Combinatorics 2017 Fall
week4 note

Teaching by: Professor Xiande Zhang

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Polya Counting Theorem:

Group
Finite group
Subgroups
Coset
You can find them in any Algebra book. We just give some examples.

Group: $(\mathbb{R},\cdot)$,$(\mathbb{Z},\cdot)$,$(\mathbb{Z}_{>0},+)$,$(\mathbb{Z}_{>0},\cdot)$.

Finite groups: $\mathbb{Z}/n\mathbb{Z} := 0,1,2,\cdots,n-1$, $((\mathbb{Z}/n\mathbb{Z})^{\times},+)$,$S_n =$ {all permutations over $[n]$}.

Subgroup: $(2\mathbb{Z},+) \leq (\mathbb{Z},+)$.$(2\mathbb{Z},\cdot) \nleq (\mathbb{R},\cdot)$.

Coset: $G = (\mathbb{Z},+), H = (3\mathbb{Z},+), H \leq G$.

$$0 + H = \{\cdots,-6,-3,0,3,6,\cdots\} = 3 + H.$$
$$1 + H = \{\cdots,-5,-2,1,4,7,\cdots\} = 4 + H.$$
$$2 + H = \{\cdots,-4,-1,2,5,8,\cdots\} = 5 + H.$$

Fact: $H \leq (G,\cdot)$

(1) $\forall s \in G, sH = H$. iff $s \in H$

(2) $\forall s,t \in G$ *either* $sH = tH$ *or* $(sH) \cap (tH) = \phi$.

(3) $G = g_1H \cup g_2H \cup \cdots \cup g_nH$

(4) $\forall s \in H, |sH| = |H|$. just for finite group.

(5) $n \cdot |H| = |G|$ denote $n = [G : H]$.just for finite group.

Group action: finite group $(G, \cdot)$,X is a set,a group action is a map,denote by (G, X).
$* : G \times X \to X$,s.t.

(1) $e * x = x$,where e is id of $G, \forall x \in X$.

(2) $\forall g, h \in G, \forall x \in X, g * (h * x) = (g \cdot h) * x$.

E.g.

(1) $(S_n, \circ), [n]$, define a group action by $\sigma * (i) = \sigma(i)$.

(2) $(G, \cdot)$, set G,define a group action by $g * h = g \cdot h, \forall g, h \in G$.

Def:$(G, \cdot), X$ group action $*, \forall x \in X$.

1. orbit of $x, Orb(x) = \{g * x : \forall g \in G\}$.
2. stabilizer of $x, Stab(x) = \{g \in G : g * x = x\}$.
3. fixed points of $g \in G, Fix(g) = \{x \in X : g * x = x\}$.

Ex:$Stab(x) \leq (G, \cdot)$.

Fact:$X = Orb(x_1) \cup \cdots \cup X = Orb(x_k)$. where $Orb(x_i) \cap Orb(x_j) = \phi, i \neq j$.
E.g.$X = [5], Z = (123)(45)$ then $Z^2 = (132)(4)(5), Z^3 = (1)(2)(3)(45), Z^4 = (123)(4)(5), Z^5 = (132)(45), Z^6 = (1)(2)(3)(4)(5).G =< Z >$.
$Orb(1) = \{1, 2, 3\}, Orb(4) = Orb(5) = \{4, 5\}, Orb(1) = Orb(2) = Orb(3) = \{1, 2, 3\}$.

Thm1:finite group actio$(G, X), \forall x \in X$, then $|G| = |Orb(x)| \cdot |Stab(x)|$.

proof: Let $G_x = Stab(x)$, denote $G/G_X = \{g_i G_x : i \in [n]\}$. we have $|G| = n|G_x| = |G/G_x| \cdot |G_x|$.
define a function $f : G/G_x \to Orb(x)$ by $f(g_i G_x) = g_i * x$.

(1) injection:if $g_i * x = g_j * x$, then $(g_j^{-1} \cdot g_i) * x = x \Rightarrow g_j^{-1} \cdot g_i \in G_x$. that is $g_i G_x = g_j G_x$.

(2) surjection:take any $h * x \in Orb(x)$,we have $h \in g_i G_x$ for some i, then $h = g_i s$, where $s \in G_x$. then
$h * x = (g_i s) * x = g_i * (s * x) = g_i * x = f(g_i G_x)$.

by (1)&(2), $|Orb(x)| = |G/G_x|$ so $|G| = |Orb(x)| \cdot |Stab(x)|$. □

Note:if $x_1 \neq x_2 \in Orb(x)$, then $|Stab(x_1)| = |Stab(x_2)|$.

Thm2:(Burnside Lemma)finite group action (G, X). Let $N(G)$ be number of distinct orbits of X, then $N(G) = \frac{1}{|G|} \sum_{g \in G} |Fix(g)|$.

proof:(Double Counting)Consider the set $S = \{(g, x) : g * x = x\}$.
group(g, x) by the first element

$$|S| = \sum_{g \in G} |\{x : g * x = x\}| = \sum_{g \in G} |Fix(g)|.$$

group(g, x) by the second element

$$\begin{aligned} |S| &= \sum_{x \in X} |\{g : g * x = x\}| = \sum_{x \in X} |Stab(x)| \\ &= \sum_{i=1}^{N(G)} \sum_{x \in Orb(x_i)} |Stab(x_i)| = \sum_{i=1}^{N(G)} |Orb(x_i)| \cdot |Stab(x_i)| \\ &= N(G) \cdot |G|. \end{aligned}$$

□

Thm3:(Weighted Burnside Lemma) finite group action (G, X) Let $\Omega_i = Orb(x_i), i \in [N(G)]$ be all distinct orbits.Let $\omega : X \to R$ be a weighted function,where $\omega(x) = \omega(y)$ if $x, y \in \Omega_i$, for some $i \in [N(G)]$. define $\omega(\Omega_i) = \omega(x_i)$, then

$$\sum_{i=1}^{N(G)} \omega(\Omega_i) = \frac{1}{|G|} \sum_{g \in G} \sum_{x \in Fig(x)} \omega(x).$$

proof: Let $S = \{\{g, x\} : g * x = x\}$.
compute $M = \sum_{(g,x) \in S} \omega(x)$, Then

1. $$M = \sum_{g \in G} \sum_{x \in Fix(g)} \omega(x).$$

2. $$\begin{aligned} M &= \sum_{x \in X} \sum_{g \in Stab(x)} \omega(x) = \sum_{x \in X} \omega(x)|Stab(x)| \\ &= \sum_{i=1}^{N(G)} \sum_{x \in \Omega_i} \omega(x)|Stab(x)| = \sum_{i=1}^{N(G)} \omega(x_i)|\Omega_i||Stab(x_i)| \\ &= |G| \sum_{i=1}^{N(G)} \omega(\Omega_i). \end{aligned}$$

$\square$

Fact: any finite group can be embedded in S_n.

Polya Thm

finite group action (G, X), a coloring of X in m colors is a function $f : X \to C$, where C is a set of m colors.
Let C^X ={all coloring of X in C}.
define a group action (G, C^X) as

$$(g * f)(x) = f(g^{-1} * x)$$

E.g. $X = [6], C = \{r, b, g\}$.
$f_1 : 1 \to r \quad 2 \to b \quad 3 \to b \quad 4 \to r \quad 5 \to g \quad 6 \to b$

$\tau = (123)(45)(6) \qquad f_2 = \tau * f_1$
$f_2 : 2 \to r \quad 3 \to b \quad 1 \to b \quad 5 \to r \quad 4 \to g \quad 6 \to b$

Ploya Thm4:$|X| = n, |C| = m, G$ is a permutation group of X
$\mathscr{F}$ is a collection of different orbits of C^X then

$$|\mathscr{F}| = \frac{1}{|G|} \sum_{g \in G} m^{c(g)}$$

where $c(g)$ is the number of cycles of g.

proof:to apply Burnside Lemma,we need to compute $|Fix(g)|$ with group action (G, C^X),that is the number of coloring functions fixed by g By definition of (G, C^X),a coloring is fixed by g iff the color along with cycle of g is constant.so $|Fix(g)| = m^{c(g)}$. $\square$

E.g.The number of vertices colorings of a square with colors $\{r, g\}$ under rotation.

Sol:the permutation group $G = \{(1)(2)(3)(4), (1432), (13)(24), (1234)\}$ using the Thm4,so the number is $\frac{1}{4}(2^4 + 2^1 + 2^2 + 2^1) = 6$. $\square$

Ex: What is the number of coloring under rotation and reflection?

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cycle type: $\sigma \in S_n$,Let $\lambda_i(\sigma)$ be the number cycles length i of σ.
$(\lambda_1(\sigma), \lambda_2(\sigma), \cdots, \lambda_n(\sigma))$ is $type(\sigma)$.
E.g. $\sigma = (123)(4)(5), type(\sigma) = (2, 0, 1, 0, 0)$.

cycle index:$G \leq S_n$, the cycle index of G is

$$P_G(x_1, x_2, \cdots, x_n) = \frac{1}{|G|} \sum_{g \in G} x_1^{\lambda_1(g)} x_2^{\lambda_2(g)} \cdots x_n^{\lambda_n(g)}.$$

Corollary(Polya Thm):$\frac{1}{|G|} \sum_{g \in G} m^{c(g)} = P_G(m, m, \cdots, m)$.
Promblem:$\sigma_n =< (1, 2, \cdots, n) > \leq S_n$, Compute $P_G(x_1, x_2, \cdots, x_n)$.

sol:$\tau = (1, 2, \cdots, n), \sigma_n = \{\tau, \tau^2, \cdots, \tau^n\}$. Consider $type(\tau^k)$.
$\forall i \in [n], \tau^k(i) \equiv i + k \pmod n$ suppose the cycle containing i has length lThen the cycle is $(i, i+k, \cdots, i+(l-1)k)$. and $l = \frac{n}{(k,n)}$ independent on i.
so all cycles of τ^k have length $\frac{n}{(k,n)}$.
i.e.

$$\lambda_i(\tau^k) = \begin{cases} 0 & i \neq \dfrac{n}{(k,n)}; \\ (k,n) & i = \dfrac{n}{(k,n)}. \end{cases}$$

so,

$$\begin{aligned} P_{\sigma_n}(x_1, x_2, \cdots, x_n) &= \frac{1}{n} \sum_{i=1}^{n} (x_{\frac{n}{(k,n)}})^{(k,n)} \\ &= \frac{1}{n} \sum_{j|n} \sum_{k:(k,n)=j} (x_{\frac{n}{j}})^j \end{aligned}$$

let

$$\begin{aligned} r(j) &= \#\{k, s.t.\ (k,n) = j, 1 \leq k \leq n.\} \\ &= \#\{\frac{k}{j} : (\frac{k}{j}, \frac{n}{j}) = 1, 1 \leq \frac{k}{j} \leq \frac{n}{j}.\} \\ &= \varphi(\frac{n}{j}). \end{aligned}$$

so

$$P_{\sigma_n}(x_1, x_2, \cdots, x_n) = \frac{1}{n}\sum_{j|n}\varphi(\frac{n}{j})(x_{\frac{n}{j}})^j$$
$$= \frac{1}{n}\sum_{d|n}\varphi(d)x_d^{\frac{n}{d}}.$$

$\square$

E.g.compter $C_m(n)$

sol:a cycle of length n over $[m]$ corresponds to a coloring of vertices of cycle of length n in m colors.
so,

$$C_m(n) = \#different\ colorings\ under\ \sigma_n,$$
$$= P_G(m, m, \cdots, m) = \frac{1}{n}\sum_{d|n}\varphi(d)m^{\frac{n}{d}}$$
$$= \frac{1}{n}\sum_{d|n}\varphi(nd)m^d.$$

$\square$

<u>Graphs:</u>simple graph $G = (V, E), V$ vertex set.$E \subseteq \binom{V}{2}$ edge set.
$V(G) = |V|, e(G) = |E|$. if $\{i, j\} \in E$,then $\{i.j\}$ is incident to i and j.and i, j is adjacent.

<u>Def:</u>$V(G) = |V|, \varphi : V \to V$, if $\forall u, v \in V$ we have $\{u, v\} \in E \Leftrightarrow \{\varphi(u), \varphi(v) \in E\}$.
Then say φ is an automorphism of G.

Let $Aut(G)$ ={all automorphism of G}$\leq Sym(V)$.

Ex:$Aut(G) = D_n$.

Problem: Show

$$P_{D_n}(x_1,x_2,\cdots,x_n)=\begin{cases}\frac{1}{2}P_{\sigma_n}(x_1,x_2,\cdots,x_n)+\frac{1}{2}x_ix_2^{\frac{n-1}{2}} & n\ odd;\\ \frac{1}{2}P_{\sigma_n}(x_1,x_2,\cdots,x_n)+\frac{1}{4}(x_2^{\frac{n}{2}}+x_i^2x_2^{\frac{n-2}{2}}) & n\ even.\end{cases}$$

sol: Since $D_n=\sigma_n\cup H_n$, where $\tau=(1,2,\cdots,n),\sigma_n=\{\tau,\tau^2,\cdots,\tau^n\}.\tau=(1,2,\cdots,n),H_n=\{\pi.\pi^2,\cdots,\pi^n\}$.
and we know that $\pi^k(i)\equiv -i+k \pmod n$.

$$\begin{aligned}P_{D_n}(x_1,x_2,\cdots,x_n)&=\frac{1}{|D_n|}(\sum_{i=1}^{n}x_1^{\lambda_1(\tau^i)}\cdots x_n^{\lambda_n(\tau^i)}\sum_{i=1}^{n}x_1^{\lambda_1(\pi^i)}\cdots x_n^{\lambda_n(\pi^i)})\\&=\frac{1}{2}P_{\sigma_n}+\frac{1}{2n}\sum_{i=1}^{n}x_1^{\lambda_1(\pi^i)}\cdots x_n^{\lambda_n(\pi^i)}\end{aligned}$$

since each π^i is a reflection,i.e.$(\pi^i)^2=id$, that is π^i is a product of cycles of length 1 and cycles of length 2.

Assume there are m_i cycles of length 1 in π^i.
then there are $\frac{n-m_i}{2}$ cycles of length of 2.

so $\frac{1}{2n}\sum_{i=1}^{n}x_1^{\lambda_1(\pi^i)}\cdots x_n^{\lambda_n(\pi^i)}=\frac{1}{2n}\sum_{i=1}^{n}x_1^{mi}x_2^{\frac{n-m_i}{2}}$.

since $m_i=|Fix(\pi^i)|$ if $a=\pi^i(a)\equiv -a+i \pmod n$
$m_i=\#$ solutions of $2a\equiv i \pmod n$.
If n is odd,$(2,n)=1$, 2 has inverse in ring $(\mathbb{Z}_n,\cdot)$.
so$m_1=m_2=\cdots=m_n=1$,then $\frac{1}{2n}\sum_{i=1}^{n}x_1^{mi}x_2^{\frac{n-m_i}{2}}=\frac{1}{2}x_1x_2^{\frac{n-1}{2}}$.
If n is even,then $2a\equiv i \pmod n$. has no solutions if i is odd.
if i is even,then two solutions $a=\frac{i}{2}$ and $a=\frac{n}{2}+\frac{i}{2}$.
so

$$m_i=\begin{cases}0 & i\ odd;\\ 2 & i\ even.\end{cases}$$

then,

$$\begin{aligned}\frac{1}{2n}\sum_{i=1}^{n}x_1^{mi}x_2^{\frac{n-m_i}{2}}&=\frac{1}{2n}(\frac{n}{2}x_2^{\frac{n}{2}}+\frac{n}{2}x_i^2x_2^{\frac{n-2}{2}})\\&=\frac{1}{4}(x_2^{\frac{n}{2}}+x_i^2x_2^{\frac{n-2}{2}}).\square\end{aligned}$$

Corollary: The number of cycles of length n over $[m]$ under rotations and reflections

$$= \frac{1}{2} C_m(n) + \begin{cases} \frac{1}{2} m^{\frac{n+1}{2}} & n \ odd; \\ \frac{1}{2} m^{\frac{n}{2}}(m+1) & n \ even. \end{cases}$$

Ex.consider coloring of a cube in m colors.
The number of vertex colorings $= \frac{1}{24}(m^8 + 17m^4 + 6m^2)$.
cycle index $= \frac{1}{24}(x_1{}^8 + 9x_2{}^4 + 6x_4{}^2 + 8x_1{}^2 x_3{}^2)$.

The number of edges colorings $= \frac{1}{24}(m^12 + 6m^7 + 3m^6 + 8m^4 + 6m^3)$.
cycle index $= \frac{1}{24}(x_1{}^12 + 3x_2{}^6 + 6x_4{}^3 + 6x_1{}^2 x_2{}^5 + 8x_3{}^4)$.

The number of faces colorings $= \frac{1}{24}(m^6 + 3m^4 + 12m^3 + 8m^2)$.
cycle index $= \frac{1}{24}(x_1{}^6 + 3x_1{}^2 x_2{}^2 + 6x_1{}^2 x_4 + 6x_2{}^3 + 8x_3{}^2)$.

[Hint:]$|V| = 8, |E| = 12, |F| = 6$, since rotation of the cube induces different automorphism of V, E and F.