Combinatorics 2017 Fall
week8 note

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P-P(pigeonhole principle) and Poset

$P = (X, \leq)$ say x is an immediate predecessor of y denoted by $x \lhd y$ if $(i) x < y$ $(ii) \nexists t, s.t. x < t < y$.

<u>Fact:</u>$\forall x, y \in (X, \leq), x < y$ iff $\exists x_1, \cdots, x_k \in X$. s.t.$x \lhd x_1 \lhd \cdots \lhd x_k \lhd y$.

<u>proof:</u>" $\Leftarrow$ " trivial.
" $\Rightarrow$ "$\forall x < y$. Let $M_{xy} = \{t \in X : x < t < y\}$. prove by induction on $|M_{xy}|$. If $|M_{xy}| = 0 \Rightarrow x \lhd y$. Suppose it holds for $x < y$ with $|M_{xy}| < n$. consider $x < y$ with $|M_{xy}| = n \geq 1$. Pick $t \in M_{xy}$. consider M_{xt}, M_{ty}. since $M_{xt} \subsetneq M_{xy}$, and $M_{ty} \subsetneq M_{xy}$, by induction on $M_{xt}, M_{ty}, \exists x_1, \cdots, x_k \in X$. and $y_1, \cdots, y_l \in X$, s.t.$x \lhd x_1 \lhd \cdots \lhd x_k \lhd t$. and $t \lhd y_1 \lhd \cdots \lhd y_l \lhd y$. then $x \lhd x_1 \lhd \cdots \lhd x_k \lhd t \lhd y_1 \lhd \cdots \lhd y_l \lhd y$. □

<u>Def:</u>Hassa diagram of $P = (X, \leq)$ in the plane.

(1) Each $x \in X$ is drawn as a node in the plane.

(2) Each $x, y \in X$ with $x \lhd y$ is connected with a line segment.

(3) If $x \lhd y$,then x must appear lower than y in the plane.

<u>Fact:</u> $x < y$ iff exist path in the Hassa diagram from $x \to y$, strictly from bottom to top.

Definition:

(i) $A \subset P = (X, \leq)$ is a chain if any two of A are comparable.Let $\omega(P)$ be the maximum size of a chain of P.

(ii) $A \subset P = (X, \leq)$ is a antichain if any two of A are incomparable.Let $\alpha(P)$ be the maximum size of a antichain of P.

In Hassa diagram,$\omega(P)$ is called the height of P. and $\alpha(P)$ is called the width of P.

Fact:the set of minimal(maximal) element of $P = (X, \leq)$ forms an antichain of P.

Thm1:$\forall$ finite poset $P = (X, \leq)$,we have $\alpha(P) \cdot \omega(P) \geq |X|$.

proof: Let $P_1 = P = (X, \leq), X_1 = X, M_1 =$\{minimal elements of P_1.\} let $X_2 = X_1 - M_1, P_2 = (X_2, \leq), M_2 =$\{minimal elements of P_2\}.Inductively,define a sequence of posets $P_i = (X_i, \leq)$, and $M_i \subset P_i$ be the set of all minimal elements of $P_i, X_i = X - \cup_{j=1}^{i-1} M_j$. each M_i is an antichain of P_i,thus an antichain of P.so $|M_i| < \alpha(P)$. We keep doing this until $X_{l+1} = \phi$.

Claim: $\forall x \in M_{i+1} \exists y \in M_i, s.t. y < x$.
By definition of M_i. so we can find a chain $x_1 < x_2 < \cdots < x_l$ in P. since $X = M_1 \cup M_2 \cup \cdots \cup M_l$.
$\Rightarrow |X| = \sum_{i=1}^{l} |M_i| \leq \alpha(P) \cdot l \leq \alpha(P) \cdot \omega(P)$. □

Def: A chain is maximal if it cannot be prolonged by adding new element.

Thm2:(Dilworth) For any $P = (X, \leq)$ with $|X| \geq sr + 1$, there exists a chain of length $s + 1$ or an antichain of length $r + 1$.

proof: By contradiction,if $\alpha(P) \leq r$ and $\omega(P) \leq s$, then by Thm1, $|X| \leq \alpha(P) \cdot \omega(P) \leq sr$. a contradiction. □

THE ORDER FROM DISORDER!

Def: Let $A = (a_1, \cdots, a_n)$ be a sequence of different real numbers. $B = (a_{i_1}, \cdots, a_{i_k})$ with $i_1 < \cdots < i_k$ is a subsequence. consider the length of the increasing subsequence or decreasing subsequence.

Thm3: (Erdös–Szekeres)Let $A = (a_1, \cdots, a_n)$ be a sequence of n different real numbers. If $n \geq sr + 1$, then there exists an increasing subsequence of $s+1$ terms or a decreasing subsequence of $t+1$ terms.

proof: Define a poset.Let $X = A$,define $P = (X, \preccurlyeq), a_i \preccurlyeq a_j$ iff $a_i \leq a_j$ and $i \leq j$.
veify:

(1) $a_i = a_i, i \in [n]$.

(2) if $a_i \precneqq a_j$ then no $a_j \precneqq a_i$.

(3) if $a_i \preccurlyeq a_j$ and $a_j \preccurlyeq a_l$,then $a_i \preccurlyeq a_l$.

In $P = (X, \preccurlyeq)$,a chain $a_{i_1} \precneqq \cdots \precneqq a_{i_k}$ is an increasing subsequence. since $a_{i_j} < a_{i_{j+1}}$ and $i_j < i_{j+1}, j \in [k-1]$. an antichain $\{b_{i_1}, \cdots, b_{i_k}\}$, assume $i_1 < \cdots < i_k$, since $b_{i_j}, b_{i_{j+1}}$ are incomparable,we have $b_{i_j} < b_{i_{j+1}}, j \in [k-1]$. so an antichain is a decreasing subsequence,
Then by **Thm2**,we can get this result. □

Rational approximation(Dirichlet 1879)

Thm4: Given integer $n > 0$, for any $x \in \mathbb{R}, \exists$ a rational number p/q with $1 \leq q \leq n$ and $|x - p/q| < \frac{1}{nq} \leq \frac{1}{n}$.

proof: Let $\{x\} := x - \lfloor x \rfloor$ be the fractional part of x.consider $\{ax\}, a = 1, \cdots n+1$, which are $n+1$ real numbers in $[0, 1)$.Put these numbers into n pigeonholes$[0, \frac{1}{n}), [\frac{1}{n}, \frac{2}{n}), \cdots [\frac{n-1}{n}, 1)$. By P-P,there are 2 numbers say $\{ax\}$ and $\{bx\}$ with $a > b$ are in the same interval.then $|\{ax\} - \{bx\}| < \frac{1}{n}$.
since $ax - bx = (\lfloor ax \rfloor - \lfloor bx \rfloor) + \{ax\} - \{bx\}$.
we have $\{ax\} - \{bx\} = (a-b)x - (\lfloor ax \rfloor - \lfloor bx \rfloor)$.
Let $q = a - b$,with $1 \leq q \leq n$, and $p = \lfloor ax \rfloor - \lfloor bx \rfloor$,

we have $|qx-p| < \frac{1}{n}, i.e. |x-p/q| < \frac{1}{nq}$. $\square$

subset without divisors

Question: How large a subset $S \subset [2n]$ can be such that for $\forall i, j \in S$, we have $i \nmid j$ and $j \nmid i$?
e,g. $S = \{n+1, n+2, \cdots 2n\}$ with $|S| = n$.

Thm5: For any $S \subset [2n]$ with $|S| \geq n+1, \exists i, j \in S, a.t. i|j$.

proof: For each $k \in [n]$, Let $S_k = \{x \in S : x$ can be written as $x = 2^i(2k-1)$ for some $i\}$. Then S can be partition into $S_1, S_2, \cdots S_n$.
since $|S| \geq n+1$, by P-P $\exists |S_k| \geq 2$.
Assume $i, j \in S_k \subset S$, then either $i|j$ or $j|i$ $\square$

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Decomposition in chains and antichains

Fact: If a finite poset P has a chain(antichain) of size r then it cannot be partitioned into fewer than r antichains (chains).

Thm1: If $\alpha(P)=r$, Then P can be partitioned into r antichains.

proof: Let $A_i=\{x\in P:$ the longest chain with greatest element x has size $i\}$.Then since $\alpha(P)=r$, we have $A_i=\phi$ if $i\geq r+1$. so $P=A_1\cup A_2\cup\cdots\cup A_r$ is a partition (some of them may be empty).

show each A_i is an antichain.If not,say $x,y\in A_i, x<y$. then the largest chain with greatest element $x, x_1<x_2<\cdots<x_i=x$ can be prolonged to a longer chain $x_1<x_2<\cdots<x<y$,then the largest chain with greatest element y has size $\geq i+1$,i.e.$y\notin A_i$ $\square$

Thm2:(Dilworth's Decomposition Thm) If $\omega(P)=r$, then P can be partitioned into r chains.[This implies If $|P|\geq sr+1$, then either $\omega(P)\geq r+1$ or $\alpha(P)\geq s+1$.]

proof: By induction on $|P|$. Let a be a maximal element of P. let $s=\omega(P\prime)$, where $P\prime=P\backslash\{a\}$. Then by assumption, $P\prime$ is union of s disjoint chains $C_1,C_2,\cdots C_s$.
We will show that either $\omega(P)=s+1$ or P is union of s disjoint chains.
Note that every antichain of $P\prime$ of size s consists one element form each C_i.Let a_i be the maximal element of C_i,which belongs to some antichain of size s in $P\prime$.i.e.a_i is the greatest element in$\{x\in C_i, x$ belongs to some antichain of size $s\}\subset C_i$
Then $A=\{a_1,\cdots a_s\}$ is an antichain of $P\prime$.If there is only one antichain of size s,then A is the antichain. If not,say $a_i<a_j$,and $a_i\in B_i, a_j\in B_j$, where B_i,B_j are two different antichains of size s in $P\prime$.Since B_j contains one element x from C_i, by the maximality of a_i,we have $x<a_i$. Hence $x<a_j$ contradiction.
If $A\cup\{a\}$ is an antichain of P,i.e.$\omega(P)=s+1$, we have P is a union of $s+1$ chains $C_1,C_2,\cdots,C_s,\{a\}$.
Otherwise,we have $a>a_i$ for some i.Then $K=\{a\}\cup\{x\in C_i: x\leq a_i\}$ is a chain in P.and $\omega(P\backslash K)=s-1$.[since a_i was the maximal element of C_i in a $s-$element antichain],i.e.$K\supset\{x,x\in C_i, x$ be-

longs to same antichain of size $s\}$
So,$P\backslash K$ is a union of $s-1$ chains,and P is a union of s chains. $\square$

Second proof of Hall's Thm: Suppose $S_1, S_2, \cdots S_m$ satisfies $|S(I)| \geq |I|, \forall I \subset [m]$, where $S(I) = \cup_{i \in I} S_i$.construct a poset $P = (Y, \leq)$ with $Y = S_1 \cup \cdots \cup S_m \cup \{y_1, y_2, \cdots, y_m\}$.with $x < y_i$ iff $x \in S_i$.and no other comparable.
Let $X = S_1 \cup \cdots \cup S_m$. then X is an antichain of P. we claim that $\omega(P) = |X|$. In fact,if A is an antichain, let $I = \{i : y_i \in A\}$,then A contains no point of $S(I)$,for $x < y_i$ if $x \in S_i$.so $|A| \leq |I| + |X| - |S(I)| \leq |X|$ as claimed.
By Dilworth's Decomposition Theorem, P can be partition into $|X|$ chains,and each of these chains contain a point of X.
Let the chain through y_i be $\{x_i, y_i\}$,then $(x_1, x_2, \cdots, x_m)$ is a S-DR.since $x_i \in S_i$ and $x_i \neq x_j$(since chains are disjoint). $\square$

Sperner's Thm:
A set system (i.e.family of subsets)$\mathscr{F}$ is an antichain(or sperner system) if $A, B \in \mathscr{F}$ and $A \neq B$ then $A \not\subseteq B$.

e.g. antichain over $[n]$,such as $\binom{[n]}{k}, k = 0, 1, \cdots, n$.

Sperner's Thm: Let $\mathscr{F}$ be a set system over $[n]$,If $\mathscr{F}$ is an antichain,then $|\mathscr{F}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$.

LYM Inequality:$\mathscr{F}$ is an antichain over $[n]$.then $\sum\limits_{A \in \mathscr{F}} \binom{n}{|A|}^{-1} \leq 1$.

LYM Inequality $\Rightarrow$ Sperner's Thm:since $\binom{n}{|A|} \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$
$1 \geq \sum\limits_{A \in \mathscr{F}} \binom{n}{|A|}^{-1} \geq |\mathscr{F}| \cdot \binom{n}{\lfloor \frac{n}{2} \rfloor}^{-1}$.

Poof of LYM Inequality:$\forall A \subset [n]$, Let's say a permutation$(x_1, \cdots, x_n)$ of $[n]$ contains A if $\{x_1, \cdots, x_{|A|}\} = A$. then A is contained in exactly $|A|!(n-|A|)! \leq n!$ permutations. Since $\mathscr{F}$ is an antichain,then each permutation contains at most one $A \in \mathscr{F}$, Then

$$\sum_{A\in\mathscr{F}} |A|!(n-|A|)! \le n! \Rightarrow \sum_{A\in\mathscr{F}} \binom{n}{|A|}^{-1} \le 1.\square$$

Bollobás Thm: Let $A_1, \cdots, A_m$ and $B_1 \cdots, B_m$ be two sequence of sets,such that $A_i \cap B_j = \phi$ iff $i = j$.
Then $\sum_{i=1}^{m} \binom{a_i+b_i}{a_i}^{-1} \le 1$, where $|A_i| = a_i, |B_i| = b_i, i \in [m]$. In particular,if $a_i = a$ and $b_i = b$ for all $i \in [m]$, then $m \le \binom{a+b}{a}$.

Fact: Bollobás Thm $\Rightarrow$ LYM Inequality $\Rightarrow$ Sperner's Thm.

proof of Bollobás Thm: Let $X = (\cup_{i=1}^{m} A_i) \cup (\cup_{i=1}^{m} B_i)$, suppose $|X| = n. \forall A, B \subset X$, and $A \cap B = \phi$,say a permutation $(x_1, \cdots, x_n)$ of X separates the pair (A, B) if no element of B precedes an element of A,i.e.if $x_k \in A, x_l \in B$ then $k < l$.
we claim that each permutation separates at most one pair $(A_i, B_i), i \in [m]$. Indeed, suppose $(x_1, \cdots, x_n)$ separates $(A_i, B_i), (A_j, B_j), i \ne j$. and assume $\max\{k : x_k \in A_i\} \le \max\{k : x_k \in A_j\}$ since $(x_1, \cdots, x_n)$ separates (A_j, B_j),we have
$\min\{k : x_k \in B_j\} > \max\{k : x_k \in A_j\} \ge \max\{k : x_k \in A_i\}$
$\Rightarrow A_i \cap B_j = \phi$, a contradiction.
Now count the number N_i of permutations separated one fixed pair say (A_i, B_i), First choose $a_i + b_i$ positions for $A \cup B$. there are $\binom{n}{a_i+b_i}$ choices.Then A occupy the first a_i positions,giving $a_i!$ choices for the order of A,and $b_i!$ choices for the order of B,the remaining elements can be arranged in $(n - a_i - b_i)!$ ways.So $N_i = \binom{n}{a_i+b_i} a_i! b_i! (n-a_i-b_i)! = n! \binom{a_i+b_i}{a_i}^{-1}$.
Summing over all pairs,we have

$$\sum_{i=1}^{m} n! \binom{a_i+b_i}{a_i}^{-1} \le n! \Rightarrow \sum_{i=1}^{m} \binom{a_i+b_i}{a_i}^{-1} \le 1.\square$$