# Combinatorics 2017 Fall

## week8 note

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**Reference:**
Extremal Combinatorics with applications in Computer Science. 2nd Edition.Stasys Jukna,Springer.

### *2017/11/14*

**Intersecting Family**

**<u>Definition:</u>** $\mathcal{F} \subset 2^X$, if $A, B \in \mathcal{F}, A \cap B \neq \emptyset$.

**Example**

- fix $a \in X, \mathcal{F} = \{A \subset X : a \in A\}.|\mathcal{F}| = 2^{n-1}$.
- $|X| = n$, n is odd. $\mathcal{F} = \{A \subset X : |A| > \frac{n}{2}\}, |\mathcal{F}| = \sum_{i=\frac{n+1}{2}}^{n} \binom{n}{i}$.

**Fact:** For any intersecting family $\mathcal{F} \subset 2^X, |\mathcal{F}| \leq 2^{n-1}$.

**k-uniform:** $\mathcal{F} \subset \binom{X}{k}$, intersecting family.

**Example**

- $\mathcal{F} = \{A \subset X : a \in A, |A| = k\}.|\mathcal{F}| = \binom{n-1}{k-1}$.
- $n < 2k, |\mathcal{F}| = \binom{X}{k}$.

**<u>Theorem1</u>**(Erdős-Ko-Rado *EKR*) If $n \geq 2k.\mathcal{F} \subset \binom{X}{k}$ is intersecting family, then $|\mathcal{F}| \leq \binom{n-1}{k-1}$.

Proof: consider a cyclic permutation $\pi = (a_1, a_2, \cdots, a_n)$. in total $(n-1)!$ cyclic permutation. say $\pi$ contain a subset A if A appears as consecutive elements in $\pi$. Suppose $A \in \mathcal{F}$ and A is contained in $\pi$. $\exists 2(k-1)$ subsets B contain in $\pi$. $B \neq A, B \cap A \neq \emptyset$. These 2(k-1) subsets can be partitioned into k-1 pairs of disjoint subsets and $\mathcal{F}$ contains at most one subset from each pair. $N = \#(\pi, A), s.t.\ A \in \mathcal{F}$. Fix A, $\#\{\pi\ contain\ A\} = k!(n-k)!$. $N = \sum_{A\in\mathcal{F}} k!(n-k)! = |\mathcal{F}|k!(n-k)! \leq \sum_{\pi} k = (n-k)!k$. $\square$

**<u>Theorem2</u>**(EKR) If $n > 2k.$, the intersecting family $\mathcal{F} \subset \binom{X}{k}$ with $|\mathcal{F}| = \binom{n-1}{k-1}$ must be a star.

Proof:

- (1) $\forall \pi$ contain exactly k subsets of $\mathcal{F}$.
- (2) if $\pi = (a_1, a_2, \cdots, a_n).\pi$ contains $A_j \in \mathcal{F}$. where $A_j = (a_j, \cdots, a_{j+k-1})$. Fix $\pi$, let $A_1 \cap A_2 \cap \cdots \cap A_k = \{a_k\}$. If $A_0 \in \mathcal{F}, a_k \in A_0$. Aim: $\mathcal{F} = \binom{A_1 \cup A_k}{k}$

  **<u>Claim1</u>**:$\forall B \in \binom{A_1 \cup A_k}{k}$ and $a_k \in B$, then $B \in \mathcal{F}$.

  Proof: $B \cap A_1 = \{a_k, b_1, \cdots, b_s\}, B \cap A_k = \{a_k, c_1, \cdots, b_t\}, s+t+1 = k.\pi = (A_1 \setminus B, b_1, \cdots, b_s, c_1, \cdots, b_t, A_k \setminus B \cdots)$. By (2), $B \in \mathcal{F}$. $\square$

  **<u>Claim2</u>**:$A_0 \subseteq (A_1 \cup A_k) \setminus \{a_k\}$.

  Proof: If $A_0 \nsubseteq A_1 \cup A_k, |A_0 \cap (A_1 \cup A_k)| \leq k-1$. $\exists B \in A_1 \cup A_k, |B| = k, a_k \in B$. By Claim1, $B \in \mathcal{F}$. But $A_0 \cap B = \emptyset$. contradiction. $\square$

  **<u>Claim3</u>**:$\binom{A_1 \cup A_k}{k} \subset \mathcal{F}$.

  Proof: $\forall C \in \binom{A_1 \cup A_k}{k}, a_k \notin C, C \cap A_0 = \{b_1, b_2, \cdots, b_s\}$. $\pi = (\cdots b_1, \cdots, b_s \cdots), C \in \mathcal{F}$. $\square$

  **<u>Claim4</u>**: $\mathcal{F} = \binom{A_1 \cup A_k}{k}$.

  Proof:$\exists B \in \mathcal{F}. B \notin A_1 \cup A_k. |B \cap (A_1 \cup A_k)| \leq k-1$. Then $\exists C \subset A_1 \cup A_k, |C| = k, C \cup B = \emptyset$. But by Claim3, $C \in \mathcal{F}$. contradiction. $\square$

**Fisher Inequality:** Let $A_1, \cdots, A_m \in 2^X$. be distinct subsets. if $|A_i \cap A_j| = k.\exists i \neq j$. then $m \leq |X| = n$.

Proof: $\forall$ vectors $x, y \in \mathbb{R}$. Let $\langle x, y\rangle = x_1y_1 + \cdots + x_ny_n$. let $v_1, \cdots, v_m$ be columns of the incidence matrix of $A_1, \cdots, A_m$, then it suffices to show that $v_1, \cdots, v_m$ are linearly independent over the reals. Assume the contrary,i.e.$\exists \sum_{i=1}^{m} \lambda_i v_i = 0$ with not all coefficients being zero. Since $\langle v_i, v_j\rangle = \begin{cases} |A_i|, i = j \\ k, i \neq j. \end{cases}$ then

$$0 = \langle \sum_{i=1}^{m} \lambda_i v_i, \sum_{j=1}^{m} \lambda_j v_j \rangle = \sum_{i=1}^{m} \lambda_i^2 \langle v_i, v_i \rangle + \sum_{1\leq i \neq j \leq m} \lambda_i \lambda_j \langle v_i, v_j \rangle$$
$$= \sum_{i=1}^{m} \lambda_i^2 |A_i| + \sum_{1\leq i \neq j \leq m} \lambda_i \lambda_j (|A_i| - k) + k(\sum_{i=1}^{m} \lambda_i^2)^2$$

Note that $|A_i| \geq k$ for all i and $|A_i| = k$ for at most one i. since otherwise the intersection condition would not be satisfied.i.e if $\exists |A_i| \geq k, |A_j| = k$, then $|A_i \cap A_j| < k$. contradiction. If only one $\lambda_i \neq 0$, then $k(\sum_{i=1}^{m} \lambda_i^2)^2 > 0$. Here RHS$> 0$. IF$\geq 2, \lambda_i \neq 0$, then $(\sum_{i=1}^{m} \lambda_i^2)^2(|A_i| - k) > 0$. Both cases we have RHS$> 0$. a contradiction. $\square$

***2017/11/17***

Preliminaries A real symmetric matrix of order n.

- $\lambda_1 \neq \lambda_2$ different eigenvalues, $Au = \lambda_1 u, Av = \lambda_1 v$, then $(u, v) = 0$.
- if $\lambda$ is an eigenvalue of A, then $\lambda$ is a real number.
- if U is A-invariant subspace of $\mathbb{R}^n$, then $U^{\perp}$ is also A-invariant.
- if $0 \neq U$ is A-invariant subspace of $\mathbb{R}^n$, then U contains a real eigenvalue of A.
- $\mathbb{R}^n$ has an orthogonal basis consisting of eigen vectors of A.

**Definition:** Kneser graph KG(n,k) for $n \geq 2k$ is a graph with vertex set $\binom{[n]}{k} s.t. A, B \in \binom{[n]}{k}, A \sim B$ iff $A \cap B = \emptyset$.

**Fact:** an intersecting family $\mathcal{F} \subset \binom{[n]}{k}$ is an independent set in KG(n,k). So EKR Theorem $\Leftrightarrow \alpha(KG(n,k)) \leq \binom{n-1}{k-1}$.

**Definition:** adjacency matrix $A_G = (a_{ij})_{m \times n}$ of an n-vertex graph G is defined by $a_{ij} = \begin{cases} 1, if\ i \sim j\ in\ G \\ 0, oterwise. \end{cases}$

**Definition:** The eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ of $A_G$ is called the eigenvalues of the graph G, The eigenvalues $v_1, v_2, \cdots, v_n$ of $A_G s.t. A_G v_i = \lambda_i v_i, \|v_i\| = 1, v_i \perp v_j$ are called orthogonal eigenvectors of G.

**Definition:** A graph G is regular if all vertices have the same degree.

**Theorem1**(Hoffman's Theorem) If an n-vertex graph G is regular with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, then $\alpha(G) \leq n\frac{-\lambda_n}{\lambda_1 - \lambda_n}$.

**Proof:** Let $v_1, \cdots, v_n$ be the corresponding to orthogonal eigenvectors of $\lambda_1, \cdots, \lambda_n$. let I be an independent set od G with $|I| = \alpha(G)$, let $e_I$ be the column indicator vector of I, write $e_I = \sum_{i=1}^{n} \alpha_i v_i$, Then $|I| = \langle e_I, e_I \rangle = \sum_{i=1}^{n} \alpha_i^2$. and $\alpha_i = \langle e_I, v_i \rangle$. since G is regular(all degree d), we have $\lambda_1 = d$ and $v_1 = (\frac{1}{\sqrt{n}}, \cdots, \frac{1}{\sqrt{n}})^T$. so $\alpha_1 = \langle e_I, v_1 \rangle = \frac{|I|}{\sqrt{n}}$. since I is an independent set of G, $e_I^T e_I = \sum_{i,j} x_i a_{ij} x_j = 0$. where $e_I = (x_1, \cdots, x_n)^T$.

$$
\begin{aligned}
0 = e_I^T A_G e_I = e_I^T \sum_{i=1}^{m} \alpha_i x_i v_i &= \sum_{i=1}^{m} \alpha_i \lambda_i \langle e_I, v_i \rangle \\
&= \sum_{i=1}^{n} \alpha_i^2 \lambda_i \geq \alpha_1^2 \lambda_1 + (\alpha_2^2 + \cdots + \alpha_n^2)\lambda_n \\
&= (\frac{|I|}{\sqrt{n}})^2 \lambda_1 + (|I| - \frac{|I|^2}{\sqrt{n}})\lambda_n
\end{aligned}
$$

$$
\Longrightarrow 0 \geq \frac{|I|^2}{\sqrt{n}}\lambda_1 + (|I| - \frac{|I|^2}{\sqrt{n}})\lambda_n = |I|(\frac{|I|}{\sqrt{n}}\lambda_1 + \lambda_n - \frac{|I|}{\sqrt{n}}\lambda_n)
$$

$$
\Longrightarrow \frac{|I|}{\sqrt{n}}\lambda_1 + \lambda_n - \frac{|I|}{\sqrt{n}}\lambda_n \leq 0 \Longrightarrow \frac{|I|}{\sqrt{n}}(\lambda_1 - \lambda_n) \leq -\lambda_n
$$

$$
\Longrightarrow |I| \leq n\frac{-\lambda_n}{\lambda_1 - \lambda_n}
$$

$\square$

**<u>Lemma2:</u>** The eigenvalues of Kg(n,k) are $u_j = (-1)^j \binom{n-k-j}{k-j}$ of multiplicity $\binom{n}{j} - \binom{n}{j-1}$ for every $0 \leq j \leq k$.

**Recall:** Any intersecting family $\mathcal{F}$ is an independent set of KG(n,k). Let $\alpha(G) = max|I|$ over all independent set I of G. Thus, EKR Theorem $\Longleftrightarrow \alpha(KG(n,k)) \leq \binom{n-1}{k-1}$. The second proof of EKR Theorem: consider the eigenvalues of KG(n,k), say $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{\binom{n}{k}}$, where $\lambda_1 = \binom{n-k}{k} = u_0, \lambda_{\binom{n}{k}} = -\binom{n-k-1}{k-1} = u_1$. By Hoffman's bound, $\alpha(KG(n,k)) \leq$

$$
\binom{n}{k}\frac{-\lambda_{\binom{n}{k}}}{\lambda_1 - \lambda_{\binom{n}{k}}} = \binom{n}{k}\frac{\binom{n-k-1}{k-1}}{\binom{n-k}{k} + \binom{n-k-1}{k-1}} = \binom{n-1}{k-1}.
$$

Let $X = [v]$ be a set of points.

**<u>Definition:</u>** A $(v, k, \lambda)$ design over X is a collection D of distinct subsets of X(called blocks) such that:

- (1) $\forall B \in D, |B| = k$.
- (2) $\forall$ pair of distinct points is contained in exactly $\lambda$ blocks. Denote $b = |D|$, if we replace (2) by

- (2') every t-subset of X is contained in exactly $\lambda$ blocks.

then D is called a $t-(v,k,\lambda)$ design. If $\lambda = 1$, a $t-(v,k,\lambda)$ design is called a Steiner system S(t,k,v). If $b = v$, then a $t-(v,k,\lambda)$ design is called symmetric. A family of sets $\mathcal{F}$ is called r-regular if every point lies in exactly r sets, r is the replication number of $\mathcal{F}$.

**Theorem3:** D is a $(v,k,\lambda)$ design with b blocks, then D is r-regular satisfying $r(k-1) = \lambda(v-1)$ and $bv = vr$.

Proof: For any fixed $a \in X$, assume a occurs in $r_a$ blocks, double count the cordinality of $S = \{(x,B) : B \in D; a, x \in B; x \neq a\}$

- there are v-1 possibilities of $x(x \neq a)$, and for each x there are exactly $\lambda$ blocks B containing both x and a, Hence $|S| = (v-1)\lambda$.
- for each of the $r_a$ blocks B containing a, there are k-1 ways to choose an element $x \in B \setminus a$. so $|S| = r_a(k-1)$.

Hence $r_a(k-1) = (v-1)\lambda, r_a$ is independent of a i.e. D is regular. To prove $bv = vr$, double count $T = \{(X,B) : B \in D, x \in B\}$.

- $\forall x \in X, B$ can be chosen in r ways, so $|T| = vr$.
- $\forall B \in D, x$ can be chosen in k ways, so $|T| = nk$.

□

A finite linear space over a set X is a family $\mathcal{L}$ of its subsets, called lines, such that:

- every line contains at least two points.
- any two points are on exactly one line.

**Theorem4**: If $\mathcal{L}$ is a linear space over X, then $|\mathcal{L}| \geq |X|$, with equality holds iff any two lines share exactly one point. Proof:(Conway) Let $b = |\mathcal{L}| \geq 2$, and $v = |X|, \forall x \in X$, let $r_x$ be the replication number,i.e. the number of lines in $\mathcal{L}$ containing x. If $x \notin L$, then $r_x \geq |L|$ since there are $|L|$ lines joining x to the points on $L$, suppose $b \leq v$, for $x \notin L$, we have $b(V - |L|) = bv - b|L| \geq bv - v|L| \geq v(b - r_x)$. Hence

$$b = \sum_{L \in \mathcal{L}} 1 = \sum_{L \in \mathcal{L}} \sum_{x : x \notin L} \frac{1}{v - |L|} \leq \frac{b}{v} \sum_{L \in \mathcal{L}} \sum_{x : x \notin L} \frac{1}{b - r_x}$$
$$= \frac{b}{v} \sum_{x : x \notin L} \sum_{L \in \mathcal{L}} \frac{1}{b - r_x} = \frac{b}{v} \sum_{x \in X} 1 = b.$$

This implies all inequalities are equalities so that b=v. and $v_x = |L|$ whenever $x \notin L$,i.e. any line containing x share one point with L. □