Combinatorics 2017 Fall

week14 note

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Reference:

Extremal Combinatorics with applications in Computer Science. 2nd Edition.Stasys Jukna,Springer.

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Theorem1 Let G be a graph on n vertices such that every vertex has at least one neighbor and at most d non-neighbors. Then $cc(G) = O(d^2 \ln n)$.

Proof: consider the following way for choosing a clique of $G(V, E)$. First, pick every vertex $v \in V$ independently with probability $p = \frac{1}{d+1}$, to get a set $W \subset V$. Then remove from W all vertices having at least one non-neighbors in W. Then we get a clique of G. Applying this way independently t times to get t cliques $H_1, \cdots, H_t$ of G, where t is to be determined later. Let X be the number of edges of G not covered by any H_i, and for $\forall$ edge $u\,w \in E(G)$, let X_{uw} be the indicator random variable of the event "none of H_i covers uw". Then $X = \sum_{u\,w} X_{uw}$. Note that if H_i covers uw, if both uw are chosen and none of their$\leq$neighbors are chosen. Hence

$$\begin{aligned} Pr[H_i\ cover\ uw] &\geq p^2(1-p)^{2d} = \frac{1}{(1+d)^2}(1 - \frac{1}{1+d})^{2d} \\ &\geq \frac{1}{(1+d)^2}[e^{-\frac{1}{1+d} - \frac{1}{2(1+d)^2}}]^{2d} \geq \frac{1}{e^2(1+d)^2} \end{aligned}$$

Since H_i are chosen independently.

$$Pr[none\ of\ H_i\ cover\ uw] = \prod_{i\in[t]} Pr[H_i\ does\ not\ cover\ uw] \leq (1 - \frac{1}{e^2(1+d)^2})^t \leq e^{-\frac{t}{e^2(1+d)^2}}$$

Then by linearity of expectation.

$$E[X] = \sum_{u\,w} E[X_{uw}] = \sum_{u\,w} Pr[none\ of\ H_i\ cover\ uw] \leq \binom{n}{2} e^{-\frac{t}{e^2(1+d)^2}}$$

Take $t = \lfloor cd^2 \ln n \rfloor$ for sufficiently large constant c, we have $E[X] < 1$. Hence, there is at least one choice of t cliques that form a covering of G and $cc(G) \leq t$. $\square$

Markov's Inequality

Let $X \geq 0$ be a random variable and $t > 0$, then $p(X \geq t) \leq \frac{E[X]}{t}$.

Proof: Let $I_{\{X \geq a\}}$ be the indicator random variable which have value 1 if $X \geq a$ and 0 otherwise. Then $aI_{\{X\geq a\}} \leq X$. In fact if $X < a, I_{\{X\geq a\}} = 0$ and so $aI_{\{X\geq a\}} = 0 \leq X$. If $X \geq a, I_{\{X\geq a\}} = 1$ and so $aI_{\{X\geq a\}} = a \leq X$. Since E is a monotonically increasing function, taking expectation of both sides, we have $E[aI_{\{X\geq a\}}] \leq E[X]$. Since $E[aI_{\{X\geq a\}}] = aE[I_{\{X\geq a\}}] = aPr[X \geq a] \leq E[X]$. $\square$

Definition: The random graph $G(n,p)$ for $0 \leq p \leq 1$ is a graph with vertex set [n], where each of potential $\binom{n}{2}$ edges appears with probability p independent of other edges.

Corollary: Let $X_n \geq 0$ be integer value random variable in $(\Omega_n, P_n), n \in \mathbb{Z}_{\geq 0}$. If $E[X_n] \longrightarrow 0$ as $n \longrightarrow +\infty$, then $Pr[X_n = 0] \longrightarrow 1$ as $n \longrightarrow +\infty$, we say $X_n = 0$ almost surely occur.

Proof: Let $t = 1, Pr[X_n \geq 1] \leq \frac{E[X_n]}{1} \longrightarrow 0$ as $n \longrightarrow +\infty$. $\square$

Theorem For a random graph $G(n,p)$ for some fixed $p \in (0,1)$. then $Pr[\alpha(G) \leq \left\lceil \frac{2\ln n}{p} \right\rceil] \longrightarrow 1$ as $n \longrightarrow \infty$.

Proof: Let $k = \left\lceil \frac{2\ln n}{p} \right\rceil$. For $\forall S \in \binom{[n]}{k+1}$, Let A_S be the event that S is an independent set, Let $X_n = \sum_{S \in \binom{[n]}{k+1}} I_{A_S}$ be the number of independent set of size $k+1$, here I_{A_S} be the indicator random variable of the event "A_S occurs". Then

$$\begin{aligned} E[X_n] &= \sum_{S \in \binom{[n]}{k+1}} E[I_{A_S}] = \sum_{S \in \binom{[n]}{k+1}} Pr[A_S] \\ &= \binom{n}{k+1}(1-p)^{\binom{k+1}{2}} \leq \frac{n^{k+1}}{(k+1)!} e^{-p\binom{k+1}{2}} \\ &= \frac{1}{(k+1)!}(ne^{-p\frac{k}{2}})^{k+1} \leq \frac{1}{(k+1)!} \longrightarrow 0 \end{aligned}$$

Then $Pr[X_n = 0] \longrightarrow 1$ as $n \longrightarrow \infty \Longrightarrow Pr[\alpha(G) \leq \left\lceil \frac{2\ln n}{p} \right\rceil] \longrightarrow 1$. $\square$

Definition: The girth of G denoted by $g(G)$ is the length of a shortest cycle of G.

Recall: $\chi(G)\alpha(G) \geq n = V(G)$.

Theorem For any $k \in N^+$, there exists a graph G with $\chi(G) \geq k$ and $g(G) \geq k$.

Proof: Consider $G(n,p)$ where p will be determined later. let $X =$ be the number cycles of length less than k in G, and X_i be the number of cycles of length $i, i \in [k-1]$. Then

$$E[X] = \sum_{i=3}^{k-1} E[X_i] = \sum_{i=3}^{k-1} \frac{n(n-1)\cdots(n-i+1)}{2i} p^i.$$

where $\frac{n(n-1)\cdots(n-i+1)}{2i}$ is the number of of cycles of length i in K_n. Then $E[X] \leq \sum_{i=3}^{k-1} \frac{n^i p^i}{2i} \leq$

$\sum_{i=0}^{k-1}(np)^i = \frac{(np)^k-1}{np-1}$. By Markov's Inequality

$$Pr[X \geq \frac{n}{2}] \leq \frac{E[X]}{n/2} \leq \frac{2[(np)^k-1]}{n(np-1)}.$$

Let $p = n^{-\frac{k-1}{k}}, \Longrightarrow Pr[X \geq \frac{n}{2}] \leq \frac{2(n-1)}{n(n^{\frac{1}{k}}-1)} \longrightarrow 0$ as $n \longrightarrow +\infty$. Let $t = \left\lceil \frac{2\ln n}{p} \right\rceil \leq 3\ln n \cdot n^{\frac{k-1}{k}}$, recall $\alpha(G) \leq t$ almost surely. Then $\exists G$ on n vertices, $\alpha(G) \leq t$ and with$\leq \frac{n}{2}$ cycles of length less than k. By deleting from each cycle of length less than k, we have a graph $G' \subset G$, with No cycles of length less than k, and $|V(G')| \geq n - \frac{n}{2} = \frac{n}{2}.\alpha(G') \leq \alpha(G) \leq 3\ln n \cdot n^{\frac{k-1}{k}}$. So $\chi(G') \geq \frac{|V(G')|}{\alpha(G')} \geq \frac{n/2}{3\ln n \cdot n^{\frac{k-1}{k}}} = \frac{n^{\frac{1}{k}}}{6\ln n} >> k$, where n is sufficiently large. $\square$

2017/12/29

The Lovasz Sieve

Recall union bound $Pr[A \cup B] \leq Pr[A] + Pr[B]$. Let $B_1, \cdots, B_m$ be bad events, and $A = \overline{B_1} \cap \cdots \cap \overline{B_m}$, then

$$Pr[A] = Pr[\overline{B_1} \cap \cdots \cap \overline{B_m}] = Pr[\overline{B_1 \cup \cdots \cup B_m}] = 1 - Pr[B_1 \cup \cdots \cup B_m] \geq 1 - \sum_{i=1}^{m} Pr[B_i].$$

- If $\sum_{i=1}^{m} Pr[B_i] < 1$, then $Pr[A] > 0$.
- If $B_1, \cdots, B_m$ are independent, then

$$Pr[A] = Pr[\overline{B_1} \cap \cdots \cap \overline{B_m}] = \prod_{i=1}^{m} Pr[\overline{B_i}] = \prod_{i=1}^{m}(1 - Pr[B_i])$$

  So if $Pr[B_i] < 1, i \in [m]$, then $Pr[A] > 0$.

- What if some B_i, B_j are not independent?

  Conditional probability: $Pr[A|B] = Pr[A \cap B]/Pr[B]$.

  A and B are independent iff $Pr[A \cap B] = Pr[A]Pr[B] \Longleftrightarrow Pr[A|B] = Pr[A]$.

$$Pr[A|(B \cap C)] = \frac{Pr[A \cap B \cap C]}{Pr[B \cap C]} = \frac{Pr[A \cap B \cap C]}{Pr[C]} / \frac{Pr[B \cap C]}{Pr[C]} = Pr[(A \cap B)|C]/Pr[B|C].$$

$$Pr[A|(B \cap C)] \cdot Pr[B|C] \cdot Pr[C] = \frac{Pr[A \cap B \cap C]}{Pr[B \cap C]} \cdot \frac{Pr[B \cap C]}{Pr[C]} \cdot Pr[C] = Pr[A \cap B \cap C].$$

Definition: An event A is mutually independent of events $B_1, \cdots, B_m$ if $Pr[A|(C_1 \cap \cdots \cap C_m)] = Pr[A], C_i \in \{B_i, \overline{B_i}\}, i \in [m]$.

Note: The fact A is mutually independent with each of $B_1, \cdots, B_m$ does not mean that A is mutually independent of $B_1, \cdots, B_m$.

Eg:consider flipping a fair coin twice, let $B_i, i = 1, 2$ be the event that the i-th flip is a head, and let A be the event that both flips are the same. Then A is independent with each B_i. But $Pr[A|B_1 \cap B_2] = 1$.

Definition: Let $A_1, \cdots, A_n$ be events, $G = ([n], E)$ is called a dependency graph if $\forall i, A_i$ is mutually independent of all events A_j with $j \in [n]\backslash\{i\}$ and $\{i, j\} \notin E$. The smallest degree of among all such graphs G is called the degree of dependency of $A_1, \cdots, A_n$.

Lemma: Let $A_1, \cdots, A_n$ be events with $Pr[A_i] \leq p < 1, \forall i$, and let d be the degree of dependency. If $4pd \leq 1$, then $Pr[\overline{A_1} \cap \cdots \cap \overline{A_n}] > 0$.

Proof: Fix a dependency graph G of $A_1, \cdots, A_n$ of degree d. Prove by induction on m or$S, S \subset [n], |S| = s < n, \forall i \notin S, Pr[A_i|\cap_{i \in S} \overline{A_j}] \leq 2p$ that ($Claim$)for any m events of $A_1, \cdots, A_n$, say $A_1, \cdots, A_m$ for convenience, $Pr[A_1|\overline{A_2} \cap \cdots \cap \overline{A_m}] \leq 2p$. $m = 1$ is true. Let $2, \cdots, k \in [2, m]$ which are adjacent to 1 in G. Then $Pr[A_1|\overline{A_2} \cap \cdots \cap \overline{A_m}] = \frac{Pr[(A_1 \cap \overline{A_2} \cap \cdots \cap \overline{A_k})|(\overline{A_{k+1}} \cap \cdots \cap \overline{A_m})]}{Pr[(\overline{A_2} \cap \cdots \cap \overline{A_k})|(\overline{A_{k+1}} \cap \cdots \cap \overline{A_m})]}$. Since A_1 is mutually independent of $A_{k+1}, \cdots, A_m$, the numerator

$$Pr[(A_1 \cap \overline{A_2} \cap \cdots \cap \overline{A_k})|(\overline{A_{k+1}} \cap \cdots \cap \overline{A_m})] \leq Pr[A_1|(\overline{A_{k+1}} \cap \cdots \cap \overline{A_m})] = Pr[A_1] \leq p.$$

The denominator

$$\begin{aligned}
&Pr[(\overline{A_2} \cap \cdots \cap \overline{A_k})|(\overline{A_{k+1}} \cap \cdots \cap \overline{A_m})] \\
=&1 - Pr[(A_2 \cup \cdots \cup A_k)|(\overline{A_{k+1}} \cap \cdots \cap \overline{A_m})] \\
\geq&1 - \sum_{i=2}^{k} Pr[A_i|(\overline{A_{k+1}} \cap \cdots \cap \overline{A_m})] \\
\geq&1 - (k-1)2p \geq \frac{1}{2}
\end{aligned}$$

(since $k - 1 \leq d, 2pd \leq \frac{1}{2}$)

Then $Pr[A_1|\overline{A_2} \cap \cdots \cap \overline{A_m}] \leq p/(\frac{1}{2}) = 2p$. Finally,

$$Pr[\overline{A_1} \cap \cdots \cap \overline{A_n}] = \prod_{i=1}^{n} Pr[\overline{A_i}|(\overline{A_{i+1}} \cap \cdots \cap \overline{A_n})] = \prod_{i=1}^{n}(1 - Pr[A_i|(\overline{A_{i+1}} \cap \cdots \cap \overline{A_n})]) \geq (1-2p)^n > 0,$$

when $A_i's$ probability are very different. □

Lemma: Let $G = (V, E)$ be a dependency graph of event $A_1, \cdots, A_n$. Suppose there exist real numbers $x_1, \cdots, x_n, 0 \leq x_i \leq 1$, so that for all $i, Pr[A_i] = x_i \cdot \prod_{\{i,j\} \in E}(1 - x_j)$. Then $Pr[\overline{A_1} \cap \cdots \cap \overline{A_n}] \geq \prod_{i=1}^{n}(1 - x_i) > 0$.

Proof: Similar to the above lemma. Replace the claim by for any m events $Pr[A_1|\overline{A_2} \cap \cdots \cap \overline{A_m}] \leq x_1$, we also prove it by induction on m. Using same notations, we have

$$Pr[A_1|\overline{A_2} \cap \cdots \cap \overline{A_m}] = \frac{Pr[(A_1 \cap \overline{A_2} \cap \cdots \cap \overline{A_k})|(\overline{A_{k+1}} \cap \cdots \cap \overline{A_m})]}{\prod_{j=2}^{k} Pr[\overline{A_j}|(\overline{A_{j+1}} \cap \cdots \cap \overline{A_m})]}$$

the numerator $Pr[(A_1 \cap \overline{A_2} \cap \cdots \cap \overline{A_k})|(\overline{A_{k+1}} \cap \cdots \cap \overline{A_m})] \leq Pr[A_1] \leq x_1 \cdot \prod_{\{1,j\} \in E}(1 - x_j)$.

the denominator

$$\prod_{j=2}^{k} Pr[\overline{A_j}|(\overline{A_{j+1}}\cap\cdots\cap\overline{A_m})] = \prod_{j=2}^{k}(1-Pr[A_j|(\overline{A_{j+1}}\cap\cdots\cap\overline{A_m})]) \geq \prod_{j=2}^{k}(1-x_j) \geq \prod_{\{1,j\}\in E}(1-x_j)$$

$\Longrightarrow Pr[A_1|(\overline{A_2}\cap\cdots\cap\overline{A_m})] \leq x_1$. Then $Pr[\overline{A_1}\cap\cdots\cap\overline{A_m}] \geq \prod_{i=1}^{n}(1-x_i)$. □

Corollary: Let $A_1,\cdots,A_n$ be events with $Pr[A_i]\leq p$, degree of dependency is d. If $ep(d+1)\leq 1$, then $Pr[\overline{A_1}\cap\cdots\cap\overline{A_n}]>0$.

Proof: Let $x_i=\frac{1}{d+1}$, then

$$x_i(1-x_i)^d = \frac{1}{d+1}(1-\frac{1}{d+1})^d > \frac{1}{d+1}\frac{1}{e} \geq p \geq Pr[A_i].$$

Since $\ln(1+t)=t-\frac{t^2}{2}+\frac{t^3}{3}-\cdots, -1<t\leq 1$. we have

$$\begin{cases} 1-t>e^{-t-\frac{t^2}{2}}, if\ 0<t<1; \\ 1-t<e^{-t-\frac{t^2}{2}}, if\ -1<t<0. \end{cases}$$

Then $(1-\frac{1}{d+1})^d > e^{-d(\frac{1}{d+1}+\frac{1}{2(d+1)^2})} > e^{-1}$. Or since $(1-\frac{1}{d+1})^d = (\frac{d}{d+1})^d = \frac{1}{(1+|frac1d)^d} > \frac{1}{e}$. □

Recall: Let $A_1,\cdots,A_n$ be events, $Pr[A_i]=p, i\in[n]$. If all events are mutually independent, then $Pr[\cup_{i=1}^n A_i] = 1-Pr[\cap_{i=1}^n A_i] = 1-(1-p)^n \geq 1-e^{-pn}$. But if not, we have two more cases.

(1) k-wise independent, i.e. $Pr[\cap_{i\in I}A_i]=p^{|I|}$ for all $|I|\leq k$.

(2) k-wise independent, i.e. let $\delta<1$. $|Pr[\cap_{i\in I}A_i]-p^{|I|}|\leq\delta$ for all $|I|\leq k$.

Recall Inclusion-Exclusion Principle: $|A_1\cup A_2\cup\cdots\cup A_n| = \sum_{\emptyset\neq I\subset[n]}(-1)^{|I|+1}|A_I|$.

Lemma Bonferroni inequality, for each even $k\geq 2$,

$$\sum_{v=1}^{k}(-1)^{v+1}\sum_{|I|=v}A_I \leq |\cup_{i=1}^n A_i| \leq \sum_{v=1}^{k+1}(-1)^{v+1}\sum_{|I|=v}A_I.$$

Proof: By induction on n, EX.