Combinatorics 2017 Fall

week5 note

Teaching by: Professor Xiande Zhang

Reference:

Extremal Combinatorics with applications in Computer Science. 2nd Edition.Stasys Jukna,Springer.

2017/10/20

A system of distinct representatives(SDR) for a sequence of set $S_1, S_2, \cdots, S_m$(are not necessarily distinct) is a sequence of distinct elements $x_1, \cdots, x_m$, s.t. $x_i \in S_i, i \in [m]$.

Theorem1(**Hall's marriage Theorem**) The sets of $S_1, S_2, \cdots, S_m$ has a SDR iff $|\cup_{i\in I} S_i| \geq |I|$ for $I \subset [m]$(Hall's condition) i.e. for $\forall k \in [m]$, the union of any k sets has at least k elements.

Proof: "$\Longrightarrow$" If $S_1, S_2, \cdots, S_m$ have a SDR $x_1, \cdots, x_m$, then $\forall I \subset [m], |\cup_{i\in I} S_i| \geq |\{x_i, i \in I\}| = |I|$.

$\Longleftarrow$ Prove by induction on m. The case $m = 1$ is clear. Assume the claim holds for any collection with $< m$ sets.

- if for all $I \subset [m], |\cup_{i\in I} S_i| > |I|$, take $x \in S_1$ as its representative, let $S_i' = S_i \backslash \{x\}, i = 2, \cdots, m$. then for all $I \subset [2, m], |\cup_{i\in I} S_i'| \geq |I|$. by assumption $S_2', \cdots, S_m'$ have a SDR $x_2, \cdots, x_m$. then $x_1, \cdots, x_m$ is a SDR of $S_1, \cdots, S_m$.
- if for some $I \subset [m], |\cup_{i\in I} S_i| = |I| = k$ for some k. by recording $S_i, i \in [m]$, we may assume $|\cup_{i=1}^k S_i| = k$. by assumption, $S_1, \cdots, S_k$ have a SDR $x_1, \cdots, x_k$. Let $S_i' = S_i \backslash \{x_1, \cdots, x_k\}, i \in [k+1, m]$. then for all $I \subset [k+1, m], |\cup_{i\in I} S_i'| \geq |I|$, if not, $|(\cup_{i\in I} S_i) \cup (\cup_{j=1}^k S_j)| < |I| + k$, a contradiction. So by assumption $S_{k+1}', S_{k+2}', \cdots, S_m'$ have a SDR $x_{k+1}, \cdots, x_m,$, then $x_1, x_2, \cdots, x_m$ is a SDR of $s_1, s_2, \cdots, s_m$.

Corollary2:X is a set of size n,$|S_i| = r, S_i \subset X, i \in [m].s.t.|\{i : x \in S_i\}| = d$ for all $x \in X$. If $m \leq n$, then $S_1, \cdots, S_m$ have a SDR.

Proof: Consider the incidence matrix $M = (m_{x,i})$ of $S_i, i \in [m]$. That is M is a 0-1 matrix with $|X|$ rows labeled by elements $x \in X$, and with m colors labeled by $i \in [m]$, such that $m_{x,i} = 1$ iff $x \in S_i$, count the number of 1 in M. we have $dn = mr$, then $m \leq n$ implies $d \leq r$. Suppose

$S_1, \cdots, S_m$ don't have a SDR, by Hall's Theorem, $\exists k$ sets $S_{i_1}, \cdots, S_{i_k}$ for some $k \in [m]$,

$$|Y| = |S_{i_1} \cup \cdots \cup S_{i_k}| < k$$

$\forall x \in Y$, let $d_x = |\{i \in [k] : x \in S_{i_j}\}| \leq d$. count the number of 1 in rows labeled by $x \in Y$ and columns $i_j, j \in [k]$.

$$rk = \sum_{j=1}^{k} |S_{ij}| = \sum_{x \in Y} d_x \leq d|Y| < dk.$$

a contradiction. $\square$

<u>Theorem3</u>: Suppose elements in X are colored either in red or in blue. $S_i \subset X, i \in [m]$, Then $S_1, \cdots, S_m$ have a SDR with $\leq t$ red elements iff $S_1, \cdots, S_m$ have a SDR and $\forall I \subset [m], \cup_{i \in I} S_i$ has $\geq t$ blue elements.

Proof: "$\Longrightarrow$" Let $x_1, \cdots, x_m$ be a SDR of $S_1, \cdots, S_m$ with $\leq t$ red elements. then $\forall I \subset [m], \{x_i, i \in I\} \subset \cup_{i \in I} S_i$ has at least $|I| - t$ blue elements.

"$\Longleftarrow$" Let R be the set of red elements in X. If $|R| \leq t$, trivial. Assume $|R| > t$, let $S_{m+1} = S_{m+2} = \cdots = S_{m+r} = R$, where $r = |R| - t$. then $S_1, \cdots, S_m$ have a SDR with $\leq t$ red elements $\Longleftrightarrow S_1, \cdots, S_m, S_{m+1}, \cdots, S_{m+r}$ have a SDR. So we need to check Hall's condition for $S_1, \cdots, S_{m+r}$, let $Y = \cup_{i \in I} S_i$, if $I \subset [m]$, then $|Y| \geq |I|$ since $S_1, \cdots, S_m$ have a SDR. if $I = J_1 \cup J_2$, where $J_1 \subset [m], J_2 \subset [m+1, m+r]$, then $|J_2| \leq |R| - t, |Y| = |\cup_{i \in J_1} (S_i \backslash R)| + |R| \geq |J_1| - t + |R| = |J_1| + (|R| - t) \geq |J_1| + |J_2| = |I|$. $\square$

Application

Definition: A $r \times n (r \leq n)$ Latin rectangle is $r \times n$ matrix over $[n]$ s.t.numbers $1, 2, \cdots, n$ occurs once in each row and $\leq$once in each column. A Latin square is an $n \times n$ Latin rectangle.

Evans conjecture: If fewer than n cells of an $n \times n$ matrix are filled, then one can always complete it into a Latin square.

<u>Theorem4</u>: If $r < n$, then any given $r \times n$ Latin rectangle can be extended to an $(r+1) \times n$ Latin rectangle.

Proof: Let R be $r \times n$ LR, For $j \in [n]$, let S_j be the set of integers in $[n]$ which don't occur in the j-th column. Then it suffices to prove $S_1, \cdots, S_n$ have a SDR. Since $|S_j| = n - r$, and each $i \in [n], i$ occurs in $n - r$ sets S_j, by `Corollary2`, $S_1, \cdots, S_n$ have a SDR. $\square$

Definition: An $n \times n$ matrix $A = \{A_{ij}\}$ with $a_{ij} \geq 0$ is called doubly stochastic if $\sum_{j=1}^{n} a_{ij} = \sum_{i=1}^{n} a_{ij} = 1$ for $\forall i, j \in [n]$. If $a_{ij} = 0$ or 1, then it is a permutation matrix.

<u>Thoeorem5</u>: Every doubly stochastic matrix A is a convex combination of permutation matrixes, that is, $\exists$ permutation matrixes $P_1, \cdots, P_s$ and non-negative reals $\lambda_1, \cdots, \lambda_s$ s.t.$A = \sum_{i=1}^{s} \lambda_i P_i$ and $\sum_{i=1}^{s} \lambda_i = 1$.

Proof: Let A be an $n \times n$ doubly stochastic matrix, let m be the number of non-zero entries in A, then $m \geq n$. prove by induction on m. If $m = n$, then each non-zero entry is 1, so A itself is a permutation matrix. If $m > n$ and the results holds for matrices with $< m$ non-zero entries.

Define $S_i = \{j : a_{ij} > 0\}, i \in [n]$. If for some of the sets $S_{i_1}, S_{i_2}, \cdots, S_{i_k}, |\cup_{i=1}^{k} S_{ik}| \leq k-1$. that is all non-zero entries in rows $i_1, \cdots, i_k$ occupy at most $k-1$ columns of, say columns $j_1, \cdots, j_{k-1}$, if count by rows, we have the sum is k, but if count by columns, the sum is at most $k-1$,a contradiction. By Hall's Theorem, there is a SDR $j_1 \in S_1, j_2 \in S_2, \cdots, j_n \in S_n$. Take a permutation matrix $P_1 = (P_{ij})$ with entries $p_{ij} = 1$ iff $j = j_1$. Let $\lambda_1 = min\{a_{1j_1}, \cdots, a_{nj_n}\}$. and consider $B_1 = A - \lambda_1 P_1$. By definition of S_i, we have $\lambda_1 > 0$, matrix B_1 has at most $m-1$ non-zero entries, and the row sum and column sum of B_1 is $1-\lambda_1$. Let $A_1 = \frac{1}{1-\lambda_1} B_1$, then A_1 is a doubly stochastic matrix with less than m non-zero entries. By assumption $A_1 = \mu_2 P_2 + \cdots + \mu_s P_s$ a convex combination. Hence, $A = \lambda_1 P_1 + (1-\lambda_1)A_1 = \lambda_1 P_1 + (1-\lambda_1)\mu_2 P_2 + \cdots + (1-\lambda_1)\mu_s P_s$. Since $\sum_{i=2}^{s} \mu_i = 1$, we have $\lambda_1 + (1-\lambda_1)(\sum \mu_i) = 1$. $\square$