Combinatorics 2017 Fall
week14 note

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Monochromatic

Thm: If k is sufficiently large, then there exists a $k-$uniform family $\mathscr{F}$ s.t.$|\mathscr{F}| \leq k^2 2^k$ and $\mathscr{F}$ is not $2-$colorable

proof:$r = \lfloor \frac{k^2}{2} \rfloor$, Let $\mathscr{F} = \{A_1, \cdots, A_b\}$ be a random $k-$uniform family over $[r]$,i.e. each A_i is chosen with probability $1/\begin{pmatrix} r \\ k \end{pmatrix}$, independently of other choice.Let B be the event that $\mathscr{F}$ is $2-$colorable. For each $2-$coloring $\chi : [r] \to \{red, blue\}$,let B_χ be the event that no subset $A_i \in \mathscr{F}$ is monochromatic under χ, then $B = \cup_{\chi \in 2^{[r]}} B_\chi$. Suppose χ colors a red points and $r - a$ blue points, Then

$$\begin{aligned}
&Pr[A_i \in \mathscr{F} \text{is monochromatic under} \chi] \\
&= Pr[A_i \text{ is red}] + Pr[A_i \text{ is blue}] \\
&= \frac{\begin{pmatrix} a \\ k \end{pmatrix} + \begin{pmatrix} r-a \\ k \end{pmatrix}}{\begin{pmatrix} r \\ k \end{pmatrix}} \\
&\overset{\text{Jensen's Inequality}}{\geq} \frac{2\begin{pmatrix} r/2 \\ k \end{pmatrix}}{\begin{pmatrix} r \\ k \end{pmatrix}} := p \\
&\approx e^{-1} 2^{1-k}.
\end{aligned}$$

since A_i are chosen independently,Then

$$
\begin{aligned}
Pr[B_\chi] &= Pr[\cap_{i=1}^{b}(A_i \text{ is not monochromatic})] \\
&= \prod_{i=1}^{b}(1 - Pr(A_i \text{ is monochromatic})) \leq (1-p)^b.
\end{aligned}
$$

Then $Pr[B] \leq \sum_{\chi \in 2^{[r]}} Pr[B_\chi] \leq 2^r(1-p)^b < e^{r\ln 2 - pb}$.

when $b = (r\ln 2)/p = (1+o(1))k^2 2^{k-2} e \ln 2$, $Pr[B] < 1$.
$\Rightarrow Pr[\mathscr{F} \text{ is not 2-colorable}] = 1 - Pr[B] > 0. \quad b \leq k^2 2^k.$ $\square$

Let $B(k)$ be the minimum possible number of sets in a $k-$uniform family which is not $2-$colorable. Then $2^{k-1} \leq B(k) \leq k^2 2^k$.

The linearity of expectation

- $E[X+Y] = E[X] + E[Y], \forall X, Y$
- $Pr[X \geq E[X]] > 0, Pr[X \leq E[X]] > 0$

Def:subset A of an addictive group is ***sum-free***,if $\forall x, y \in A, x+y \notin A (allow\ x = y)$. E.g.

- $A = \{n+1, n+2, \cdots, 2n\} \subset \mathbb{Z}$ is ***sum-free***.
- $A = \{\text{odd integers}\} \subset \mathbb{Z}$ is ***sum-free***.

Thm: For any set A of non-zero integers, there is a ***sum-free*** set $B \subset A$ with $|B| \geq |A|/3$.

proof: Let $p = 3k+2$ be a prime,s.t. $p > 2\max_{a\in A}|a|$

[such a prime exists by Dirichlet's prime number theorem:$\forall a, b\ s.t.\ (a,b) = 1, \exists$ infinity many primes of the form $a + nb$]

Let $S = \{k+1, k+2, \cdots 2k+1\}, |S| = k+1$. and S is ***sum-free*** in $\mathbb{Z}_p$, we proceed by reducing the original problem to $\mathbb{Z}_p$,For $x \in \mathbb{Z}_p^* = \mathbb{Z}_p \backslash \{0\}$. Let $A_x = \{a \in A : (ax \mod p) \in S\} \subset A$.

Note that A_x is ***sum-free*** in A, because $\forall a, b \in A_x, (ax \mod p) \in S, (bx \mod p) \in S, then\ (ax + bx \mod p) \notin S$ by ***sum-free*** of S.Then we want to find some $x \in \mathbb{Z}_p^*$. s.t.$|A_x| \geq |A|/3$. Choose

$x \in \mathbb{Z}_p^*$ uniformly at random, We compute $E[|A_x|]$.

$\forall a \in A$, Let $|A_{a,x}|$ be a random variable $\mathbb{Z}_p^* \to \{0,1\}$, where $|A_{a,x}|$ maps x to 1 if $(ax \mod p) \in S$, and 0 otherwise.
Note that $|A_x| = \sum_{a\in A} |A_{a,x}|$, by linearity of expectation, $E[|A_x|] = \sum_{a\in A} E[|A_{a,x}|] = \sum_{a\in A} Pr[(ax \mod p) \in S]$.
observe that for fixed $a \in A$,running over all $x \in \mathbb{Z}_p^*$, then $(ax \mod p)$ also run over all $\mathbb{Z}_p^*$, so $Pr[(ax \mod p) \in S] = \dfrac{|S|}{p-1} > \frac{1}{3}$.
Then $E[|A_x|] = \sum_{a\in A} Pr[(ax \mod p) \in S] > \dfrac{|A|}{3}$.
Then, there must exists some $x \in \mathbb{Z}_p^*, s.t. |A_x| \geq E[|A_x|] > \dfrac{|A|}{3}$. where A_x is ***sum-free***. $\square$

<u>**Def:**</u> A dominating set of vertices in a graph $G = (V, E)$ is a subset $A \subset V(G)$ such that every $v \in V \backslash A$ has a neighbor in A.

<u>**Thm:**</u>Let $G = (V, E)$ be a graph on n vertices and with minimum degree $\delta > 1$.Then G contains a dominating set of at most $\dfrac{1 + \ln(1+\delta)}{1+\delta} n$ vertices.

<u>**proof:**</u>For $p \in (0, 1)$, which will be determined later, We pick each vertex in $V(G)$ with probability p uniformly at random. Let X be the random set of vertices picked. Let Y be the random set of vertices $y \in V \backslash X$, which has no neighbors in X. That is, $y \in Y$ iff y is not picked and al neighbors of y are not picked. So $P(y \in Y) = (1-p)^{1+deg(y)} \leq (1-p)^{1+\delta} \leq e^{-p(1+\delta)}$.

for any fixed $y \in V$ Let $I_{\{y\in Y\}}$ be a random variable, $2^{[v]} \to \{0, 1\}$, which maps Y to 1 if $y \in Y$, and 0 otherwise. i.e.

$$I_{\{y\in Y\}} = \begin{cases} 1 & \text{event: } y \in Y. \text{ occur} \\ 0 & \text{event: } y \notin Y. \text{ occur} \end{cases}$$

Then $|Y| = \sum_{y\in V} I_{\{y\in Y\}}$,by linearity of expectation.

$E[|Y|] = \sum_{y \in Y} E[|I_{\{y \in Y\}}|] = \sum_{y \in Y} Pr[y \in Y] \leq n \cdot e^{-p(1+\delta)}$.

Claim: $X \cup Y$ is a dominating set of G. Why?(Exercise)

Since $E[|X \cup Y|] = E[|X|] + E[|Y|] \leq np + e^{-p(1+\delta)} = n(p + e^{-p(1+\delta)})$.

check: when $p = \frac{\ln(1+\delta)}{1+\delta}$ to get $E[|X \cup Y|] \leq \frac{1+\ln(1+\delta)}{1+\delta} n$.

Then there exists a dominating set of size at most $\frac{1+\ln(1+\delta)}{1+\delta} n$.$\square$

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Recall:$\alpha(G) = \max|I|$ over all independent set $I \subset V$.

Thm: For any graph $G, \alpha(G) \geq \sum_{v\in V} \frac{1}{1+deg(v)}$.

Thm: Let $V(G) = [n]$. For $i \in [n]$, Let N_i be the neighborhoods of i in G, i.e. $N_i = \{j \in [n] : j \sim i\}$, let S_n be the symmetric group over $[n]$.

For given $\pi \in S_n$ we say a vertex $i \in [n]$ is $\pi-$dominating, if $\pi(i) < \pi(j)$ for all $j \in N_i$. let M_π ={all $\pi-$dominating vertices.}

Claim:$\forall \pi \in S_n, M_\pi$ is a independent set.

proof of claim: Suppose not,then $\exists i, j \in M_\pi$ with $i \sim j$. let $\pi(i) < \pi(j) \Rightarrow j$ is not $\pi-$dominating, i.e. $j \notin M_\pi$, a contradiction.

Pick an $\pi \in S_n$ uniformly at random, compute $E[|M_\pi|]$.

For any fixed $i \in [n]$, let $I_{\{i \text{ is } \pi-\text{dominating}\}}$ be a random variable, $S_n \to \{0,1\}$, which maps π to 1 if i is $\pi-$dominating. and 0 otherwise. i.e.

$$I_{\{i \text{ is } \pi-\text{dominating}\}} = \begin{cases} 1 & \text{event: } i \text{ is } \pi - \text{dominating. occur} \\ 0 & \text{event: } i \text{ is not } \pi - \text{dominating. occur} \end{cases}$$

Then $|M_\pi| = \sum_{i\in[n]} I_{\{i \text{ is } \pi-\text{dominating}\}}$,By linearity of expectation.

$$\begin{aligned} E[|M_\pi|] &= \sum_{i\in[n]} E[|I_{\{i \text{ is } \pi-\text{dominating}\}}] \\ &= \sum_{i\in[n]} Pr[i \text{ is } \pi - \text{dominating}] \end{aligned}$$

By definition, i is $\pi-$dominating iff $\pi(i)$ is minimum over $\pi(\{i\} \cup N_i)$ Since π is random,every vertex in $\{i\} \cup N_i$ has the equal probability

to achieve the minimum in π, which is $\frac{1}{1+deg(i)}$. Thus
$E[|M_\pi|] = \sum_{i\in[n]} \frac{1}{1+deg(i)} = \sum_{v\in V} \frac{1}{1+deg(v)}$.

proof of $Pr[i \text{ is } \pi - \text{dominating}] = \frac{\#\{\pi : i \text{ is } \pi - \text{dominating}\}}{|S_n|} = \frac{\binom{n}{deg(i)+1} \cdot deg(i)! \cdot (n-deg(i)-1)!}{n!} = \frac{1}{deg(i)+1}$. $\square$

Def: A biclique covering of a graph G is a set $H_1, \cdots, H_t$ of its biclique subgraphs such that each edge of G belongs to at least one of those subgraphs. The weight of such a covering is $\sum_{i=1}^{t} |V(H_i)|$. Let $bc(G)$ be the smallest weight of a biclique covering of G.

Thm:

$$bc(K_n) \geq n \log_2 n$$

proof: $K_{A_i,B_i}, i \in [t]$ be a covering of K_n, For $v \in V(K_n)$,Let m_v be the number of these cliques containing v. By double-counting, $\sum_{i=1}^{t}(|A_i|+|B_i|) = \sum_{i=1}^{t} m_v$

So,it is enough to show that $\sum_{i=1}^{t} m_v \geq n \log_2 n$.
To do this, throw a fair $0-1$ coin for each $i \in [t]$, if outcome is 0 remove all vertices in A_i from K_n, else remove all vertices in B_i from K_n.

Let X_v be the indicator variable for the event "vertex survives". and $X = X_1 + \cdots + X_n$, which is the number of vertices survive at the end. since any two vertices are adjacent in K_n, and each edge occurs in at least one biclique, we can have at most one vertex survive at the end,i.e. $E[X] \leq 1$.

Further $Pr[v \text{ survives}] = \left(\frac{1}{2}\right)^{m_v}$, By linearity expectation $1 \geq E[X] = \sum\limits_{v=1}^{n} E[X_v] = \sum\limits_{v=1}^{n} Pr[v \text{ survives}] = \sum\limits_{v=1}^{n} \left(\frac{1}{2}\right)^{m_v}$.

By inequality $\frac{1}{n}\sum\limits_{v=1}^{n} a_v \geq (\prod\limits_{v=1}^{n} a_v)^{\frac{1}{n}}$, let $a_v = 2^{-m_v}$,we have

$$\frac{1}{n} \geq \frac{1}{n}\sum_{v=1}^{n} 2^{-m_v} \geq (\prod_{v=1}^{n} 2^{-m_v})^{\frac{1}{n}} = 2^{-\sum_{v=1}^{n} m_v/n}$$

Then $2^{\sum\limits_{v=1}^{n} m_v/n} \geq n$, hence $\sum\limits_{v=1}^{n} m_v \geq n\log_2 n$. $\square$

Ex:Show $bc(K_n) = n\log_2 n$. if n is a power of two.

The deleting method

idea:A random structure doesn't always have the directed property, and may have some very few "blemishes". After deleting all blemishes, we will obtain the wanted structure.

Recall: For any G, $\alpha(G) \geq \sum\limits_{v\in V} \frac{1}{1+deg(v)}$.

Corollary:$\forall G$ with m edges and n vertices, then $\alpha(G) \geq \frac{n^2}{2m+n}$.
If $m = nd/2$ where d is average degree, then $\alpha(G) \geq \frac{n}{1+d}$.

proof: By double-counting $\sum\limits_{v\in V} deg(v) = 2m$, then $\sum\limits_{v\in V} deg(v) + 1 = 2m + n$.

By the fact that harmonic mean is less than arithmetic mean, i.e. $\frac{n}{\sum\limits_{i=1}^{n} 1/x_i} \leq \sum\limits_{i=1}^{n} x^i/n$. we have

$$\alpha(G) \geq \sum_{v\in V} \frac{1}{1+deg(v)} \geq \frac{n^2}{\sum_{v\in V} deg(v)+1} = \frac{n^2}{2m+n}. \qquad \square$$

Next, we show the half-way of the previous result by a short argument.

Thm: Let G be a graph on n vertices and with average degree d. Then $\alpha(G) \geq \frac{n}{2d}$.

proof: Let $S \subset V(G)$ be a random set, where for $\forall v \in V, Pr(v \in S) = p$ and value of p will be determined later. Let $X = |S|$ and $Y = \#$edges in S. Then $E[X] = np, E[Y] = |E(G)| \cdot p^2 = \frac{nd}{2}p^2$. Then $E[X-Y] = np - \frac{nd}{2}p^2 = n(p - \frac{d}{2}p^2)$.

By choosing $p = \frac{1}{d}$, we have $E[X-Y] = \frac{n}{2d}$ which is maximum.
So there is a particular set S such that $|S| - |Y| \geq E[X-Y] = \frac{n}{2d}$.

Now,deleting one vertex form each edge of S. leaving a set $S\prime$. This set $S\prime$ is independent and has at least $\frac{n}{2d}$ vertices. $\square$

Remark: If let $E[Y] < 1, i.e. p < \sqrt{\frac{2}{nd}}$ we can get another bound. $E[X] < n \cdot \sqrt{\frac{2}{nd}} \Rightarrow \alpha(G) \geq n \cdot \sqrt{\frac{2}{nd+1}}$.
Recall: If $\binom{n}{k} 2^{1-\binom{k}{2}} < 1$, then $R(k,k) > \frac{1}{e\sqrt{2}} k 2^{k/2}$.

Thm: $\forall n, R(k,k) > n - \binom{n}{k} 2^{1-\binom{k}{2}}$

proof: Consider a random $2-$edge-coloring of K_n, where each edge is colored by red or blue with probability $\frac{1}{2}$, independent of other choices. For $A \in \binom{[n]}{k}$, Let X_A be the indicator random variable of the event that A is monochromatic.

Let $X = \sum_{A \in \binom{[n]}{k}} E[X_A] = \binom{n}{k} 2^{1-\binom{k}{2}}$.

Then there exists a $2-$edge-coloring of K_n, s.t. $X \leq E[X] = \binom{n}{k} 2^{1-\binom{k}{2}}$. Fix such a coloring, remove one vertex from each monochromatic $k-$subset. This will delete $X \leq \binom{n}{k} 2^{1-\binom{k}{2}}$ vertices, which has No monochromatic K_k. So $R(k,k) > n - \binom{n}{k} 2^{1-\binom{k}{2}}$. $\square$