Combinatorics, 2017 Fall, USTC

Week 3 Note 2

2017.9.29,Friday

Given $\{a_n\}$, we know GF $\sum a_n x^n$ and EGF $\sum \frac{a_n}{n!} x^n$.

Def: Dirichlet series of $\{a_n\}_{n=1}^{\infty}$ is $a(x) = \sum\limits_{n \geq 1} \frac{a_n}{n^x} = \frac{a_1}{1} + \frac{a_2}{2^x} + \frac{a_3}{3^x} + \cdots$.

Let $b(x) = \sum\limits_{n \geq 1} \frac{b_n}{n^x}$, $c(x) = a(x)b(x) = \sum\limits_{n \geq 1} \frac{c_n}{n^x}$. Then $c_n = \sum\limits_{rs = n} a_r b_s = \sum\limits_{d \mid n} a_d b_{\frac{n}{d}}$, where $\sum\limits_{d \mid n}$ means d runs over all positive factors of n.

Dirichlet Convolution

Def: The Dirichlet Convolution of $f = \{f(n)\}_1^{\infty}$ and $g = \{g(n)\}_1^{\infty}$ is a sequence $f \odot g$, where $f \odot g(n) = \sum\limits_{rs = n} f(r)g(s) = \sum\limits_{d \mid n} f(\frac{n}{d})g(d)$.

Facts:

(1) $\odot$ is commutative, associative and distributive with $+$.

(2) All real sequences form a ring under $\odot$ and $+$.

(3) Identity is $I = \{I(n)\}$ with $I(n) = \lfloor \frac{1}{n} \rfloor = \begin{cases} 1, & n = 1 \\ 0, & n > 1 \end{cases}$. Check $f \odot I = I \odot f = f, \ \forall f$.

(4) If $f \odot g = I$, we say f is the *D-inverse of* g.

Lemma 1. *$f = \{f(n)\}$ is D-invertible iff $f(1) \neq 0$.*

Proof: "$\Rightarrow$" f is D-invertible means $\exists g = \{g(n)\}$ such that $f \odot g = I$. So $1 = I(1) = (f \odot g)(1) = f(1)g(1) \Rightarrow \ f(1) \neq 0$.

"$\Leftarrow$" $f(1) \neq 0$. We want to prove $\exists g = \{g(n)\}$ such that $f \odot g = I$, i.e.

$$\sum_{d \mid n} f(\frac{n}{d})g(d) = \begin{cases} 1, & n = 1 \\ 0, & n > 1 \end{cases}.$$

That is

$$\begin{cases} f(1)g(1) = 1 \\ \sum\limits_{\substack{d \mid n \\ d \neq n}} f(\frac{n}{d})g(d) + f(1)g(n) = 0, \quad n > 1 \end{cases}.$$

We can define $g(n)$ inductively, $g(1) = \frac{1}{f(1)}$ and

$$g(n) = -\frac{1}{f(1)} \sum_{\substack{d \mid n \\ d \neq n}} f(\frac{n}{d}) g(d), \ n > 1.$$

$\square$

Invertible Formula: Assume $a \odot b = I$, then

$$f(n) \equiv \sum_{d|n} a(\frac{n}{d}) g(d) \Longleftrightarrow g(n) \equiv \sum_{d|n} b(\frac{n}{d}) f(d)$$

Proof: $f = a \odot g \Longleftrightarrow g = b \odot f.$ $\square$

Möbius Function: $\mu = \{\mu(n)\}_1^\infty$, where

$$\mu(n) = \begin{cases} 1 & n = 1 \\ (-1)^k & n \text{ is a product of } k \text{ distinct primes} \\ 0 & others \end{cases}.$$

Theorem 1. *Let $e = \{e(n)\}_1^\infty = \{1\}_1^\infty$. Then $\mu \odot e = I$, that is*

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & n = 1 \\ 0 & n > 1 \end{cases}.$$

Proof: $n = 1$, true.

$n > 1$, write $n = p_1^{k_1} \cdots p_r^{k_r}$, where $p_1, \ldots, p_r$ are distinct primes, $k_i > 0$. If $d \mid n$ but $d \nmid \prod_{i=1}^r p_i$, then $\mu(d) = 0$. So

$$\sum_{d|n} \mu(d) = \sum_{d \mid \prod_{i=1}^r p_i} \mu(d) = \sum_{I \subseteq [r]} (-1)^{|I|} = \sum_{i=0}^r \binom{r}{i} (-1)^i = 0$$

$\square$

Möbius Inverse Formula :

For any two sequences $\{f(n)\}$ and $\{g(n)\}$, we have

$$f(n) \equiv \sum_{d|n} g(d) \Longleftrightarrow g(n) \equiv \sum_{d|n} \mu(\frac{n}{d}) f(d)$$

Recall: Euler function $\varphi(n) = \sharp\{m \in [n] : \gcd(m, n) = 1\}$.

Write $n = p_1^{k_1} \cdots p_r^{k_r}$, then $\varphi(n) = n \cdot \prod_{i=1}^r (1 - \frac{1}{p_i})$.

Theorem 2. *Let $N = \{N(n)\} = \{1, 2, 3, \cdots\}$. Then*

(1) $\varphi = N \odot \mu$, i.e. $\varphi(n) = \sum_{d|n} \frac{n}{d}\mu(d)$.

(2) $N = \varphi \odot e$, i.e. $n = \sum_{d|n} \varphi(d)$.

Proof: (1) $n = p_1^{k_1} \cdots p_r^{k_r}$, then

$$\frac{1}{n}\varphi(n) = \prod_{i=1}^{r}(1 - \frac{1}{p_i}) = \sum_{I \subseteq [r]} (-1)^{|I|} \frac{1}{\prod_{i \in I} p_i},$$

$$\sum_{d|n} \frac{\mu(d)}{d} = \sum_{d \mid \prod_{i=1}^{r} p_i} \frac{\mu(d)}{d} = \sum_{I \subseteq [r]} (-1)^{|I|} \frac{1}{\prod_{i \in I} p_i}$$

(2) By *Möbius Inverse Formula.*

□

Def(Itemwise Product): The itemwise product of $a = \{a_n\}$ and $b = \{b_n\}$ is a sequence ab, where $(ab)(n) = a(n)b(n)$.

E.g. $NI = I$, $Ne = N$.

Lemma 2. *(1) $\forall$ two sequences f, g, $(Nf) \odot (Ng) = N(f \odot g)$.*

(2) If $f \odot g = I$, then $(Nf) \odot (Ng) = I$.

Theorem 3. *$N \odot (N\mu) = I$, $\varphi \odot (N\mu \odot e) = I$.*

Arrangements in a cycle(without seat number)

If no repetition, it is $\frac{n!}{n} = (n-1)!$.

Question: Let $C_m(n) = \sharp$ cycles of length n over $[m]$. Find $C_m(n)$.

Consider how many lines of length n corresponds to the same n-cycle. Suppose we have an n-cycle of the smallest period p(here $p \mid n$):

$$a_1 a_2 \ldots a_p a_1 a_2 \ldots a_p \ldots a_1 a_2 \ldots a_p.$$

Cut it into lines, then we have p different lines:

$$\begin{gathered} a_1 a_2 \ldots a_p a_1 a_2 \ldots a_p \ldots a_1 a_2 \ldots a_p \\ (a_2 \ldots a_p a_1)(a_2 \ldots a_p a_1) \ldots (a_2 \ldots a_p a_1) \\ \vdots \\ (a_p a_1 \ldots a_{p-1})(a_p a_1 \ldots a_{p-1}) \ldots (a_p a_1 \ldots a_{p-1}) \end{gathered}$$

Let $L(p)$ be $\sharp$lines of period p, $M(p)$ be $\sharp$cycles of period p. Then

$$L(p) = pM(p).$$

Theorem 4. $C_m(n) = \sum_{p|n} \frac{1}{p} \sum_{d|p} \mu(\frac{p}{d}) m^d$.

Proof: $C_m(n) = \sum_{p|n} M(p)$, $m^n = \sum_{p|n} L(p)$, then use *Möbius Inverse Formula*. $\square$

Theorem 5. $C_m(n) = \frac{1}{n} \sum_{d|n} \varphi(\frac{n}{d}) m^d$.

Proof: Let $C = \{C_m(n)\}$, $M = \{M(n)\}$, $L = \{L(n)\}$, $f = \{f(n)\} = \{m^n\}$. Then $C = M \odot e$, $f = L \odot e \implies L = f \odot \mu$, $L = NM$. So $NC = N(M \odot e) = (NM) \odot (Ne) = L \odot N = f \odot \mu \odot N = f \odot \varphi$. $\square$

Exercise: Compute $C_{10}(9) = 111111340$.