Combinatorics, 2017 Fall, USTC
Week 3 Note 3

2017.9.30, Saturday

<u>**Definition**</u> **1.** *A partially ordered set(**poset**) $P = (X, \leq)$ is a set X with a relation "$\leq$" on X, s.t.*

(1) Reflectivity: $x \leq x$;

(2) Antisymmetry: If $x \leq y$ and $y \leq x$, then $x = y$;

(3) Transitivity: If $x \leq y$ and $y \leq z$, then $x \leq z$.

E.g.

(1) $(\mathbb{Z}_{\geq 0}, <)$, "less than" relation.

(2) $(\mathbb{Z}_{>0}, \leq)$, divisor poset, $a \leq b \Leftrightarrow a \mid b$.

(3) $(2^X, \leq)$, inclusion relation, $A \leq B \Leftrightarrow A \subseteq B$.

<u>**Definition**</u> **2.** *$P = (X, \geq)$, the incidence algebra of P is*

$$\mathbb{A}(P) = \{f : P^2 \to \mathbb{R} | f(x, y) = 0, whenever\ x \nleq y\}$$

E.g.

(1) $0(x, y) = 0$.

(2) Kronecker delta function: $\delta(x, y) = \left\{ \begin{array}{ll} 1, & x = y \\ 0, & x \neq y \end{array} \right.$.

(3) Zeta function: $\zeta(x, y) = \left\{ \begin{array}{ll} 1, & x \leq y \\ 0, & x \nleq y \end{array} \right.$.

<u>**Facts**</u>**:**

(1) $f, g \in \mathbb{A}(P) \Rightarrow f + g \in \mathbb{A}(P)$.

(2) $f \in \mathbb{A}(P) \Rightarrow cf \in \mathbb{A}(P), \forall c \in \mathbb{R}$.

Definition 3. *Let $f, g \in \mathbb{A}(P)$, the Dedekind convolution of f and g is $f * g \in \mathbb{A}(P)$, where $(f * g)(x, y) = \sum_{z\,:\,x \leq z \leq y} f(x, z)g(z, y)$.*

E.g.

(1) $(f * 0)(x, y) = 0$.

(2) $(f * \delta)(x, y) = f(x, y)\delta(y, y) = f(x, y) = (\delta * f)(x, y)$, so δ is the identity.

(3) $(f * \zeta)(x, y) = \sum_{z\,:\,x \leq z \leq y} f(x, z)$

Exercise: $*$ is commutative?(N) associative?(Y) distributive?(Y)

Definition 4. *If $f * g = \delta$, we say f, g is invertible.*

$f * g_1 = g_2 * f = \delta \Rightarrow g_1 = g_2$.

Theorem 1. *$f \in \mathbb{A}(P)$, then f is invertible $\Longleftrightarrow f(x, x) \neq 0$, $\forall x \in P$.*

Proof. *"$\Rightarrow$" $\exists g \in \mathbb{A}(P)$, s.t. $f * g = \delta$. $0 \neq 1 = \delta(x, x) = (f * g)(x, x) = f(x, x)g(x, x)$.*

"$\Leftarrow$" Find $g \in \mathbb{A}(P)$, s.t. $f * g = \delta$.

$$\begin{cases} f(x,x)g(x,x) = \delta(x,x) = 1 & \Longrightarrow \quad g(x,x) = \frac{1}{f(x,x)}, \forall x \\ \sum\limits_{\substack{z \\ x \leq z \leq y}} f(x,z)g(z,y) = \delta(x,y) = 0 & \qquad\quad x \neq y \end{cases}.$$

Then

$$f(x,x)g(x,y) + \sum_{\substack{z \\ x < z \leq y}} f(x,z)g(z,y) = 0,$$

i.e. $g(x, y) = -\frac{1}{f(x,x)} \left(\sum\limits_{\substack{z \\ x < z \leq y}} f(x, z)g(z, y) \right) = 0$, *by recursion.* □

Note: If use $g * f = \delta$, then $g(x, y) = -\frac{1}{f(y,y)} \left(\sum\limits_{\substack{z \\ x \leq z < y}} g(x, z)f(z, y) \right)$.

Definition 5. *Möbius Function over P is $\mu_P = \zeta^{-1}$, where*

$$\mu_P(x,y) = \begin{cases} 1, & x = y \\ -\sum\limits_{x < z \leq y} \mu_P(z,y) = -\sum\limits_{x \leq z < y} \mu_P(x,z), & x < y \\ 0, & else \end{cases}.$$

Theorem 2. ***(Inverse Formula I)*** *Suppose P has a unique minimal element. Let $e: P \to \mathbb{R}$ be a function. If we have $n: P \to \mathbb{R}$ s.t.*

$$n(y) = \sum_{z \le y} e(z), \ \forall y \in P,$$

then

$$e(y) = \sum_{z \le y} n(z)\mu_P(z, y).$$

The converse is also true.

Proof. *Let m be the minimal element of P. Define $f, g \in \mathbb{A}(P)$ as the following:*

$$f(x,y) = \begin{cases} e(y), & x = m \\ 0, & x \neq m \end{cases}, \ g(x,y) = \begin{cases} n(y), & x = m \\ 0, & x \neq m \end{cases}.$$

We want to prove $g = f * \zeta$.

$g(m,y) = n(y) = \sum_{z \le y} e(z) = \sum_{z \le y} f(m,z) = \sum_{m \le z \le y} f(m,z)\zeta(z,y) = (f * \zeta)(m,y)$. $\forall x \neq m,\ g(x,y) = 0 = (f * \zeta)(x,y) \Longrightarrow g = f * \zeta \Longrightarrow f = g * \mu_P$ $\Longrightarrow e(y) = f(m,y) = \sum_{m \le z \le y} g(m,z)\mu_P(z,y) = \sum_{z \le y} n(z)\mu_P(z,y)$. □

Theorem 3. *$\Omega = (\mathbb{Z}_{>0}, \le)$ is the divisor poset, then $\mu_\Omega(x,y) = \mu(\frac{y}{x})$ if $x \mid y$.*

Proof. *(1) Show $\mu_\Omega(x,y) = \mu_\Omega(1, \frac{y}{x})$ if $x \mid y$. Prove by induction on the ♯prime factors of $\frac{y}{x}$.*

If $\frac{y}{x} = 1$, $\mu_\Omega(x,y) = \mu_\Omega(1,1) = 1$. Assume $\mu_\Omega(x,y) = \mu_\Omega(1, \frac{y}{x})$ if $\frac{y}{x}$ has $\le k$ prime factors. Take $x \mid y$ and $\frac{y}{x}$ has $k+1$ prime factors.

$$\mu_\Omega(x,y) = -\sum_{x \le z < y} \mu_\Omega(x,z) = -\sum_{x \le z < y} \mu_\Omega(1, \frac{z}{x}),$$

$$\mu_\Omega(1, \frac{y}{x}) = -\sum_{1 \le z < \frac{y}{x}} \mu_\Omega(1,z) = -\sum_{x \le zx < y} \mu_\Omega(1,z).$$

(2) Show $\mu_\Omega(1,d) = \mu(d)$. Prove by induction on the ♯prime factors of d.

$$\mu_\Omega(1,1) = \mu(1),$$
$$\mu_\Omega(1,p) = -\sum_{1 \le z < p} \mu_\Omega(1,z) = -1 = \mu(p),$$
$$\mu_\Omega(1,p_1p_2) = -(\mu_\Omega(1,1) + \mu_\Omega(1,p_1) + \mu_\Omega(1,p_2)) = 1 = \mu(p_1p_2),$$
$$\mu_\Omega(1,p^2) = 0 = \mu(p^2).$$

Assume $\mu_\Omega(1,d) = \mu(d)$ if d has $\le k$ prime factors.

Case 1: $d = p_1 \cdots p_{k+1}$

$$\mu_\Omega(1,d) = -\sum_{1 \le z < d} \mu_\Omega(1,z) = -\sum_{i=0}^{k} \binom{k+1}{i}(-1)^i = (-1)^{k+1} = \mu(d).$$

Case 2: $d = p_1^{k_1} \cdots p_r^{k_r},\ k_1 + \cdots k_r = k+1$ *and* $k_i \geq 2$ *for some* $i \in [r]$

$\mu_\Omega(1,d) = -\sum\limits_{1 \leq z < d} \mu_\Omega(1,z) \xlongequal{\text{by induction}} -\sum\limits_{1 \leq z \leq p_1 \cdots p_r} \mu_\Omega(1,z) = 0 = \mu(d).$ □

Corollary 1. $g(n) = \sum\limits_{d|n} f(d) \Longleftrightarrow f(n) = \sum\limits_{d|n} \mu(\frac{n}{d}) g(d).$

Theorem 4. ***(Inverse Formula II)*** *Suppose P has a unique maximal element. Let $e:\ P \to \mathbb{R}$ be a function. If we have $n:\ P \to \mathbb{R}$ s.t.*

$$n(x) = \sum_{x \leq z} e(z),$$

then

$$e(x) = \sum_{x \leq z} \mu_P(x,z) n(z).$$

Proof. *Exercise!* □

Theorem 5. $|X| = n$ $P_n = (2^X, \leq)$ *with "$\subseteq$" relation. Then*

$$\mu_{P_n}(A,B) = (-1)^{|B \setminus A|},\ \textit{if } A \subseteq B.$$

Proof. *Prove by induction on $|B \setminus A|$(Exercise!)* □

Corollary 2. ***(IEP)*** $|A_1^c \cap \cdots \cap A_n^c| = \sum\limits_{I \subseteq [n]} (-1)^{|I|} |A_I|.$

Proof. *Let $X = [n], P_n = (2^X, \leq)$ with "$\subseteq$" relation.*
Define

$$n:\ P_n \to \mathbb{R} \textit{ as } n(|I|) = |A_I| = |\bigcap_{i \in I} A_i|,$$

$$e:\ P_n \to \mathbb{R} \textit{ as } n(|I|) = |A_I \cap (\bigcap_{j \notin I} A_j^c)|.$$

Then $n(J) = \sum\limits_{J \subseteq I \subseteq [n]} e(I)$. Since P_n has a unique maximal element $[n]$, by ***Theorem 4***

$$e(J) = \sum_{J \subseteq I \subseteq [n]} \mu_{P_n}(J,I) n(I) = \sum_{J \subseteq I \subseteq [n]} (-1)^{|I \setminus J|} |A_I|.$$

Particularly, $e(\emptyset) = |A_1^c \cap \cdots \cap A_n^c| = \sum\limits_{I \subseteq [n]} (-1)^{|I|} |A_I|.$ □