Combinatorics 2017 Fall
week1 note

Teaching by: Professor Xiande Zhang

Reference:
Extremal Combinatorics with applications in Computer Science. 2nd Edition.Stasys Jukna,Springer.

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Notation:

(1) A set X (a collection of distinct elements)
$|X| :=\#$ elements of X

(2) for integer $n > 0$ $[n] := \{1, 2, \cdots, n\}$

(3) A vector(or string) of length n over X
$(x_1, \cdot, x_n).x_i \in X, i \in [n]$

(4) i.e. that is.

$\exists$ exist.

$\Rightarrow$ imply.

s.t. such that.

! unique.

Counting:

Function.For sets A,B. a function $A \to B$ is a relation between A and B. $(a, f(a))$ s.t. for each $a \in A, \exists b \in B$ satisfying $b = f(a)$

Injection(one to one function): if $a \neq a'$,then $f(a) \neq f(a')$.

Surjection(onto) $\forall b \in B, \exists a \in A, s.t. f(a) = b$.

Bijection(one to one correspondence) Injection & Surjection.

Porpostion1. For sets A,B

(1) if $\exists$ injection $f : A \to B$ then $|A| \leq |B|$.

(2) if $\exists$ surjection $f : A \to B$ then $|A| \geq |B|$.

(3) if $\exists$ bijection $f : A \to B$ then $|A| = |B|$.

proof:

(1) Assume $|B| < |A|$ since $\exists$ function$f : A \to B$ by definition $\forall a \in A, \exists! b \in B$ s.t.$f(a) = b$. since $|B| < |A|$ then $\exists a \neq a\prime$ s.t. $f(a) = f(a\prime)$(by pigeonhole principal) This is contradiction with the fact that f is a injection.

(2) Assume $|B| > |A|$ since $\exists$ function$f : A \to B$ by definition $\forall a \in A, \exists! b \in B$ s.t.$f(a) = b$. $\exists b \in B$ s.t.there does not exist $a \in A$ satisfying $f(a) = b$ This is contradiction with the fact that f is a surjection.

(3) (1)&(2) $\Rightarrow$ (3). $\square$

Porpostion2:

For sets X,Y.$|X| = n, Y = [r]$. Let $X^Y := \{all\ functions\ f : Y \to X\}$.Then $|X^Y| = n^r$.

proof: Let B=$\{$all vectors of length r over $X\}$. define a function

$$g : X^Y \to B.\ f \mapsto (f(1), \cdots, f(r))$$

g is a injection since $f \neq f' \Rightarrow (f(1), \cdots, f(r)) \neq (f'(1), \cdots, f'(r))$. g is a surjection.since $\forall (b_1, \cdots, b_r) \in B$.define $f : Y \to X, i \mapsto b_i (i.e. f(i) = b_i)$ That is easy to get $g(f) = (b_1, \cdots, b_r)$.
$\Rightarrow g$ is a bijection.$\Rightarrow |X^Y| = |B| = n^r$. □

Porpostion3:

Let $Y = [r], 2^Y$:=$\{$all subsets of $Y\}$.then $|2^Y| = 2^r$

proof: Let B =$\{$all vectors of length r over $\{$0,1$\}\}$. define function

$$f : 2^Y \to B, A \to f(A) = (b_1, \cdots, b_r)$$

$$\text{where b}_i = \begin{cases} 0, if\ i \notin A; \\ 1, if\ i \in A. \end{cases}$$

claim:f is a injection.

$\forall A \neq A', \Rightarrow \exists i \in [r]$ s.t.$i \in A, i \notin A'$ or $i \notin A, i \in A'$.
$\Rightarrow f(A) \neq f(A')$.

claim:f is a surjection.

$\forall (b_1, \cdots, b_r) \in B$ define $A = \{i | b_i = 1\}$

$$\Rightarrow f(A) = (b_1, \cdots, b_r)$$

$\Rightarrow f$ is a bijection.$\Rightarrow |Y| = |B| = 2^r$.

[f is called as indicator(characteristic function) A is called as support of $f(A) = (b_1, \cdots, b_r)$]. □

Binomial Coefficient:

- $\binom{n}{k}$:= $\#k-$elements subsets of an n elements set.
- a$|X| = n$, $\binom{X}{k}$:={all $k-$subsets of X}.$\left|\binom{X}{k}\right| = \binom{n}{k}$.
- $n! = n(n-1)\cdots 2\cdot 1$.

  $(n)_r := n(n-1)\cdots(n-r-1)$.

  $0! = 1$.

  $\binom{n}{0} = 1$.

  $\binom{n}{k} = 0$ if $k > n$.

Porposition4: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

proof: Let B ={all vectors of length k over $[n]$ consisting k different elements.}
(Double Counting)
Way1:just directly count the number of vectors.$|B| = \frac{n!}{k!(n-k)!}$.
Way2:There are $\binom{n}{k}$ ways to choode $k-$subset of X.For each $k-$subset ,there are $k!$ ways to order it to a vector.
$\Rightarrow |B| = \binom{n}{k}k!$
$\Rightarrow \binom{n}{k} = \frac{n!}{k!(n-k)!}$. $\square$

Porposition5:

(1) $\binom{n}{k} = \binom{n}{n-k}$.

(2) (Pascal Triangle) $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

<u>proof:</u>

(1) trivial.(Hint: you can construct a bijection.)

(2) Separating $\binom{X}{k}$ into two parts. and find a fixed element $t \in X$.

- #{all $k-$subsets containing t}=$\binom{n-1}{k-1}$.
- #{all $k-$subsets avoiding t}=$\binom{n-1}{k}$.

Combinating the two situations.then we prove (2). $\square$

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Slections with repetition

Porposition6: # {integer solutions to $x_1 + \cdots + x_n = k$. where $x_i > 0$}= $\binom{k-1}{n-1}$.

<u>proof:</u> The question is equivalent to How many ways of distributing k sweets to n children.Such that each child has at least one sweet.

Lay out the sweets in a single row of length k,cut it into n pieces.then given yhe sweets in the ith piece to child i.So we need $n-1$ cuts from $k-1$ possibles.

$$\Rightarrow \left(\begin{array}{c} k-1 \\ n-1 \end{array} \right).$$

$\square$

{integer solutions to $x_1+\cdots+x_n = k$. where $x_i \geq 0$}= $\binom{n+k-1}{n-1}$.

<u>proof:</u>Let$A$={integer solutions $x_1 + \cdots + x_n = k, x_i \geq 0$}
B ={integer solutions $y_1 + \cdots + y_n = n + k, y_i > 0$}

Define $f : A \to B, (x_1, \cdots, x_n) \mapsto (y_1, \cdots, y_n)$ by $y_i = x_i+1, i \in [n]$.
Show:f is a bijection.

(1) f is weiidefined.if $(x_1, \cdots, x_n) \in A$ then $(y_1, \cdots, y_n) \in B$.

(2) injection.

(3) surjection.

$\Rightarrow |A| = |B| = \binom{n+k-1}{n-1}$ $\square$

Porposition7: $X = [n], A = \{\{a_1, \cdots, a_r\} \subseteq X, 1 \leq a_1 \leq a_2 \leq \cdots \leq a_r \leq n, and\ a_{i+1} - a_i \geq k+1, i \in [r-1]\}$.

proof: Let $B=\{$integers set $b_1+\cdots+b_{r+1}=n-1, where\ b_1\geq 0, b_i\geq k+1, i=2,\cdots,r, b_{r+1}\geq 0.\}$

$$f: A\to B, (a_1,\cdots,a_r)\mapsto(b_1,,\cdots,b_{r+1})$$

By

$$b_1=a_1-1\geq 0.$$

$$b_i=a_i-a_{i-1}\geq k+1, i=2,\cdots,r.$$

$$b_{r+1}=n-a_r\geq 0.$$

Easy to check f is a bijection.And construct a function.

$$g: B\to C$$

$C=\{$integers set $c_1+\cdots+c_{r+1}=n-1-(k+1)(r-1), where\ c_1,\cdots,c_{r+1}\geq 0,\}$

$$and\ c_1=b_1, c_i=b_i-(k+1), i=2,\cdots,r, c_{r+1}=b_{r+1}.$$

Also,check g is a bijection.Hence,

$|A|=|B|=|C|=\binom{n-1-(k+1)(r-1)+(r+1)-1}{r}=\binom{n-k(r-1)}{r}.\square$

Arrangements with Repetition

Propostion8:$X=\{x_1,\cdots,x_n\}.B=\{$all vectors of length r over X s.t.x_i occurs a_i times.$\}$ Then,$|B|=\frac{r!}{a_1!a_2!\cdots a_n!}$.

proof:just double counting. $\square$

Corollary:(Polynomial Theorem)

$$(x_1+x_2+\cdots+x_n)^r=\sum_{a_1+a_2+\cdots+a_n=r}\frac{r!}{a_1!a_2!\cdots a_n!}{x_1}^{a_1}\cdots{x_n}^{a_n}.$$

proof:

$$(x_1+x_2+\cdots+x_n)^r=\sum_{1\leq i_1,i_2,\cdots,i_r\leq n}x_{i_1}x_{i_2}\cdots x_{i_r}$$

the coefficient of ${x_1}^{a_1}\cdots{x_n}^{a_n}$ is equal to the number of vectors $(i_1,\cdots,i_n)$ over $[n]$,s.t.i occur a_i times.by proposition8,we get this theorem. $\square$

Remark:

(1) if $n = 2, x_1 = a, x_2 = b$, then $(a+b)^r = \sum_{i=0}^{r} \binom{r}{i} a^i b^{r-i}$.

(2) if $x_1 = x_2 = \cdots = x_n = 1$ then $n^r = \sum_{a_1+a_2+\cdots+a_n=r} \frac{r!}{a_1!a_2!\cdots a_n!}$.

Partition:$X = R_1 \cup R_2 \cup \cdots \cup R_n$. there are two cases: unordered partition $\{R_1, \cdots, R_n\}$,and ordered partition $(R_1, \cdots, R_n)$.

Propostion9:$|X| = r, A =$\{ordered partitions of X into n parts s.t. ith block has size a_i\}. Then $|A| = \frac{r!}{a_1!a_2!\cdots a_n!}$.

proof:Let $X = x_1, \cdots, x_n$. Let B=\{all vectors of length r over $[n]$ s.t.i occurs a_i times. $i \in [n], \sum_{i=1}^{n} ia_i = r.$\}
define $f : A \to B, (R_1, \cdots, R_n) \mapsto (b_1, , \cdots, b_r)$. [by $b_i = j$ if $i \in R_j$] check that f is a bijection.then $|A| = |B| = \frac{r!}{a_1!a_2!\cdots a_n!}$. □

Propostion9:[Exercise] $|X| = r, S =$\{unordered partitions of X, s.t. there are k_i blocks of size $i, i \in [r], \sum_{i=1}^{r} ik_i = r$\}.Then,

$$|S| = \frac{r!}{(1!)^{k_1}(2!)^{k_2}\cdots(r!)^{k_r}k_1!k_2!\cdots k_r!}.$$

Stirling Number of 2nd kind.
Def:Let $S(r,n)$ be the number of partitions of $[r]$ into n unordered non-empty sets.

$$S(r,n) = \sum_{\substack{\sum_{i=1}^{r} k_i = n \\ \sum_{i=1}^{r} ik_i = r}} \frac{r!}{(1!)^{k_1}(2!)^{k_2}\cdots(r!)^{k_r}k_1!k_2!\cdots k_r!}.$$

[Exercise]$S(r,2) = \frac{1}{2}\sum_{i=1}^{r-1} \binom{r}{i}$.

Binomial Theorem $n \geq 0$, for all x and y.

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Thm.(Vandermonder's Indentity) $\forall m, n, r \geq 0$

(1) $\binom{m+n}{r} = \sum_{i=0}^{r} \binom{n}{i}\binom{m}{r-i} = \sum_{i+j=r} \binom{n}{i}\binom{m}{j}$.

(2) $\binom{m+n}{r+m} = \sum_{i-j=r} \binom{n}{i}\binom{m}{j}$.

<u>proof:</u>

(1)

$$(1+x)^{m+n} = (1+x)^m = (1+x)^n$$

$$\Rightarrow \sum_{r=0}^{m+n} \binom{m+n}{r} x^r = \sum_{i=0}^{n} \binom{n}{i} x^i \sum_{j=0}^{m} \binom{m}{j} x^j$$

compute coefficient of x^r for both side.

$$\binom{m+n}{r} = \sum_{i+j=r} \binom{n}{i}\binom{m}{j} = \sum_{i=0}^{r} \binom{n}{i}\binom{m}{r-i}.$$

(2)

$$\sum_{i-j=r} \binom{n}{i}\binom{m}{j} = \sum_{i-j=r} \binom{n}{i}\binom{m}{m-j} = \sum_{i+(m-j)=m+r} \binom{n}{i}\binom{m}{m-j} = \binom{m+n}{r+m}. \square$$

[Exercise]

(1) $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$.

(2) $\sum_{k=0}^{n} \binom{m}{k}\binom{n}{p+k} = \binom{n+m}{p+m}$.

(3) $\sum_{k=1}^{m} \binom{m}{k}\binom{n-1}{k-1} = \binom{n+m-1}{n}$.

(4) $\binom{n}{k}\binom{k}{m} = \binom{n}{m}\binom{n-m}{k-m} \quad (n \geq k \geq m)$.

(5) $\sum_{k=0}^{n} \binom{n}{k}\binom{k}{m} = \binom{n}{m} 2^{n-m}$.