Cycles with consecutive odd lengths

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Abstract

In this paper we prove that there exists an absolute constant $c > 0$ such that for every natural number k, every non-bipartite 2-connected graph with average degree at least ck contains k cycles with consecutive odd lengths. This implies the existence of the absolute constant $d > 0$ that every non-bipartite 2-connected graph with minimum degree at least dk contains cycles of all lengths modulo k, thus providing an answer (in a strong form) to a question of Thomassen in [14]. Both results are sharp up to the constant factors.

1 Introduction

The research of cycles has been fundamental since the beginning of graph theory. One of various problems on cycles which have been considered is the study of cycle lengths modulo a positive integer k. Burr and Erdős [6] conjectured that for every odd k, there exists a constanct c_k such that every graph with average degree at least c_k contains cycles of all lengths modulo k. In [2], Bollobás resolved this conjecture by showing that $c_k = 2[(k + 1)^k - 1]/k$ suffices. Thomassen [14, 15] strengthened the result of Bollobás by proving that for every k (not necessarily odd), every graph with minimum degree at least $4k(k + 1)$ contains cycles of all even lengths modulo k, which was improved to the bound $2k - 1$ by Diwan [5]. Note that in case k is even, any integer congruent to l modulo k has the same parity with l, and thus we cannot expect that there are cycles of all lengths modulo k in bipartite graphs (even with sufficient large minimum degree). On the other hand, Thomassen [14] showed that for every k there exists a least natural number $f(k)$ such that every non-bipartite 2-connected graph with minimum degree at least $f(k)$ contains cycles of all lengths modulo k. Here the 2-connectivity condition cannot be further improved, as one can easily construct a non-bipartite connected graph with arbitrarily large minimum degree but containing a unique (also arbitrary) odd cycle. Thomassen [14] remarked that the upper bound for $f(k)$ obtained by him is perhaps "far too large" and asked if $f(k)$ can be bounded above by a polynomial.

Bondy and Vince [3] resolved a conjecture of Erdős by showing that every graph with minimum degree at least 3 contains two cycles whose lengths differ by one or two. Verstraëte

[17] proved that if a graph G has average degree at least $8k$ and even girth g then there are $(g/2-1)k$ cycles of consecutive even lengths in G. In an attempt to extend the result of Bondy and Vince, Fan [9] showed that every graph with minimum degree at least $3k - 2$ contains k cycles of consecutive even lengths or consecutive odd lengths. Sudakov and Verstraëte proved [11] that if a graph G has average degree $192(k + 1)$ and girth g then there are $k^{\lfloor (g-1)/2 \rfloor}$ cycles of consecutive even lengths in G, strengthening the above results in the case that k and g are large. It is natural to ask if one can pursue the analogous result for odd cycles. In this paper, we show that this indeed is the case by the following theorem.

Theorem 1. *There exists an absolute constant $c > 0$ such that for every natural number k, every non-bipartite 2-connected graph G with average degree at least ck and girth g contains at least $k^{\lfloor (g-1)/2 \rfloor}$ cycles of consecutive odd lengths.*

We shall show that $c = 456$ suffices. We point out that the non-bipartite condition here is necessary and the 2-connectivity condition cannot be improved. In view of the *Moore Bound*, our lower bound on the number of cycles is best possible (up to constant factor) for infinitely many integers k when $g \le 8$ or $g = 12$. And more generally, the well-known conjecture that the minimal order of graphs with minimal degree k and girth g is $O(k^{\lfloor (g-1)/2 \rfloor})$ would imply that our result gives the correct order of magnitude for other values of g. More details about this conjecture can be found in the survey by Exoo and Jajcay [8].

Let G be a graph as in Theorem 1. It is clear that there are at least k cycles of consecutive odd lengths in G, which assures that G contains cycles of all odd lengths modulo k (whenever k is even or odd). Together with the aforementioned result of Diwan on cycles of all even lengths modulo k, we answer the question of Thomassen by improving the upper bound of $f(k)$ to a linear function by the following corollary.

Corollary 2. *There exists an absolute constant $d > 0$ such that for every natural number k, every non-bipartite 2-connected graph with minimum degree at least dk contains cycles of all lengths modulo k.*

This bound is sharp up to the constant factor: the complete graph K_{k+1} contains cycles of all lengths but 2 modulo k and thus shows that $f(k) \ge k + 1$.

All graphs considered are simple and finite. Let G be a graph. We denote the number of vertices in G by $|G|$, the vertex set by $V(G)$, the edge set by $E(G)$ and the minimum degree by $\delta(G)$. If $S \subset V(G)$, then $G - S$ denotes the subgraph of G obtained by deleting all vertices in S (and all edges incident with some vertex in S). If $S \subset E(G)$, then $G - S$ is obtained from G by deleting all edges in S. Let A and B be subsets of $V(G)$. An (A, B)-*path* in G is a path with one endpoint in A and the other in B. If A only contains one vertex a, then we simply write (A, B)-path as (a, B)-path. We say a path P is *internally disjoint* from A, if no vertex except the endpoints in P is contained in A.

The rest of paper is organized as follows. In Section 2, we establish Theorem 1 based on the approach of [11]. Section 3 contains some remarks and open problems. We make no effort to optimize the constants in the proofs and instead aim for simpler presentation.

2   The proof

Before presenting the proof, we state the following useful lemma from [17], which will be applied multiple times and become essential in the proof of our main theorem. By *a cycle with a chord*, we mean the union of a cycle and a chord of this cycle.

Lemma 3. (Verstraëte [17], Lemma 2) *Let C be a cycle with a chord, and let (A, B) be a nontrivial partition of $V(C)$. Then C contains (A, B)-paths of every length less than $|C|$, unless C is bipartite with bipartition (A, B).*

Proof of Theorem 1. We shall show that it suffices to use $c = 456$. Let G be a non-bipartite 2-connected graph with average degree at least $456 \cdot k$ and girth g. Our goal is to show that G contains $t := k^{\lfloor (g-1)/2 \rfloor}$ cycles of consecutive odd lengths. As it holds trivially when $k = 1$, we assume that $k \geq 2$.

Let G_b be a bipartite subgraph of G with the maximum number of edges. It is easy to see that G_b is a connected spanning subgraph of G with average degree at least $228k$. Since G is non-bipartite, there exists an edge $xy \in E(G)$ such that both x and y lie in the same part of the bipartition of G_b. Let T be the breadth first search tree in G_b with root x, and let L_i be the set of vertices of T at distance i from its root x for $i \geq 0$ (so $L_0 := \{x\}$). As T is a spanning tree, it follows that $V(G) = V(G_b) = \cup_{i \geq 0} L_i$. For any two vertices a and b in the tree T, we denote the unique path in T with endpoints a and b by T_{ab}. By the choice of edge xy, $D := T_{xy} \cup xy$ is clearly an odd cycle.

By the definition of T, every edge of G_b joins one vertex in L_i to another in L_{i+1} for some $i \geq 0$. Thus, we have

$$\sum_i e(L_i, L_{i+1}) = e(G_b) \geq 114k \cdot |G_b| = 114k \cdot \sum_i |L_i| \geq 57k \cdot \sum_i (|L_i| + |L_{i+1}|).$$

So there must exist some $i \geq 0$ such that the induced (bipartite) subgraph $G_i := G_b[L_i \cup L_{i+1}]$ has average degree at least $114k$. We now use the following lemma from [12] to find a long cycle with at least one chord (in fact with many chords) in G_i.

Lemma 4. (Sudakov and Verstraëte [12], Lemma 2.1) *Let G be a graph of average degree at least $12(d + 1)$ and girth g. Then G contains a cycle C with at least $\frac{1}{3} d^{\lfloor (g-1)/2 \rfloor}$ vertices of degree at least $6(d + 1)$, each of which has no neighbors in $G - V(C)$.*

Recall that we want to find $t = k^{\lfloor (g-1)/2 \rfloor}$ cycles of consecutive odd lengths. Since $114k \geq 12(9k + 1)$ for $k \geq 2$ and the girth of G_i is at least g, this lemma shows that there is a cycle C with a chord in G_i, satisfying

$$|C| \geq \frac{1}{3} \cdot (9k)^{\lfloor (g-1)/2 \rfloor} \geq 2(t + 1). \tag{1}$$

We notice that $(V(C) \cap L_i, V(C) \cap L_{i+1})$ is the unique bipartition of C (this is for the use of Lemma 3 later).

Let T' be the minimal subtree of T whose set of leaves is precisely $V(C) \cap L_i$, and let z be the root of T' (i.e., the one at the shortest distance from x). By the minimality of T', z

has at least two branches in T'. Let the depth of T' (i.e., the distance between its root z and its leaves) be j.

Recall that D is an odd cycle in G consisting of T_{xy} and edge xy. Depending on whether $V(D)$ intersects $V(C) \cup V(T') - \{z\}$ or not, we distinguish the following two cases.

Case 1. $V(D)$ and $V(C) \cup V(T') - \{z\}$ are disjoint.

In this case, the tree T contains a path Z from z to $V(D)$, which is internally disjoint from $V(C) \cup V(T')$. Note that G is 2-connected, so there are two disjoint $(V(C) \cup V(T'), V(D))$-paths, say P and Q, in G, which are internally disjoint from $V(C) \cup V(T') \cup V(D)$. Routing P, Q through Z if necessary, we may assume that P is from z to $p \in V(D)$, and Q is from $w \in V(C) \cup V(T') - \{z\}$ to $q \in V(D) - \{p\}$.

Based on the location of w, we divide the remainder of this case into two parts. Let us first consider when $w \in V(T') - \{z\}$. Let A be the set of all leaves in the subtree of T' with root w, and let $B := V(C) - A$. As $w \neq z$, we see $B^* := V(C) \cap L_i - A$ is nonempty, which shows that (A, B) is not the bipartition of C. By Lemma 3 and the equation (1), C contains (A, B)-paths of all even lengths up to $2t + 1$, all of which in fact are (A, B^*)-paths because C is bipartite. To find t cycles of consecutive odd lengths, now it is enough to show that for any $a \in A$ and $b \in B^*$, there exists an (a, b)-path internally disjoint from C and with a fixed odd length. To see this, first note that paths T'_{aw} and T'_{bz} are disjoint and of fixed lengths; since D is an odd cycle, one can choose a (p, q)-path R in D (out of two choices) such that the (a, b)-path $T'_{aw} \cup Q \cup R \cup P \cup T'_{zb}$ in $G - E(C)$ is of a fixed odd length.

Now we consider the situation when $w \in V(C) - V(T')$. So $w \in V(C) \cap L_{i+1}$, and clearly $(\{w\}, V(C) - \{w\})$ is not a bipartition of C. By Lemma 3 and the equation (1), C contains $(w, V(C) - \{w\})$-paths of all odd lengths up to $2t + 1$, all of which are $(w, V(C) \cap L_i)$-paths because C is bipartite. For any $u \in V(C) \cap L_i$, the length of path T'_{uz} is fixed, that is the depth j of T'. Therefore, $C \cup T'$ contains (z, w)-paths of all lengths in $\{1 + j, 3 + j, \ldots, 2t + 1 + j\}$. Since D is an odd cycle, similarly as in the last paragraph one can choose a (p, q)-path R in D such that the (z, w)-path $P \cup R \cup Q$ is internally disjoint from $C \cup T'$ and of the same parity as j. Putting the above paths together, we see that G contains at least t cycles of consecutive odd lengths. This completes the proof of Case 1.

Case 2. $V(D)$ intersects $V(C) \cup V(T') - \{z\}$.

Let $w \in V(D) \cap (V(C) \cup V(T') - \{z\})$ be the vertex such that T_{wy} is the shortest path among all choices of w. Note that now $T_{wz} = T'_{wz}$ is a subpath in the cycle D.

Again we distinguish on the location of w. First we consider when $w \in V(T') - \{z\}$. Let A be the set of all leaves in the subtree of T' with root w, and $B^* := V(C) \cap L_i - A$. By the same proof as in the second paragraph of Case 1, we conclude that C contains (A, B^*)-paths of all even lengths up to $2t + 1$. We also notice that D consists of two (w, z)-paths T'_{wz} and $P := D - T'_{wz}$, whose lengths are of opposite parities. For any $a \in A$ and $b \in B^*$, $T'_{aw} \cup T'_{wz} \cup T'_{zb}$ forms an (a, b)-walk with a fixed even length, that is twice of the depth of T'. Together with the definitions of w and D, this proves that $T'_{aw} \cup P \cup T'_{zb}$ is an (a, b)-path internally disjoint from C and with a fixed odd length, which, combining with these (A, B^*)-paths with lengths $2, 4, \ldots, 2t$ in C, comprise t cycles of consecutive odd lengths in G.

We are left with the case when $w \in V(C) \backslash V(T')$. This shows that $w \in V(C) \cap L_{i+1}$ and $(\{w\}, V(C) - \{w\})$ is not the bipartition of C. By Lemma 3 as well as the equation (1), C contains $(w, V(C) - \{w\})$-paths of all odd lengths up to $2t+1$. Similarly as the previous proof, these odd paths are actually $(w, V(C) \cap L_i)$-paths. Recall that the depth of T' is j. Therefore, $C \cup T'$ contains (w, z)-paths of all lengths in $\{1 + j, 3 + j, \ldots, 2t + 1 + j\}$ and particularly the sub-path T_{wz} of D has length $1 + j$. We know D is an odd cycle, so $D - T_{wz}$ is a (w, z)-path internally disjoint from $C \cup T'$ whose length is of the same parity as j. Putting $D - T_{wz}$ and these (w, z)-paths in $C \cup T'$ together, we find at least t cycles of consecutive odd lengths. This proves Case 2, finishing the proof of Theorem 1. □

3   Concluding remarks

In [3] Bondy and Vince gave an infinite family of non-bipartite 2-connected graphs with arbitrarily large minimum degree but containing no two cycles whose lengths differ by one. This tells that Theorem 1 is sharp from another point of view. The situation changes completely when the connectivity increases. Fan [9] showed that every non-bipartite 3-connected graph with minimum degree at least $3k$ contains $2k$ cycles of consecutive lengths. A conjecture of Dean (see [4]) also considered the connectivity and asserted that every k-connected graph contains a cycle of length 0 modulo k. We observe that this is best possible for odd k (if true), as the complete bipartite graph $K_{k-1,k-1}$ is $(k-1)$-connected but has no cycle of length 0 modulo k.

Thomassen showed in [13] that graphs of minimum degree at least 3 and large girth share many properties with graphs of large minimum degree. For example, he proved that if G is a graph of minimum degree at least 3 and girth at least $2(k^2+1)(3 \cdot 2^{k^2+1} + (k^2+1)^2 - 1)$, then G contains cycles of all even lengths modulo k (while we have seen the analogous result for graphs of large minimum degree in $[2, 5, 14, 15]$). We have also seen that graphs of large minimum degree contain cycles of consecutive even lengths (e.g., results from $[11, 17]$), however to the best of our knowledge it is not known if there exists a natural number $g(k)$ such that every graph of minimum degree at least 3 and girth at least $g(k)$ contains k cycles of consecutive even lengths. Similar question can be raised with respect to Theorem 1 as well.

In the 1970s Erdős and Simonovits [7] asked to determine the *chromatic profile*

$$\delta_\chi(H, k) := \inf\{c : \delta(G) \geq c|G| \text{ and } H \not\subset G \Rightarrow \chi(G) \leq k\}$$

for every graph H (we refer interested readers to [1] for related topics). Since then, very little is known about $\delta_\chi(H, k)$ for graphs H other than K_3. Recall a result of [14] that for every k there exists a least natural number $f(k)$ such that every non-bipartite 2-connected graph with minimum degree at least $f(k)$ contains cycles of all lengths modulo k. Thomassen [16] proved that for every $c > 0$ and odd integer $l \geq 5$, every C_l-free graph G with minimum degree at least $c|G|$ has chromatic number $\chi(G)$ less than $(l + f(2l - 8))/c$. We conclude this paper with the following.

Lemma 5. *For arbitrary fixed odd integer $l \geq 5$, it holds that*

$$\Theta\left(\frac{1}{(k+1)^{4(l+1)}}\right) \leq \delta_\chi(C_l, k) \leq \Theta\left(\frac{l}{k}\right).$$

Proof. The upper bound can be obtained easily by combining Thomassen's result [16] and Corollary 2. We turn to the lower bound. Let $N(g, k)$ be the minimum $|G|$ over all graphs G with girth at least g and chromatic number at least k. The proof of [10] shows that $N(g, k) \leq \Theta(k^{4g+1})$ when $k \geq 144$.

It suffices to prove $\delta_\chi(C_l, k) > k/N(l+1, k+1)$. Let G_0 be a graph of minimum order with girth at least $l+1$ and chromatic number at least $k+1$, i.e., $|G_0| = N(l+1, k+1)$, then by the minimality it holds that $\chi(G_0) = k+1$ and $\delta(G_0) \geq k$. We then construct a graph G obtained from G_0 by replacing every vertex with an independent set of size t and every edge with a complete bipartite graph. Clearly G contains no C_l (in fact there is no odd cycle of length less than $l+1$ in G), where $\chi(G) = \chi(G_0) > k$ and $\delta(G) = t \cdot \delta(G_0) \geq \frac{k|G|}{N(l+1,k+1)}$. This completes the proof. $\square$

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