem 5. *Every k-uniform family $\mathcal{F}$ with $|\mathcal{F}| < 2^{k-1}$ is 2-colorable.*

Proof. *Let $\mathcal{F}$ be any fixed k-uniform family of subsets of some finite set X. Consider a random function $f: X \to \{$blue, red$\}$, such that each $x \in X$ is colored by blue or red with probability $\frac{1}{2}$, and the coloring of different elements are independent. Let B be the event that f is "bad", i.e. $\exists\ A \in \mathcal{F}$ is monochromatic.*

For $A \in \mathcal{F}$, let B_A be the event that A is monochromatic. So $B = \bigcup_{A \in \mathcal{F}} B_A$. $\forall A \in \mathcal{F}$, $\Pr[B_A] = 2^{1-k}$. Hence $\Pr[B] \leq \sum_{A \in \mathcal{F}} \Pr[B_A] = |\mathcal{F}| 2^{1-k} < 1$. So $\Pr[\bar{B}] > 0$, i.e. $\exists$ a 2-coloring s.t. no $A \in \mathcal{F}$ is monochromatic. $\square$

Theorem 6. *If k is sufficiently large, then there exists a k-uniform family $\mathcal{F}$ such that $|\mathcal{F}| \leq k^2 2^k$ and $\mathcal{F}$ is not 2-colorable.*

Proof. *To be continued.* $\square$

Let $B(k)$ be the minimum possible number of sets in a k-uniform family with is not 2-colorable. Then $2^{k-1} \leq B(k) \leq k^2 2^k$.