Combinatorics, 2017 Fall, USTC

Week 16 Note

2018.1.2, Tuesday

Lemma 1. *Let $n > 2k$ be an integer, $0 \le p, \delta < 1$ and let $A_1, \ldots, A_n$ be events such that for $\forall I \subset [n]$ and $|I| \le k$, $\left|\Pr[\bigcap_{i \in I} A_i] - p^{|I|}\right| \le \delta$. Then*

$$\Pr[\bigcup_{i=1}^{n} A_i] \ge 1 - \mathrm{e}^{-pn} - \binom{n}{k+1}(\delta k + p^k).$$

In particular, if $\delta = 0$, then $\Pr[\bigcup_{i=1}^{n} A_i] \ge 1 - \mathrm{e}^{-pn} - \binom{n}{k+1}p^k$.

Proof. *First consider the case when k is even. Let $B_1 \ldots, B_n$ be mutually independent events, and $\Pr[B_i] = p,\ i \in [n]$. Apply Bonferroni inequality, then*

$$\Pr[\bigcup_{i=1}^{n} A_i] \ge \sum_{v=1}^{k}(-1)^{v+1} \sum_{|I|=v} \Pr[\bigcap_{i \in I} A_i], \text{ and}$$

$$\Pr[\bigcup_{i=1}^{n} B_i] \le \sum_{v=1}^{k}(-1)^{v+1} \sum_{|I|=v} \Pr[\bigcap_{i \in I} B_i] + (-1)^{k+2} \sum_{I \in \binom{[n]}{k+1}} \Pr[\bigcap_{i \in I} B_i]$$

$$= \sum_{v=1}^{k}(-1)^{v+1} \sum_{|I|=v} p^{|I|} + \binom{n}{k+1}p^{k+1}.$$

Since $A_1, \ldots, A_n$ are almost k-wise independent,

$$\Pr[\bigcup_{i=1}^{n} A_i] \ge \sum_{v=1}^{k}(-1)^{v+1} \sum_{|I|=v} (p^{|I|} - \delta) = \sum_{v=1}^{k}(-1)^{v+1} \sum_{|I|=v} p^{|I|} - \sum_{v=1}^{k}(-1)^{v+1}\binom{n}{v}\delta$$

$$\ge \sum_{v=1}^{k}(-1)^{v+1} \sum_{|I|=v} p^{|I|} - \delta k\binom{n}{k}.$$

Since $B_1, \ldots, B_n$ are mutually independent,

$$\Pr[\bigcup_{i \in I} B_i] = 1 - (1-p)^n \ge 1 - \mathrm{e}^{-pn}.$$

Then combine all inequalities above, we have

$$\Pr[\bigcup_{i=1}^{n} A_i] \geq 1-\mathrm{e}^{-pn}-\binom{n}{k+1}p^{k+1}-\delta k\binom{n}{k} \geq 1-\mathrm{e}^{-pn}-\binom{n}{k+1}(\delta k+p^{k+1})$$

$$\geq 1-\mathrm{e}^{-pn}-\binom{n}{k+1}(\delta k+p^{k}).$$

When k is odd, use $k-1$ in the above arguments and we can get the conclusion. $\square$

Theorem 2. *Let $G=(V,E)$ be a simple directed graph with minimum out-degree δ and maximum in-degree Δ. If $\mathrm{e}(\Delta\delta+1)(1-\frac{1}{k})^{\delta}<1$, then G contains a directed simple cycle of length $\equiv 0 (\mathrm{mod}\ k)$.*

Proof. *Assume every out-degree equals δ, since otherwise consider a subgraph of G with this property. Let $f: V \to \{0,1,\ldots,k-1\}$ be a random coloring of V, each $v \in V$, $f(v) \in \{0,1,\ldots,k-1\}$ independently and uniformly. $\forall v \in V$, let A_v be the event that no $u \in V$ such that $v \to u$ and $f(u) \equiv f(v)+1 (\mathrm{mod}\ k)$. Then $\Pr[A_v]=(1-\frac{1}{k})^{\delta}$. Let $N^{+}(v)=\{w \in V: v \to w\}$. Then A_v is mutually independent of all A_u except those satisfying $N^{+}(v)\cap(\{u\}\cup N^{+}(u)) \neq \emptyset$. The number of such u is $\leq \delta\Delta$, i.e. the degree of dependence $d \leq \delta\Delta$. By* Lovász Local Lemma, $\Pr[\bigcap_{v\in V} \bar{A}_v]>0$. That is $\exists f: V \to \{0,1,\ldots,k-1\}$ s.t. $\forall v \in V$, there is a $u \in V$ with $v \to u$ and $f(u) \equiv f(v)+1 (\mathrm{mod}\ k)$.

Start at any $v=v_0 \in V$, we get a sequence $v_0v_1v_2\cdots$ of vertices of G so that $v_i \to v_{i+1}$ and $f(v_{i+1}) \equiv f(v_i)+1 (\mathrm{mod}\ k)$. Let j be the minimum integer so that there is an $l<j$ with $v_l=v_j$. Then the cycle $v_lv_{l+1}\cdots v_j=v_l$ is a directed simple cycle of G whose length is divisible by k. $\square$

Coloring: Recall

(1) $\forall \mathcal{F} \subseteq \binom{[n]}{k}$ and $|\mathcal{F}|<2^{k-1}$, then $\mathcal{F}$ is 2-colorable.

(2) $\exists\ \mathcal{F} \subseteq \binom{[n]}{k}$ and $|\mathcal{F}| \leq k^2 2^k$ such that $\mathcal{F}$ is not 2-colorable.

$\mathcal{F} \subseteq 2^{[n]}$ is 2-colorable if $\exists$ a 2-coloring of points such that no set $A \in \mathcal{F}$ is monochromatic.

Theorem 3. *Suppose $\mathcal{F} \subseteq 2^{[n]}$ satisfies $\forall A \in \mathcal{F}$, $|A| \geq 2$. If any two non-disjoint members of $\mathcal{F}$ share ≥ 2 points, then $\mathcal{F}$ is 2-colorable.*

Proof. *Color each $i \in [n]$ one by one so that each $A \in \mathcal{F}$ is not monochromatic. Suppose that $1,2,\ldots,i$ are already colored. If we can not color $i+1$ in red, it means $\exists\ A \in \mathcal{F}$ such that $A \subset [i+1]$ and $i+1 \in A$, and all points in $A\setminus\{i+1\}$ are red. Similarly, if we can not color $i+1$ in blue, then $\exists\ A \in \mathcal{F}$ such that $A \subset [i+1]$ and $i+1 \in A$, and all points in $A\setminus\{i+1\}$ are blue. But then $A\cap B=\{i+1\}$.* $\square$

Fact: If $\mathcal{F}$ is a family of m mutually disjoint subsets, then $\mathcal{F}$ is 2-colorable, no matter how large m is.

__**Theorem**__ **4.** *If every member of* $\mathcal{F} \subseteq \binom{[n]}{k}$ *intersects* $\leq 2^{k-3}$ *other members, then* $\mathcal{F}$ *is* 2*-colorable.*

Proof. *Suppose* $\mathcal{F} = \{A_1, \dots, A_m\}$*. Consider a random* 2*-coloring of* $[n]$*, each* i *is independently colored in red or blue with probability* $\frac{1}{2}$*. Let* B_i *be the event that* A_i *is monochromatic. Then* $\Pr[B_i] = 2\left(\frac{1}{2}\right)^k = 2^{1-k}$*. We aim to show* $\Pr[\bar{B_1} \cap \dots \cap \bar{B_m}] > 0$*. Define a dependency graph by joining* B_i *and* B_j *if and only if* $A_i \cap A_j \neq \emptyset$*. Then degree of dependence is* $d \leq 2^{k-3}$*. Since*

$$4pd = 42^{1-k}2^{k-3} = 1,$$

we have $\Pr[\bar{B_1} \cap \dots \cap \bar{B_m}] > 0$. □

__**Theorem**__ **5.** $\mathcal{F} \subseteq 2^{[n]}$, $\forall\ A \in \mathcal{F}$, $|A| \geq k \geq 2$*. Suppose for each point* v*,*

$$\sum_{A \in \mathcal{F}: v \in A} (1 - \frac{1}{k})^{-|A|} 2^{-|A|+1} \leq \frac{1}{k}.$$

Then $\mathcal{F}$ *is* 2*-colorable.*

Proof. *Let* $\mathcal{F} = \{A_1, \dots, A_m\}$*. Consider a random* 2*-coloring of* $[n]$*, each* i *is independently colored in red or blue with probability* $\frac{1}{2}$*. Let* B_i *be the event that* A_i *is monochromatic. Then* $\Pr[B_i] = 2^{1-|A_i|}$*. Consider the dependency graph* $G = (V, E)$*,* B_i *and* B_j *are adjacent if and only if* $A_i \cap A_j \neq \emptyset$*. Let* $x_i = (1 - \frac{1}{k})^{-|A_i|} 2^{1-|A_i|}$*,* $i \in [m]$*, then* $0 \leq x_i < 1$*. By the definition of* G*,* $\forall$ $i \in [m]$*, we have*

$$x_i \Pi_{\{i,j\} \in E} (1 - x_j) \geq x_i \Pi_{v \in A_i} \Pi_{j: v \in A_j} (1 - x_j) \geq x_i \Pi_{v \in A_i} (1 - \sum_{j: v \in A_j} x_j)$$

$$\geq x_i (1 - \frac{1}{k})^{|A_i|}.$$

Hence

$$x_i \Pi_{\{i,j\} \in E} (1 - x_j) \geq x_i (1 - \frac{1}{k})^{|A_i|} = 2^{1-|A_i|} = \Pr[B_i].$$

Hence $\Pr[\bar{B_1} \cap \dots \cap \bar{B_m}] > 0$. □