Combinatorics, 2017 Fall, USTC

Week 3 Note 1

2017.9.26,Tuesday

1 Generating Function(GF) and Selections

Let $f_j(x) = \sum_{n\geq 0} f_j(n)x^n, \ \ j = 1, 2, \cdots, k$

$$f(x) = f_1(x)f_2(x)\cdots f_k(x) = \sum_{n=0}^{\infty}\left(\sum_{n_1+\cdots+n_k=n} f_1(n_1)f_2(n_2)\cdots f_k(n_k)\right)x^n.$$

Denote $[x^n]f(x) =$ coefficient of x^n in $f(x)$.

Facts:

(1) Let $f_j(x) = \sum_{i\in I_j} x^i, \ \ I_j \subseteq \mathbb{Z}_{\geq 0}, \ \ j \in [k]$,
$f(x) = f_1 f_2 \cdots f_k = \sum_{n\geq 0} b_n x^n$. Then

$$b_n = \sum_{n_1+\cdots n_k=n, n_j\in I_j} 1,$$

i.e. $b_n = \sharp$ integer solutions to $x_1 + x_2 + \cdots + x_k = n, x_j \in I_j$.

(2) Let $I_j = \mathbb{Z}_{\geq 0}$, then $f_j(x) = 1 + x + x^2 + \cdots = \frac{1}{1-x}$.
So $f(x) = \frac{1}{(1-x)^k}$ is the generating function of $b_n = \sharp$non-negative integer solutions to $x_1 + x_2 + \cdots + x_k = n$.

(3) Let $I_j = \mathbb{Z}_{>0}$, i.e. $x_j > 0$, then $f(x) = \frac{x^k}{(1-x)^k}$. (**Exercise**)

Problem 1: How many ways to pay 20 yuan using five 1-yuan bills, four 2-yuan bills, three 5-yuan bills ? Let a_n be $\sharp$ways to pay n-yuan given unlimited numbers of three kinds of bills, find the GF of $\{a_n\}$.

Solution. (1) $f_1(x) = 1+x+x^2+x^3+x^4+x^5, \ f_2(x) = 1+x^2+x^4+x^6+x^8$, $f_3(x) = 1 + x^5 + x^{10} + x^{15}, \ f(x) = f_1f_2f_3$, then $[x^{20}]f(x)$ is the answer.

(2) $I_1 = \{0,1,2,\cdots\},\ I_2 = \{0,2,4,\cdots\},\ I_3 = \{0,5,10,15,\cdots\}$,
$f_j(x) = \sum_{n\in I_j} x^n$. Then $f(x) = f_1f_2f_3 = \frac{1}{1-x}\cdot\frac{1}{1-x^2}\cdot\frac{1}{1-x^5}$ is the generating function.

$\square$

Integer Partition: Write n as a sum of positive integers without regard to the ordering of the numbers.

Problem 2: Let p_n be the number of partitions of n. Find the GF of $\{p_n\}$.

Solution. Let n_j be $\sharp$ of the j's in such a partition of n, then

$$\sum_{j\geq 1} j\cdot n_j = n.$$

Let $i_j = j\cdot n_j$, which is the contribution of the addends j in a partition of n, then $i_j \in \{0, j, 2j, 3j, \cdots\}$. Let $f_j(x) = 1 + x^j + x^{2j} + x^{3j} + \cdots = \frac{1}{1-x^j}$, then the GF of $\{p_n\}$ is

$$P(x) = \Pi_{j\geq 1} f_j(x) = \Pi_{j\geq 1}\frac{1}{1-x^j}.$$

$\square$

E.g. Compute p_4
$(1+x+x^2+x^3+\cdots)(1+x^2+x^4+\cdots)(1+x^3+x^6+\cdots)(1+x^4+x^8+\cdots)$
$= 1 + x + 2x^2 + 3x^3 + 5x^4 + \cdots$
$4 = 4, 3+1, 2+2, 2+1+1, 1+1+1+1.$

2 Exponential GF(EGF) and Arrangements

E.g. $T_n = \sharp$ strings of length n over $\{a,b,c\}$ *s.t.* both the numbers of a and b are even. Find T_n.

A selection of e_1 a's, e_2 b's and e_3 c's with $e_1+e_2+e_3 = n$ contributes $\frac{n!}{e_1!e_2!e_3!}$ to T_n. Therefore

$$T_n = \sum_{\substack{e_1+e_2+e_3=n \\ e_1, e_2 \in 2\mathbb{Z}_{\geq 0},\, e_3\in\mathbb{Z}_{\geq 0}}} \frac{n!}{e_1!e_2!e_3!}$$

What's the GF of T_n ?

Def: The EGF of the sequence $\{a_n\}$ is $f(x) = \sum_{n\geq 0}\frac{a_n}{n!}x^n$.

Fact 1: $f_j(x) = \sum_{n\geq 0}\frac{f_j(n)}{n!}x^n$, $j\in[k]$.

$$f(x) = f_1\cdots f_k = \sum_{n\geq 0}\left(\sum_{\substack{n_1+\cdots+n_k=n \\ n_1,\cdots n_k\geq 0}} \frac{\Pi_{j=1}^k f_j(n_j)}{n_1!n_2!\cdots n_k!}\right)x^n,$$

i.e. $f(x)$ is the EGF of

$$a_n = \sum_{\substack{n_1 + \cdots + n_k = n \\ n_1, \cdots n_k \geq 0}} \frac{n! \Pi_{j=1}^k f_j(n_j)}{n_1! n_2! \cdots n_k!}.$$

<u>**Fact 2**</u>**:** $f_j(x) = \sum_{i \in I_j} \frac{x^i}{i!}$, $I_j \subseteq \mathbb{Z}_{\geq 0}$, $j \in [k]$. Let

> $b_n = \sum_{i_1 + \cdots + i_k = n} \frac{n!}{i_1! \cdots i_k!} = \sharp$strings of length n over $\{a_1, \cdots, a_k\}$ s.t. the number of occurrence of a_j is in I_j.

Then $\Pi_{j=1}^k f_j$ is the EGF of $\{b_n\}_{n \geq 0}$.

<u>**Problem 3**</u>**:** Find EGF of $\{T_n\}$.

Solution. Let

$$T(x) = \sum_{n=0}^{\infty} \frac{T_n}{n!} x^n,$$

$$f_1(x) = f_2(x) = \sum_{i \in 2\mathbb{Z}_{\geq 0}} \frac{x^i}{i!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = \frac{e^x + e^{-x}}{2},$$

$$f_3(x) = \sum_{i \in \mathbb{Z}_{\geq 0}} \frac{x^i}{i!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x.$$

Therefore,

$$T(x) = f_1 f_2 f_3 = \sum_{n \geq 0} \left(\frac{3^n + 2 + (-1)^n}{4} \right) \frac{x^n}{n!} \Rightarrow T_n = \frac{3^n + 2 + (-1)^n}{4}.$$

□

Note: GF can find the number of selections, while EGF can be used to find the number of arrangements.

<u>**Problem 4**</u>**:** Let a_n be the number of ways to send n students to 4 classes R_1, R_2, R_3, R_4, s.t. each class has at least one student.

Solution. Let b_n be the number of vectors of length n over $[4]$ s.t. each $i \in [4]$ occurs at least once. Then

$$a_n = b_n = \sum_{\substack{i_1 + i_2 + i_3 + n_4 = n \\ i_1, \cdots n_4 \geq 1}} \frac{n!}{i_1! i_2! i_3! i_4!}.$$

Let $f_i(x) = \sum_{n\geq 1} \frac{x^n}{n!} = e^x - 1,\ i \in [4]$. Then the EGF of $\{a_n\}$ is

$$f = f_1 f_2 f_3 f_4 = \sum_{n\geq 0} (4^n - 4\cdot 3^n + 6\cdot 2^n - 4)\frac{x^n}{n!} + 1.$$

$\Rightarrow a_n = 4^n - 4\cdot 3^n + 6\cdot 2^n - 4,\ n \geq 1$, and $a_0 = 0$. $\square$

Exercise: Use EGF to find $a_n = \sharp$ vectors of length n over $[k]$.