Combinatorics 2017 Fall

week5 note

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Reference:

Extremal Combinatorics with applications in Computer Science. 2nd Edition.Stasys Jukna,Springer.

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Problem: The number of cycles of length n over $[m]$ s.t. i occurs k_i times, $i \in [n]$.

Definition: (G, X), C^X colorings. Let $\omega : C \longrightarrow \mathbb{R}$ be a weight function. Induce a weight function of C^X as follows

$$\forall \in C^X,\ define \omega(f) = \Pi_{x\in X}\omega(f(x)).$$

If $f_1,\ f_2 \in C^X$, in the same orbit under $(G, C^X),\ i.e. f_2 = f_1 \circ g^{-1}$. Then $\omega(f_2) = \Pi_{x\in X}\omega(f_1(g^{-1}(x))) = \Pi_{x\in X}\omega(f_1(x)) = \omega(f_1)$. i.e.$\omega$ have the same value on each orbit.

Let $\mathcal{F}$ be the set of all orbits under (G, C^X). $\forall F \in \mathcal{F}$, define $\omega(F) = \omega(f),\ f \in F$.

Theorem 1: $(G, C^X),\ \omega : C^X \longrightarrow \mathbb{R}$ as defined above. Then

$$\sum_{F\in\mathcal{F}} \omega(F) = P_G(\sum_{c\in C}\omega(c), \sum_{c\in C}\omega(c)^2, \cdots, \sum_{c\in C}\omega(c)^n).$$

Corollary 1: $(G, C^X), |x| = n, C = \{c_1, c_2, \cdots, c_m\}, k_1 + k_2 + \cdots + k_m = n, k_i \geq 0$. Denote $N(k_1, \cdots k_m)$ be the number of colorings of C^X $s.t.$ there are k_i elements of X have color $c_i, i \in [m]$. then

$$P_G(\sum_{i=1}^m y_i, \sum_{i=1}^m y_i^2, \cdots, \sum_{i=1}^m y_i^n) = \sum_{k_1+\cdots+k_m=n} N(k_1, \cdots, k_m) y_1^{k_1} \cdots y_m^{k_m}.$$

Proof Let $\omega(c_i) = y_i, i \in [m]$, then

$$\sum_{F\in\mathcal{F}} \omega(F) = P_G(\sum_{i=1}^m y_i, \sum_{i=1}^m y_i^2, \cdots, \sum_{i=1}^m y_i^n)$$

f colors k_i elements of X with $c_i \Leftrightarrow \omega(f) = y_1^{k_1} \cdots y_m^{k_m}$. If $f \in F$, then $\omega(F) = \omega(f) = y_1^{k_1} \cdots y_m^{k_m}$. So $N(k_1, \cdots, k_m)$ is the number of orbits F s.t. $\omega(F) = y_1^{k_1} \cdots y_m^{k_m}$ i.e.$N(k_1, \cdots, k_m) = [y_1^{k_1}, \cdots, y_n^{k_m}](P_G(\sum_{i=1}^m y_i, \sum_{i=1}^m y_i^2, \cdots, \sum_{i=1}^m y_i^n))$. $\square$

<u>Proof of Theorem 1</u>:$\forall g \in G$, let $k = k(g)$ be the number of cycles of g and $A_1, \cdots, A_k$ be the set of elements of each cycle. For $f \in Fix(g)$, suppose $f(x) = c_i$, if $x \in A_i, i \in [k]$. Then $\omega(f) = \Pi_{i=1}^k(\omega(c_i)^{|A_i|})$. So

$$\sum_{f \in Fix(g)} \omega(f) = \sum_{(c_1, \cdots, c_k) \in C^k} (\Pi_{i=1}^k(\omega(c_i)^{|A_i|})) = \Pi_{i=1}^k(\sum_{c \in C} \omega(c)^{|A_i|}).$$

Denote $type(g) = (\lambda_1(g), \cdots, \lambda_n(g))$, then

$$\sum_{f \in Fix(g)} \omega(f) = \Pi_{i=1}^n(\sum_{c \in C} \omega(c)^i)^{\lambda_i(g)}$$

By weighted Burnside Lemma,

$$\begin{aligned}\sum_{F \in \mathcal{F}} \omega(F) &= \frac{1}{|G|} \sum_{g \in G} \sum_{f \in Fix(g)} \omega(f) \\ &= \frac{1}{|G|} \sum_{g \in G} (\Pi_{i=1}^n(\sum_{c \in C} \omega(c)^i)^{\lambda_i(g)}) \\ &= P_G(\sum_{c \in C} \omega(c), \sum_{c \in C} \omega(c)^2, \cdots, \sum_{c \in C} \omega(c)^n).\end{aligned}$$

$\square$

Problem: The number of different cycles of length n over $[m]$ under rotation s.t. i occurs k_i times, $i \in [m]$.

Sol: $P_{\sigma_n}(x_1, \cdots, x_n) = \frac{1}{n} \sum_{d|n} \varphi(d)(x_d)^{\frac{n}{d}}$. Then

$$P_{\sigma_n}(\sum_{i=1}^m y_i, \sum_{i=1}^m y_i^2, \cdots, \sum_{i=1}^m y_i^n) = \frac{1}{n} \sum_{d|n} \varphi(d)(\sum_{i=1}^m y_i^d)^{\frac{n}{d}}.$$

consider the coefficient $T(k_1, \cdots, k_m)$ of $y_1^{k_1} \cdots y_m^{k_m}$. So

$$T(k_1, \cdots, k_m) = [y_1^{k_1} \cdots y_m^{k_m}](\frac{1}{n} \sum_{d|(k_1, \cdots, k_m)} \varphi(d)(\sum_{i=1}^m y_i^d)^{\frac{n}{d}}) = \frac{1}{n} \sum_{d|(k_1, \cdots, k_m)} \varphi(d) \frac{(\frac{n}{d})!}{(\frac{k_1}{d})! \cdots (\frac{k_m}{d})!}$$

$\square$

<u>Definition</u>: $(G, X), (H, Y)$, where $X \cap Y = \emptyset$. Let $G \times H = \{(g, h) : g \in G, h \in H\}$, define $(g, h) : X \cup Y \longrightarrow X \cup Y$ by

$$(g, h)(a) = \begin{cases} g(a), if\ a \in X \\ h(a), if\ a \in Y. \end{cases}$$

Then $G \times H$ is permutation group of $X \cup Y$.

Lemma 1:$(G\times H, X\cup Y), |X|=n, |Y|=m$. Then

$$P_{G\times H}(x_1,\cdots,x_{n+m})=P_G(x_1,\cdots,x_n)P_H(x_1,\cdots,x_m).$$

Distribution Problem:n balls with r colors. where there are n_i balls with color i, $\sum_{i=1}^r n_i = n$, Put them into m boxes: $B_1,\cdots,B_m$, s.t. there are k_i balls in B_j, $\sum_{i=1}^m k_i = n$

- If $n_i=1, i\in[r]$ and $r=n$,then it is an ordered partition problem, s.t. the ith part is of size $k_i, i\in[m]$, so $\frac{n!}{k_1!\cdots k_m!}$
- In general, it is a coloring problem as follows. $X_1,\cdots,X_r$, where $|X_i|=n_i$ with permutation group G_i, since each ball in X_i are considered to be the same, $G_i=S_{n_i}$. So we have a group cartition $S_{n_1}\times\cdots\times S_{n_r}$ over $X_1\cup\cdots X_r$. By lemma 1, it has cycle index $\Pi_{i=1}^r P_{s_i}(x_1,\cdots,x_{n_i})$. So the number of distributions$=\Pi_{i=1}^r P_{s_i}(m,\cdots,m)$.
- If $r=1$, all balls have same color, $n_1=n$. then total the number of distributions is the number of sols to $k_1+\cdots+k_m=n, k_i\geq 0. i.e. \binom{n+m-1}{m-1}=\binom{n+m-1}{n}$. Since cycle index is $P_{s_n}(x_1,\cdots,x_n)$, so $P_{s_n}(m,\cdots,m)=\binom{n+m-1}{n}$. so $\Pi_{i=1}^r P_{s_{n_i}}(m,\cdots,m)=\Pi_{i=1}^r\binom{n_i+m-1}{n_i}$.

  Eg.
- The number of n different balls into m membered boxes.

$$\Pi_{i=1}^n P_{s_i}(m)=\Pi_{i=1}^n m = m^n.$$

  The number of ways if required k_i balls in the ith box$=[y_1^{k_1}\cdots y_m^{k_m}](\Pi_{i=1}^n P_{s_i}(\sum_{i=1}^m y_i))=[y_1^{k_1}\cdots y_m^{k_m}](\sum_{i=1}^m y_i)^n=\frac{n!}{k_1!\cdots k_m!}$
- The number of putting a,a,b,b into boxes $A,B.r=2, n_1=n_2=2$.

$$\Pi_{i=1}^2 P_{s_i}(2,2)=\binom{2+2-1}{2}^2=9.$$

- The number of putting a,a,b,b,c,c into boxes A,B,C. s.t. 3 balls in A, 2 balls in B, 1 ball in C.

$$P_{s_2}(x_1,x_2)=\frac{1}{2}(x_1^2+x_2).$$

$$P_{s_2}(y_1+y_2+y_3, y_1^2+y_2^2+y_3^2)=\frac{1}{2}[(y_1+y_2+y_3)^2+y_1^2+y_2^2+y_3^2].$$

$$[y_1^3y_2^2y_3][P_{s_2}(y_1+y_2+y_3,y_1^2+y_2^2+y_3^2)]^3=15.$$

Problem: colorings of graphs.
Key point: find $Aut(G)$.