Combinatorics, 2017 Fall, USTC
Week 7 Note

2017.11.3, Friday

Pigeonhole Principle(P-P) and Graphs

Note: here we only consider finite simple graphs: no loop, no multiple edge.

Theorem 1 (Handshaking Lemma)**.** *In any finite graph, #vertices which have odd degrees is even.*

Proof. *Double counting.* $G = (V, E)$*. Count #ordered pairs* $(x, y) \in E$,

$$2|E| = \sum_{x \in V} \deg(x) = \sum_{\substack{x \in V \\ \deg(x) \text{ is odd}}} \deg(x) + \sum_{\substack{x \in V \\ \deg(x) \text{ is even}}} \deg(x).$$

If we have odd #vertices of odd degree, then the first item is odd and the second item is even, a contradiction. □

Definition 1. G *is a graph.* ***Independent number*** $\alpha(G)$ *is the maximum number of pairwise nonadjacent vertices of* G*. Such a set of vertices is called an* ***independent set****.* ***Chromatic number*** $\chi(G)$ *is the minimum number of colors in a coloring of* $V(G)$ *s.t. no two adjacent vertices have the same color. Such a coloring is called a* ***proper coloring****.*

Propositon 2. *In any graph* G *with* n *vertices,* $n \leq \alpha(G)\chi(G)$*.*

Proof. *Given a proper coloring of* G *with* $\chi(G)$ *colors and partition* $V(G)$ *into* $\chi(G)$ *color classes. By P-P, one of the classes has size*$\geq \frac{n}{\chi(G)}$ *and these vertices are pairwise nonadjacent. Hence* $\alpha(G) \geq \frac{n}{\chi(G)}$*.* □

Definition 2. *A graph* G *is* ***connected*** *if there is a path between any two vertices, where a path is* $v_1 \sim v_2 \sim \cdots \sim v_s$ *and* $v_i \neq v_j$*,* $i, j \in [s]$*,* $i \neq j$*. If* $v_1 \sim v_s$*, we say it is a* ***cycle****.*

Propositon 3. $|V(G)| = n$*. If for any* $x \in V(G)$*,* $\deg(x) \geq \frac{n-1}{2}$*, then* G *is connected.*

Proof. *Take any different* $x, y \in V(G)$*. If* $x \sim y$*, then done.*

If $x \nsim y$*, since* $\deg(x), \deg(y) \geq \frac{n-1}{2}$*, there are at least* $n-1$ *edges joining* x, y *to* $V(G) \setminus \{x, y\}$*. Since* $|V(G) \setminus \{x, y\}| = n - 2$*, by P-P,* $\exists z \in V(G) \setminus \{x, y\}$*,* $z \sim x, z \sim y$*.* □

Remark:

(1) The condition in Proposition 3 is best possible: e.g. n even, G is the union of two vertex disjoint complete graphs of $\frac{n}{2}$ vertices, each vertex has degree $\frac{n-2}{2}$, but G is disconnected.

(2) Define the **diameter** of G is the smallest number k, s.t. every two vertices are connected by a path with at most k edges. Then Proposition 3 says G has diameter at most two.

Ramsey's Theorem

Fact(A party of six): Suppose a party has 6 participants. Participants may know each other or not. Then there must be 3 people such that any 2 know each other or any 2 don't know each other.

Proof. *Construct a graph with vertices* $[6]$, *where* $i \sim j$ *iff* i *and* j *know each other. Then we need to show that there are* 3 *vertices in* G *which form a triangle or an independent set of size* 3.

Consider vertex 1*, by P-P,* 1 *is either adjacent to* ≥ 3 *vertices or nonadjacent to* ≥ 3 *vertices.*

① *Suppose* 1 *is adjacent to* $2, 3, 4$*. If one of the pairs* $\{2, 3\}$, $\{2, 4\}$, $\{3, 4\}$ *is adjacent, then we have a* K_3*. If not,* $\{2, 3, 4\}$ *is an independent set of size* 3.

② *Suppose* 1 *is nonadjacent to* $2, 3, 4$*. Similar arguments.* □

Definition 3. *$\forall s, t \geq 1$, let $R(s, t)$ denote the smallest integer n, s.t. in any graph with n or more vertices, there exists either a clique(a complete subgraph) with s vertices or an independent set with t vertices I_t.*

Remark:

① $R(s, t) \leq L \iff$ any graph with L vertices has either a K_s or an I_t.

② $R(s, t) > M \iff \exists$ a graph with M vertices has neither K_s nor I_t.

Fact:

① $R(s, t) = R(t, s)$.

② $R(2, t) = t$ and $R(s, 2) = s$.

③ $R(3, 3) = 6$.

Propositon 4. *For $s \geq 2,\ t \geq 2$, $R(s, t) \leq R(s, t-1) + R(s-1, t)$.*

Proof. *Let G be a graph on $n = R(s,t-1) + R(s-1,t)$ vertices. Take an arbitrary vertex $x \in V(G)$. Let $T = \{y \in V(G) : x \nsim y\}$ and $S = \{y \in V(G) : x \sim y\}$. Then $V(G) \setminus \{x\} = S \cup T$, i.e. $R(s,t-1) + R(s-1,t) = |S| + |T| + 1$. By P-P, we have either $|T| \geq R(s,t-1)$ or $|S| \geq R(s-1,t)$.*

① *$|T| \geq R(s,t-1)$. Consider the induced subgraph $G[T]$: a graph on T, in which $v \sim w$ iff $v \sim w$ in G. Since $G[T]$ has at least $R(s,t-1)$ vertices, $G[T]$ has either a K_s or an I_{t-1}. Therefore $G[T \cup \{x\}]$ has either a K_s or an I_t.*

② *$|S| \geq R(s-1,t)$. Similar.* □

<u>Theorem</u> 5. *$R(s,t) \leq \binom{s+t-2}{s-1} = \binom{s+t-2}{t-1}$.*

Proof. *By induction on $s+t$. $R(2,t) = t$, $R(s,2) = s$, true.*

Assume the claim holds for $R(k,l)$ with $k + l < s + t$. Then $R(s,t) \leq R(s,t-1) + R(s-1,t) \leq \binom{s+t-3}{s-1} + \binom{s+t-3}{s-2} = \binom{s+t-2}{s-1}$. □

<u>Theorem</u> 6. *If $R(s,t-1), R(s-1,t)$ are even, then $R(s,t) \leq R(s,t-1) + R(s-1,t) - 1$.*

Proof. *Homework!* □

<u>Corollary</u> 7. *$R(3,4) = 9$.*

Proof. *Homework!* □

Remark: 2-coloring version of Ramsey's theorem. Define a r-edge-coloring of K_n to be a coloring of edges of K_n by r colors. Then $R(s,t)$ denotes the smallest integer N s.t. any 2-edge-coloring of K_N has either a blue K_s or a red K_t.

Generalized Ramsey number $R_k(s_1, s_2, \ldots, s_k)$ is the smallest integer N such that any k-edge-coloring of K_N has a K_{s_i} in color i for some $i \in [k]$.