Combinatorics, 2017 Fall, USTC

Week 6 Note

2017.10.24, Tuesday

Definition 1. *Let $S_1, \cdots, S_m \subseteq X = \{x_1, \cdots, x_n\}$, and $M = (a_{ij})$ be the corresponding incidence matrix. The* ***permanent*** *of M is*

$$Per(M) = \sum_{(i_1, \cdots, i_m) \in S_n(m)} a_{i_1 1} a_{i_2 2} \cdots a_{i_m m},$$

where $S_n(m)$ is the set of all vectors of length m over $[n]$ without repetition.

Fact: $Per(M) = \sharp$different SDR's of $S_1, \cdots, S_m$.

Definition 2. *Let A be a 0-1 matrix. Two 1's are dependent if they are in the same row or the same column, otherwise, they are independent.*

Theorem 1 (König)**.** *Let A be an $m \times n$ 0-1 matrix, then the maximum number of independent 1's $r =$ the minimum number of rows and columns R required to cover all 1's in A. ($\Longleftrightarrow A$ doesn't have a zero-submatrix of size $c \times d$ such that $c + d = 2m - r + 1$)*

Proof. *Clearly, $R \geq r$, since we can find r independent 1's and every row or column covers at most one of them.*

Now we show $r \geq R$. Assume that some a rows and b columns cover all 1's and $a + b = R$. We may assume the first a rows and the first b columns cover all the 1's. Write A as the form

$$A = \begin{pmatrix} B_{a\times b} & C_{a\times(n-b)} \\ D_{(m-a)\times b} & E_{(m-a)\times(n-b)} \end{pmatrix},$$

with no 1 in $E_{(m-a)\times(n-b)}$. If we can show that there are a independent 1's in C and b independent 1's in D, then we find at least $a + b$ independent 1's, so we have $r \geq a + b = R$.

For each $1 \leq i \leq a$, let $S_i = \{j : c_{ij} = 1\} \subseteq [n - b]$. If $S_1, \cdots, S_a$ have an SDR, then we find a independent 1's in C If not, by Hall's theorem, there are some $k \in [a]$ sets, say $S_{i_1}, \cdots, S_{i_k}$, such that $\left|\bigcup\limits_{j=1}^{k} S_{i_j}\right| < k$, i.e. the 1's in these k rows occupy at most $k - 1$ columns of C,say $j_1, \cdots, j_{k-1}$. Then the first b

columns of A, the columns $j_1, \cdots, j_{k-1}$ of C and the first a rows of A deleting the rows $i_1, \cdots, i_k$ will cover all 1's in A. So we find $b+(k-1)+a-k = a+b-1$ rows and columns cover all 1's in A, Contradiction! □

<u>Definition</u> 3. *A system of common representatives(SCR) of two sequences of sets $A_1, \cdots, A_m$ and $B_1, \cdots, B_m$ is a sequence $x_1, \cdots, x_m$(not necessarily distinct) such that $x_i \in A_i \cap B_{\pi(i)},\ i \in [m]$ for some $\pi \in S_m$.*

<u>Theorem</u> 2. *Suppose X has two partitions $X = A_1 \cup \cdots \cup A_m = B_1 \cup \cdots \cup B_m$. Then they have an SCR if and only if for any $I \subset [n]$, $\underset{i \in I}{\cup} A_i$ contains at most $|I|$ sets of $B_j,\ j \in [m]$.*

Proof. *Consider the intersection matrix of the two partitions $C = (c_{ij})$:*

$$c_{ij} = \begin{cases} 1 & \textit{if } A_i \cap B_j \neq \emptyset \\ 0 & \textit{else} \end{cases}.$$

Then $\exists$ SCR $\iff C$ has m independent 1's.

By the proof of König *theorem, $\iff C$ doesn't has an $(m-k) \times (k+1)$ zero submatrix for any $0 \leq k \leq m-1 \iff \nexists\ m-k$ sets A_i, whose union is disjoint with $k+1$ sets $B_j \iff \nexists\ k$ sets A_i, whose union contains $k+1$ sets $B_j \iff$ the union of any k sets A_i contains at most k sets B_j.* □

<u>Definition</u> 4. *Bipartite graph $G = (V, E)$, $V = A \cup B$, $i \sim j$ only if i, j are in different sets. Two edges are **disjoint** if they have no common vertex. A **matching** is a set of pairwise disjoint edges. A vertex is **matched** (or **saturated**) if it is an endpoint of one of the edges in the matching. Otherwise the vertex is **free**(or **unmatched**). If a matching matches all vertices in A, say it's a matching of A into B. A **perfect matching** is a matching of A into B when $|A| = |B|$. A vertex $x \in A$ is a **neighbor** of a vertex $y \in B$ if $\{x, y\} \in E$. Let S_x be the set of all neighbors of x, then $\deg(x) = |S_x|$.*

Fact: $\exists$ a matching of A into B if and only if $S_x,\ x \in A$ has an SDR.

<u>Theorem</u> 3. *If G is a bipartite graph with bipartitions A, B, then G has a matching of A into B iff $\forall k \in [|A|]$, every subset of k vertices from A has at least k neighbors.*

<u>Definition</u> 5. *A vertex cover in $G = (A \cup B, E)$ is a set of vertices $S \subseteq A \cup B$ such that every edge in E is incident to at least one vertex in S.*

<u>Theorem</u> 4. *The maximum size of a matching in a bipartite graph equals to the minimum size of a vertex cover.*

<u>Propositon</u> 1. *$|X| = n$. For any $k \leq \frac{n-1}{2}$, it is possible to extend every k-element subset of X to a $(k+1)$-subset such that the extensions of no two sets coincide.*

Proof. *Consider the bipartite graph* $G = (A \cup B, E)$,

$$A = \{all\ k\text{-}subsets\ of\ X\},\ \ B = \{all\ (k+1)\text{-}subsets\ of\ X\}.$$

For any $x \in A$ *and* $y \in B$, $x \sim y$ *iff* $x \subset y$. *Then we only need to prove* G *has a matching of* A *into* B. *For each* $x \in A$, $\deg(x) = n - k$, *and for any* $y \in B$, $\deg(y) = k + 1$. *For* $I \subseteq A$, *let* $S(I) = \bigcup_{x \in I} S_x$, *then*

$$|I|(n-k) \leq |S(I)|(k+1) \Longrightarrow |S(I)| \geq |I|\frac{n-k}{k+1} \geq |I|. \quad \square$$