Combinatorics 2017 Fall

week2 note

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Reference:

Extremal Combinatorics with applications in Computer Science. 2nd Edition.Stasys Jukna,Springer.

2017/09/19

Estimating

Theorem 1 For $\forall n \leq 1,\ e(\frac{n}{e}) \leq n! \leq en(\frac{n}{e})^n$.

Proof: consider $\int_1^n lnxdx$,

$$ln(n-1)! = \sum_{i=1}^{n-1} lni \leq \int_1^n lnx \leq \sum_{i=1}^{n} lni = lnn!.$$

$$\Longrightarrow ln(n-1)! \leq nlnn - n + 1 \leq lnn!.$$

raise e to the power of each side,

$$(n-1)! \leq e^{nlnn-n+1} \leq n!.$$

where $e^{nlnn-n+1} = (e^{lnn})^n e^{-n} e = \frac{n^n}{e^n} e$. □

Stirling formula: $n! \sim \sqrt{2\pi n}(\frac{n}{e})^n.\ wheref(n) \sim g(n)means \lim_{n\to\infty} \frac{f(n)}{g(n)} = 1.$

Fact: $max\{\binom{n}{k} : k = 0, 1, 2 \cdots n\} = \begin{cases} \binom{n}{\frac{n}{2}}, if\ n\ is\ even; \\ \binom{n}{\lfloor \frac{n}{2} \rfloor} = \binom{n}{\lceil \frac{n}{2} \rceil}, if\ n\ is\ odd. \end{cases}$

Corollary: $\frac{2^n}{n+1} \leq \binom{n}{\lfloor \frac{n}{2} \rfloor} \leq 2^n.$

Stirling approximation: $\binom{n}{\lfloor \frac{n}{2} \rfloor} \sim \frac{2^n}{\sqrt{n}}\sqrt{\frac{2}{\pi}}.$

Theorem 2: For$1 \leq k \leq n, (\frac{n}{k})^k \leq (\frac{n}{k}) \leq (\frac{en}{k})^k$.

Proof: For lower bound, use the fact $\frac{n}{k} \leq \frac{n-i}{k-1}, \forall i \leq k$,

$$(\frac{n}{k})^k \leq \frac{n}{k} \cdot \frac{n-1}{k-1} \cdot \cdots \cdot \frac{n-k+1}{1} = \binom{n}{k},$$

For upper bound, use the Taylor series of exponential function, $e^t > 1+t, 0<t<1$. Then

$$\binom{n}{k} \le \sum_{i=0}^{k} \binom{n}{i} \le \sum_{i=0}^{k} \binom{n}{i} \frac{t^i}{t^k} = \frac{(1+t)^n}{t^k} (Binomial\ theorem).$$

Let $t = \frac{k}{n}$,

$$\frac{(1+t)^n}{t^k} = \frac{(1+\frac{k}{n})^n}{(\frac{k}{n})^k} \le \frac{(e^{\frac{k}{n}})^n}{(\frac{k}{n})^k}.$$

□

Inclusion-exclusion Principle

Let $A_1,\ A_2,\ \cdots, A_n$ be subset of X. $I \subset [n],\ A_I := \cap_{i\in I} A_i$. $A_\emptyset = X$.

Theorem 3(IEP): $|A_1^c \cap A_2^c \cap \cdots \cap A_n^c| = \sum\limits_{I\subseteq[n]} (-1)^{|I|}|A_I|$.

Proof: rewrite the right hand side

$$\sum_{I\subseteq[n]} (-1)^{|I|}|A_I| = \sum_{I\subseteq[n]} (-1)^{|I|} \sum_{x\in A_I} \cdot 1 = \sum_{x\in X} \sum_{I\subseteq[n]:x\in A_I} (-1)^{|I|}.$$

Left$=\sum\limits_{x\in X} \delta_x$, where $\delta_x = \begin{cases} 0, if\ x \in A_1 \cup A_2 \cup \cdots \cup A_n; \\ 1, otherwise. \end{cases}$

Consider the contribution of x to both sides.

For the right hand side, when $x \notin A_1 \cup A_2 \cup \cdots \cup A_n$, $\sum\limits_{I:x\in A_I} (-1)^{|\emptyset|} = 1$;

When $x \in A_1 \cup A_2 \cup \cdots \cup A_n$, Let $J = \{j:\ x\in A_j\}$,

$$\sum_{I:x\in A_I} (-1)^{|I|} = \sum_{I\subseteq J} (-1)^{|I|} = \sum_{i=0}^{|J|} \binom{|J|}{i} (-1)^i = (1-1)^{|J|} = 0$$

by the binomial theorem. □

Theorem 4: $|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum\limits_{\emptyset\neq I\subseteq[n]} (-1)^{|I|+1}|A_I|$.

Proof: Left$= |X| - |A_1^c \cap A_2^c \cap \cdots \cap A_n^c| = |A_\emptyset| - \sum\limits_{I\subseteq[n]} (-1)^{|I|}|A_I|$,

as desired. □

Applications

A bijection $\sigma:\ [n] \longrightarrow [n]$ is called derangement if $\sigma(i) \neq i,\ \forall i$.

Proposition 1: $\sharp$ derangement of $[n]$ is $n! \sum\limits_{k=0}^{n} \frac{(-1)^k}{k!}$.

Proof: Ground set=$\{$permutation of $[n]\}$, let A_i=$\{$permutation with $\sigma(i) = i\}$, $i \in [n]$. Then $|A_i| = (n-1)!$, $|A_I| = (n-|I|)!$. The number of derangement is

$$\sum_{I\subseteq[n]} (-1)^{|I|}|A_I| = \sum_{I\subseteq[n]} (-1)^{|I|}(n-|I|)! = \sum_{i=0}^{n} \binom{n}{i} (-1)^i (n-i)! = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!}$$

putting $i = |I|$. □

Definition: $\varphi(n):\ \sharp m \in [n],\ s.t.\ gcd(m,n) = 1$.

Proposition 2: If $n = p_1^{a_1} p_2^{a_2} \cdots p_t^{a_t}$, where p_i are distinct primes, then $\varphi(n) = n \prod\limits_{i=1}^{t}(1 - \frac{1}{p_i})$.

Proof: Ground set$= [n]$, let $A_i = \{m \in [n] :\ p_i | m\}$, then $A_i = \{p_i,\ 2p_i, \cdots, \frac{n}{p_i}p_i\}$. The rest of proof are left as an exercise.

[**Hint:** $\prod\limits_{i=1}^{t}(1 - \frac{1}{p_i}) = \sum\limits_{I \subseteq [n]} (-1)^{|I|} \frac{1}{\prod p_i}$.]

Averaging Principle: Every set of numbers must contain a number$\geq$average and number$\leq$average.

Proposition 3(Jensen's Inequality): If $0 \leq \lambda_i \leq 1$, $\sum\limits_{i=1}^{n} \lambda_i = 1$ and f is *convex* , then

$$f(\sum_{i=1}^{n} \lambda_i x_i) \leq \sum_{i=1}^{n} \lambda_i f(x_i).$$

Proof: Easy induction on the number of summands n. For $n = 2$ this is true, so assume the inequality holds for the number of summands up to n, and prove it for $n+1$. For this it is enough to replace the sum of the first two terms in $\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_{n+1} x_{n+1}$ by the term

$$(\lambda_1 + \lambda_2)(\frac{\lambda_1}{\lambda_1 + \lambda_2} x_1 + \frac{\lambda_2}{\lambda_1 + \lambda_2} x),$$

and apply the induction hypothesis. $\square$

Corollary 1(Cauchy-Schwarz inequality): $\sum\limits_{i=1}^{n} a_i^2 \geq \frac{1}{n}(\sum\limits_{i=1}^{n} a_i)^2,\ a_i \geq 0$.

Corollary 2:$a_i \geq 0,\ \frac{1}{n}\sum\limits_{i=1}^{n} a_i \geq (\prod\limits_{i=1}^{n} a_i)^{\frac{1}{n}}$.

Proof: $f(x) = 2^x,\ \lambda_i = \frac{1}{n},\ x_i = log_2 a_i$. $\square$