Combinatorics 2017 Fall

week2 note

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Reference:

Extremal Combinatorics with applications in Computer Science. 2nd Edition.Stasys Jukna,Springer.

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Generating function(GF)

Definition: The ordinary GF of an infinity sequence $a_0a_1\cdots$is a power series$f(x) = \sum\limits_{n\geq 0} a_nx^n$.

(1) When $\sum\limits_{n\geq 0} a_nx^n$ converges(i.e.$\exists$ a radius $R > 0$ of convergence). View GF as a function of x, and we can do operations of calculus on it including differentiation and integration. For example: $a_n = \frac{f^n(0)}{n!}$.

(2) Not sure about the convergence, view GF as a formal object with $+$ and $\cdot$. Let $a(x) = \sum\limits_{n\geq 0} a_nx^n$ and $b(x) = \sum\limits_{n\geq 0} b_nx^n$.

$$a(x) + b(x) = \sum_{n\geq 0}(a_n + b_n)x^n,$$

$$a(x)b(x) = \sum_{n\geq 0} c_nx^n,$$

where $c_n = \sum\limits_{n\geq 0} a_ib_{n-i}$.

A key GF: the GF of $\{1,\ 1,\ 1,\cdots\}$ is $\sum\limits_{n=0}^{\infty} x^n = \frac{1}{1-x},\ |x| < 1$.

Consequences:

- GF of $\{a^0,\ a^1,\ a^2,\cdots\}$ is $\sum\limits_{n=0}^{\infty} a^nx^n = \frac{1}{1-ax}$.
- GF of $\{1,\ -1,\ 1,\ -1\cdots\}$ is $\sum\limits_{n\geq 0}^{\infty}(-1)^nx^n = \frac{1}{1+x}$.

– GF of $\{1,\ 0,\ 1,\ 0,\ 1,\ 0\cdots\}$ is $\sum\limits_{n\geq 0}^{\infty} x^{2n} = \frac{1}{1-x^2}$.

– Differentiate both sides of $\sum\limits_{n=0}^{\infty} x^n = \frac{1}{1-x}$. We get $\sum\limits_{n=1}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2} = \sum\limits_{n=0}^{\infty}(n+1)x^n$, GF of $\{1,\ 2,\ 3,\cdots\}$.

– Take the k-th derivative of $\sum\limits_{n=0}^{\infty} x^n = \frac{1}{1-x}$, we get $\sum\limits_{n=k}^{\infty} n(n-1)\cdots(n-k+1)x^{n-k} = k!\frac{1}{(1-x)^{k+1}}$. i.e.$\frac{1}{(1-x)^{k+1}} = \sum\limits_{n'=0}^{\infty} \frac{(n'+k)\cdots(n'+1)}{k!}x^{n'} = \sum\limits_{n=0}^{\infty} \binom{n+k}{k} x^n$.

  Replace $k+1$ by k, $\frac{1}{(1-x)^k} = \sum\limits_{n=0}^{\infty} \binom{n+k-1}{k-1} x^n$, which is the GF of $\{a_n\}_{n=0}^{\infty}$, where a_n is the $\sharp$integer solutions to $x_1 + x_2 + \cdots + x_k = n,\ x_i \geq 0$.

Problem 1: Let $a_0 = 1,\ a_n = 2a_{n-1},\ n \geq 1$. Find a_n.

Sol: Let $f(x) = \sum\limits_{n\geq 0} a_n x^n$ be GF. Then

$$\sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} 2a_{n-1}x^n = 2x\sum_{n=1}^{\infty} a_{n-1}x^{n-1} = 2x\sum_{n=0} a_n x^n.$$

$$\Longrightarrow f(x) = 1 + 2xf(x) \Longrightarrow f(x) = \frac{1}{1-2x} = \sum_{n=0}^{\infty} 2^n x^n \Longrightarrow a_n = 2^n.$$

□

Useful Trick: In a recurrence problem, first find GF, then expand GF to find a_n.

Problem 2: Let $A_n = \{$strings of length n over $\{a,\ b,\ c\}$ s.t. No "aa" occuring $\}$. Find $a_n = |A_n|,\ n \geq 1$.

Sol: Observe that $a_1 = 3,\ a_2 = 8$. For $n \geq 2$,

$$\begin{aligned} A_n &= \{strings\ with\ prefix\ a\} \cup \{strings\ with\ prefix\ b\} \cup \{strings\ with\ prefix\ c\} \\ &= \{prefix\ ab\ or\ ac\} \cup \{prefix\ b\ or\ c\} \\ &= \{ab|s,\ ac|s:\ s \in A_{n-2}\} \cup \{b|s,\ c|s:\ s \in A_{n-1}\}. \end{aligned}$$

So $a_n = 2a_{n-2} + 2a_{n-1}$. Set $a_0 = 1$, Then $\sum\limits_{n=2}^{\infty} a_n x^n = 2\sum\limits_{n=2}^{\infty} a_{n-2}x^n + 2\sum\limits_{n=2}^{\infty} a_{n-1}x^n$. Let $f(x) = \sum\limits_{n\geq 0}^{\infty} a_n x^n$,

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + 2x^2\sum_{n=0}^{\infty} a_n x^n + 2x(\sum_{n\geq 0}^{\infty} a_n x^n - a_0).$$

$$\Longrightarrow f(x) = 1 + 3x + 2x^2 f(x) + 2x(f(x) - 1) \Longrightarrow f(x) = \frac{1+x}{1-2x-2x^2}.$$

By Partial Fraction Decomposition

$$f(x) = \frac{1-\sqrt{3}}{2\sqrt{3}}\frac{1}{\sqrt{3}+1+2x} + \frac{1+\sqrt{3}}{2\sqrt{3}}\frac{1}{\sqrt{3}-1-2x}.$$

Expand $f(x)$, we get

$$
\begin{aligned}
f(x) &= \frac{1-\sqrt{3}}{2\sqrt{3}}\frac{1}{\sqrt{3}+1}\frac{1}{1+\frac{2}{\sqrt{3}+1}x}+\frac{1+\sqrt{3}}{2\sqrt{3}}\frac{1}{\sqrt{3}-1}\frac{1}{1+\frac{2}{\sqrt{3}-1}x} \\
&= \sum_{n\geq 0}[\frac{1-\sqrt{3}}{2\sqrt{3}}\frac{1}{\sqrt{3}+1}(\frac{-2}{\sqrt{3}+1})^n+\frac{1+\sqrt{3}}{2\sqrt{3}}\frac{1}{\sqrt{3}-1}(\frac{2}{\sqrt{3}-1})^n]x^n
\end{aligned}
$$

□

Definition: For real r and integer $k \geq 0$, let $\binom{r}{k} = \frac{r(r-1)(r-2)\cdots(r-k+1)}{k!}$.

Newton's Binomial Theorem: For any real r,

$$(1+x)^r = \sum_{k=0}^{\infty}\binom{r}{k}x^k$$

holds for $x \in (-1,1)$.

Problem 3: Compute $\binom{\frac{1}{2}}{n}$.

sol:

$$
\begin{aligned}
\binom{\frac{1}{2}}{n} &= \frac{(\frac{1}{2})(\frac{1}{2}-1)(\frac{1}{2}-2)\cdots(\frac{1}{2}-n+1)}{n!} \\
&= \frac{\frac{1}{2}\cdot\frac{-1}{2}\cdot\frac{-3}{2}\cdots\frac{-(2n-3)}{2}}{n!} \\
&= \frac{(-1)^{n-1}}{2^n}\frac{1\cdot 3\cdot 5\cdots(2n-3)}{n!} \\
&= \frac{(-1)^{n-1}}{2^n}\frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdots(2n-2)}{n!(n-1)!\cdot 2^{n-1}} \\
&= \frac{(-1)^{n-1}}{2^{2n-1}}\frac{(2n-2)!}{n!(n-1)!} \\
&= \frac{(-1)^{n-1}}{2^{2n-1}}\frac{1}{n}\binom{2n-2}{n-1} \\
&= \frac{(-1)^{n-1}\cdot 2}{4^n}\binom{2n-2}{n-1}.
\end{aligned}
$$

□

Catalan Sequence: $c_0 = 0,\ c_1 = 1,\ c_n = \sum_{k=0}^{n-1} c_k c_{n-k}$.

Problem 4: Given a product $a_1 a_2 \cdots a_n$ of n letters, how many ways can we calculate the product by multiplying two factors at a time?

Sol: Denote the solution by c_n. Suppose at the last step of multiplication, we have bc, where $b = a_1 a_2 \cdots a_k,\ c = a_{k+1}a_{k+2}\cdots a_n,\ 1 \leq k \leq n-1$. Notice that there are c_k ways to put brackets in b, andc_{n-k} ways to put brackets in c. Thus for a given j, there are $c_k c_{n-k}$ ways. Then

$$c_n = \sum_{k=1}^{n-1} c_k c_{n-k},\ n \geq 2.$$

Set $c_0 = 0,\ c_1 = 1$

$$\Longrightarrow \sum_{n=2}^{\infty} c_n z^n = \sum_{n=2}^{\infty} \sum_{k=1}^{n-1} c_k c_{n-k} z^n = \sum_{n=2}^{\infty} \sum_{k=0}^{n} c_k c_{n-k} z^n.$$

Let $C(z) = \sum_{n\geq 0}^{\infty} c_n z^n$. Then $C(z)^2 = \sum_{n=0}^{\infty} \sum_{k=0}^{n} c_k c_{n-k} z^n$.

$$C(z) - z = C(z)^2.$$

Then $C(z) = \frac{1}{2} \pm \frac{1}{2}\sqrt{1-4z}$. Since $C(0) = 0,\ C(z) = \frac{1}{2} - \frac{1}{2}\sqrt{1-4z}$. By Newton's Binomial Theorem

$$\begin{aligned} C(z) &= \frac{1}{2} - \frac{1}{2} \sum_{n\geq 0}^{\infty} \binom{\frac{1}{2}}{n} (-4z)^n \\ &= \sum_{n\geq 1}^{\infty} \frac{1}{2} \frac{(-1)^{n+1} 4^n z^n}{2} \\ &= \sum_{n\geq 1}^{\infty} \frac{1}{n} \binom{2n-2}{n-1} z^n. \end{aligned}$$

So $c_n = \frac{1}{n}\binom{2n-2}{n-1},\ c_{n+1} = \frac{1}{n+1}\binom{2n}{n}$(Catalan number). $\square$