Graphics&Geometric Computing Lab @USTC

Delaunay Triangulation and Mesh Generation

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Outline

- Concepts
- Delaunay Triangulation
- Optimal Delaunay Triangulation (ODT)
- Centroidal Vonoroi Tessellation (CVT)
- 3D mesh generation

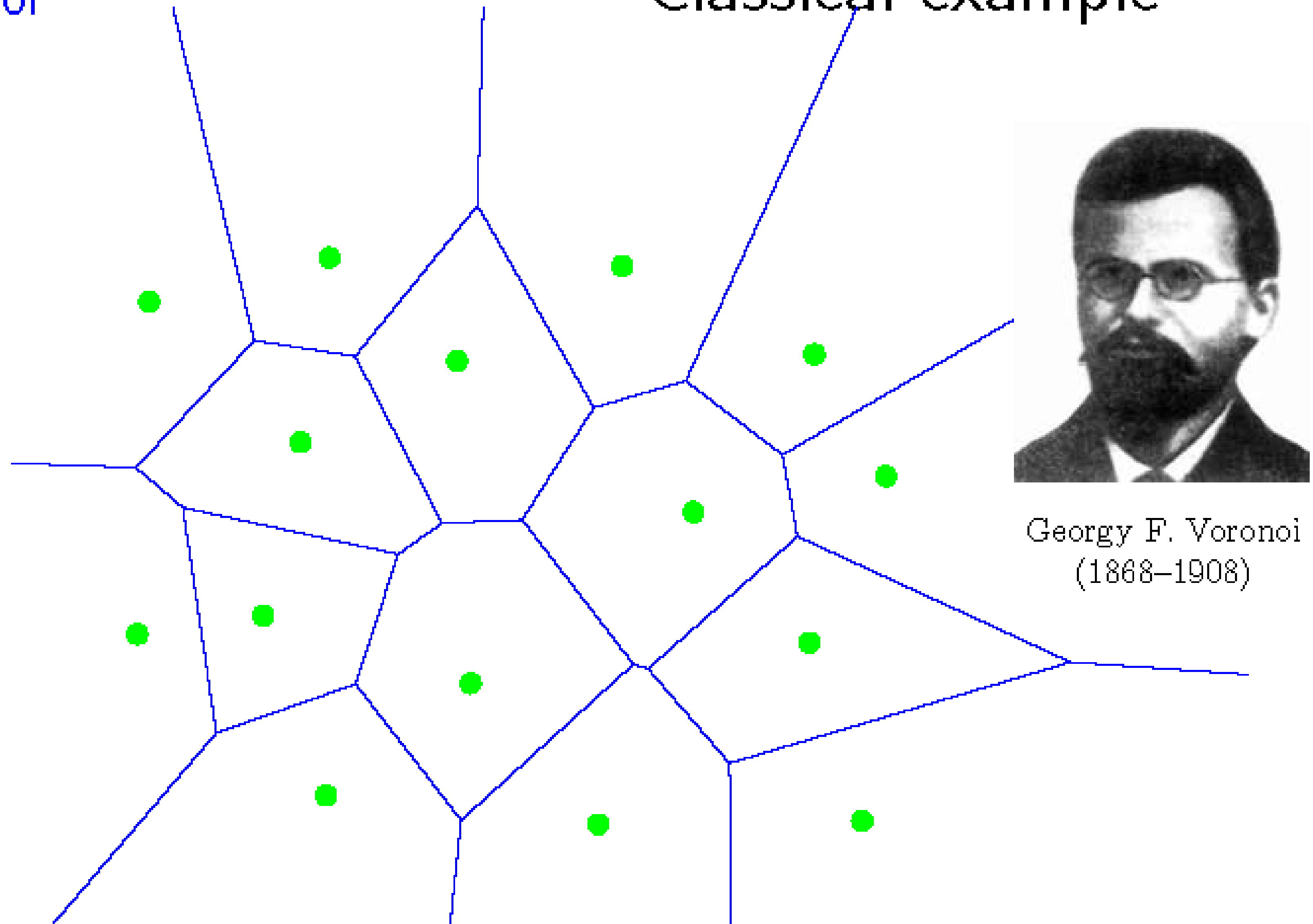
Boris N. Delaunay

- Russian mathematician
- March 15, 1890 - July 17, 1980
- Introduce Delaunay triangulation in 1934

Georgy F. Voronoi

- Russian mathematician
- April 28, 1868 - November 20, 1908

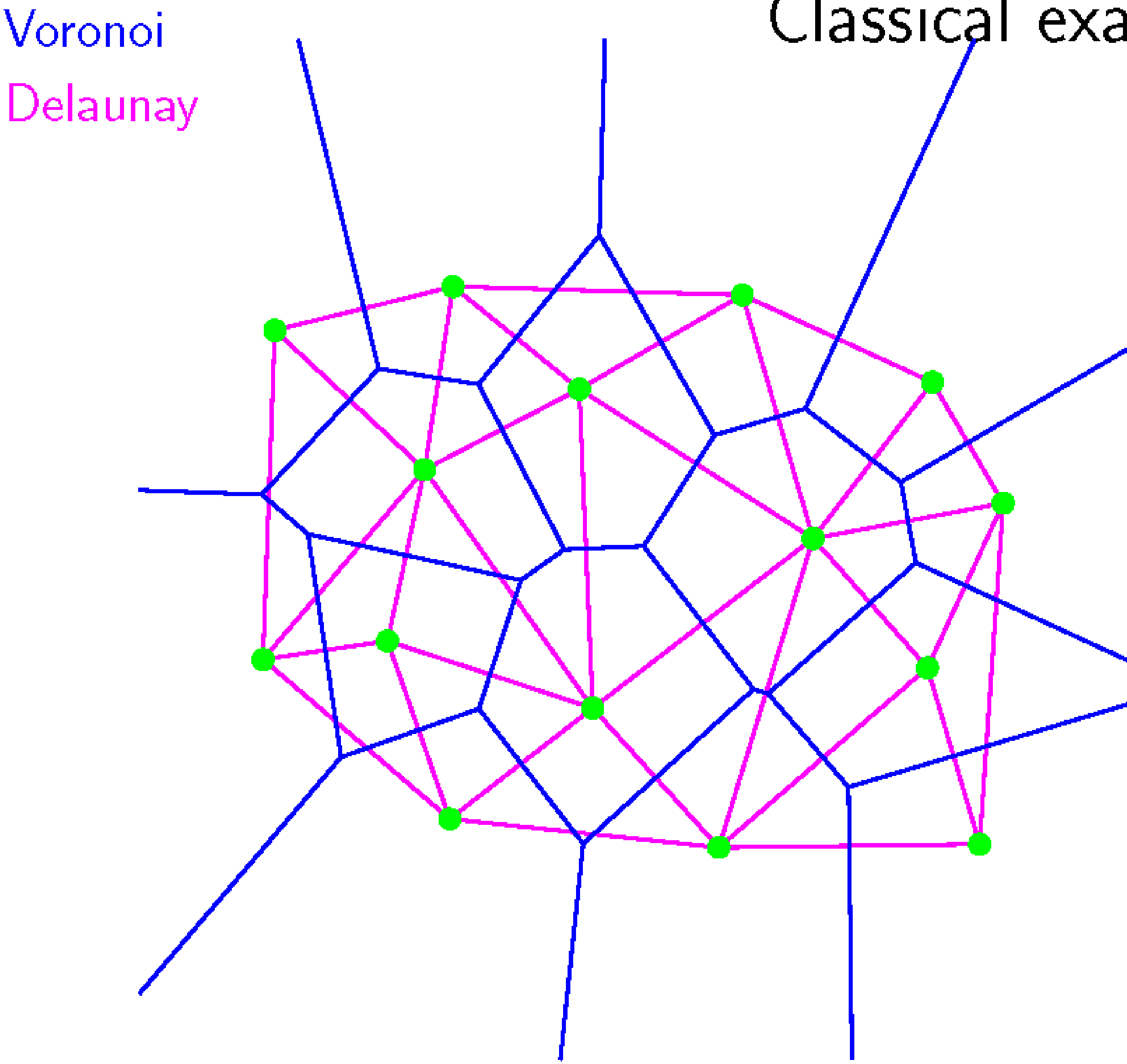
Classical example



Georgy F. Voronoi
(1868–1908)

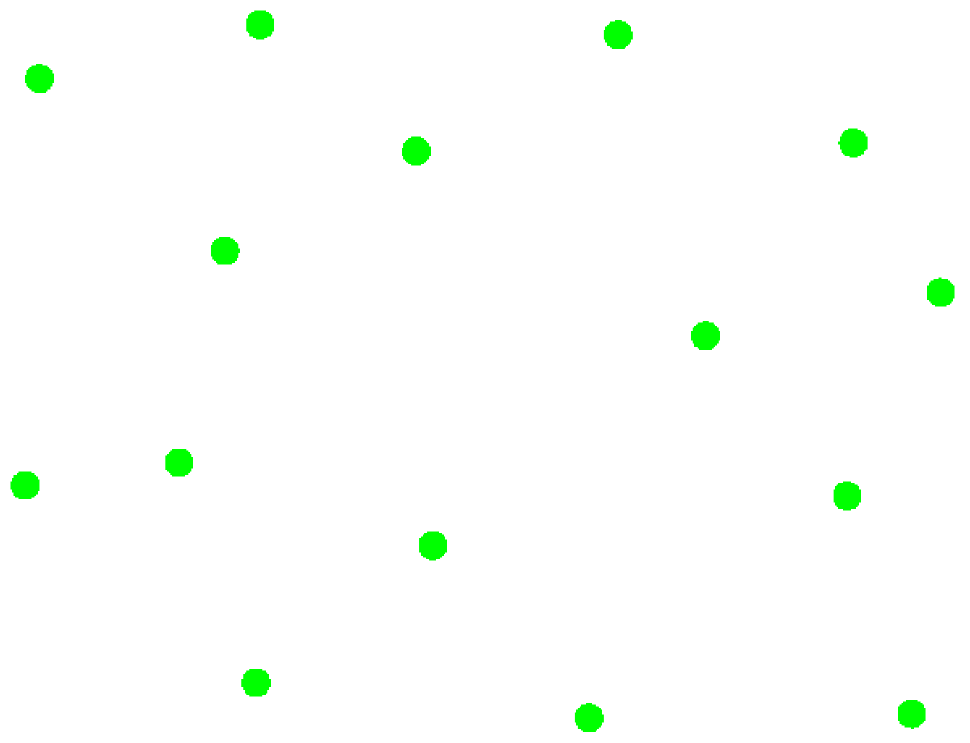
$$V_i = \{q; \forall j \neq i \, |qp_i| \leq |qp_j|\}$$

Classical example



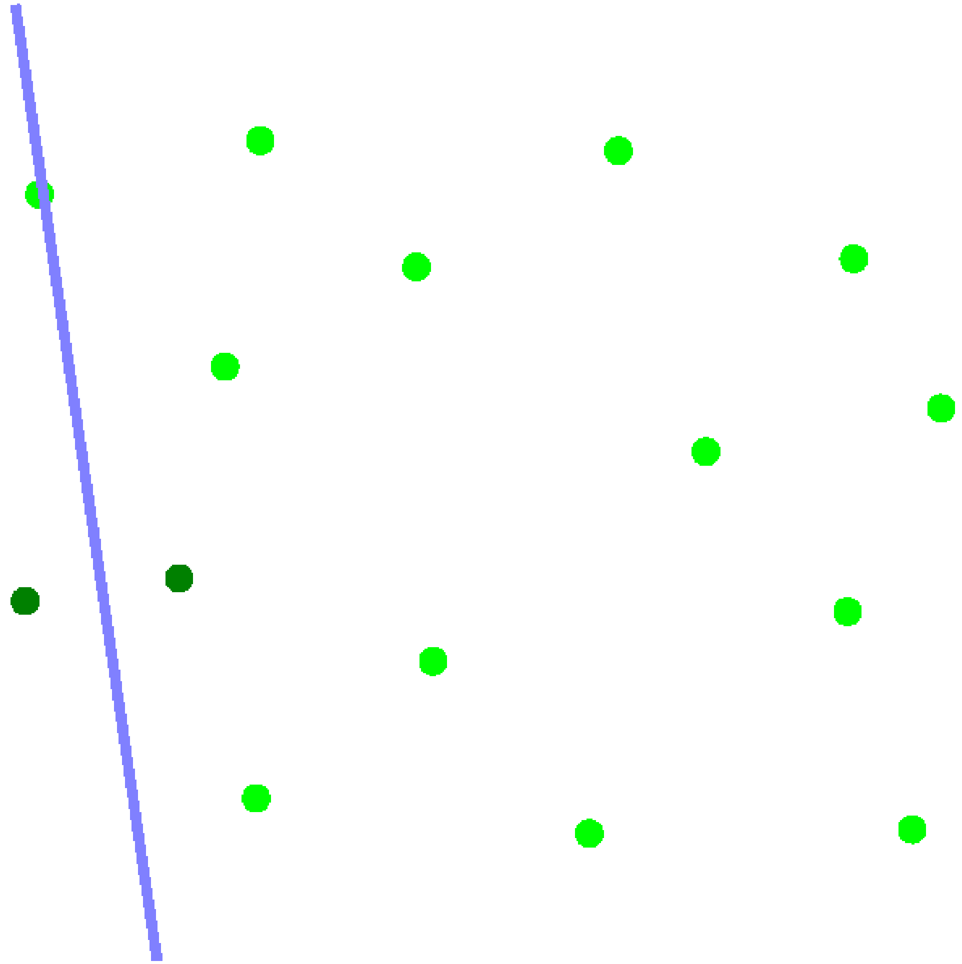
Boris N. Delaunay
(1890–1980)

Voronoi



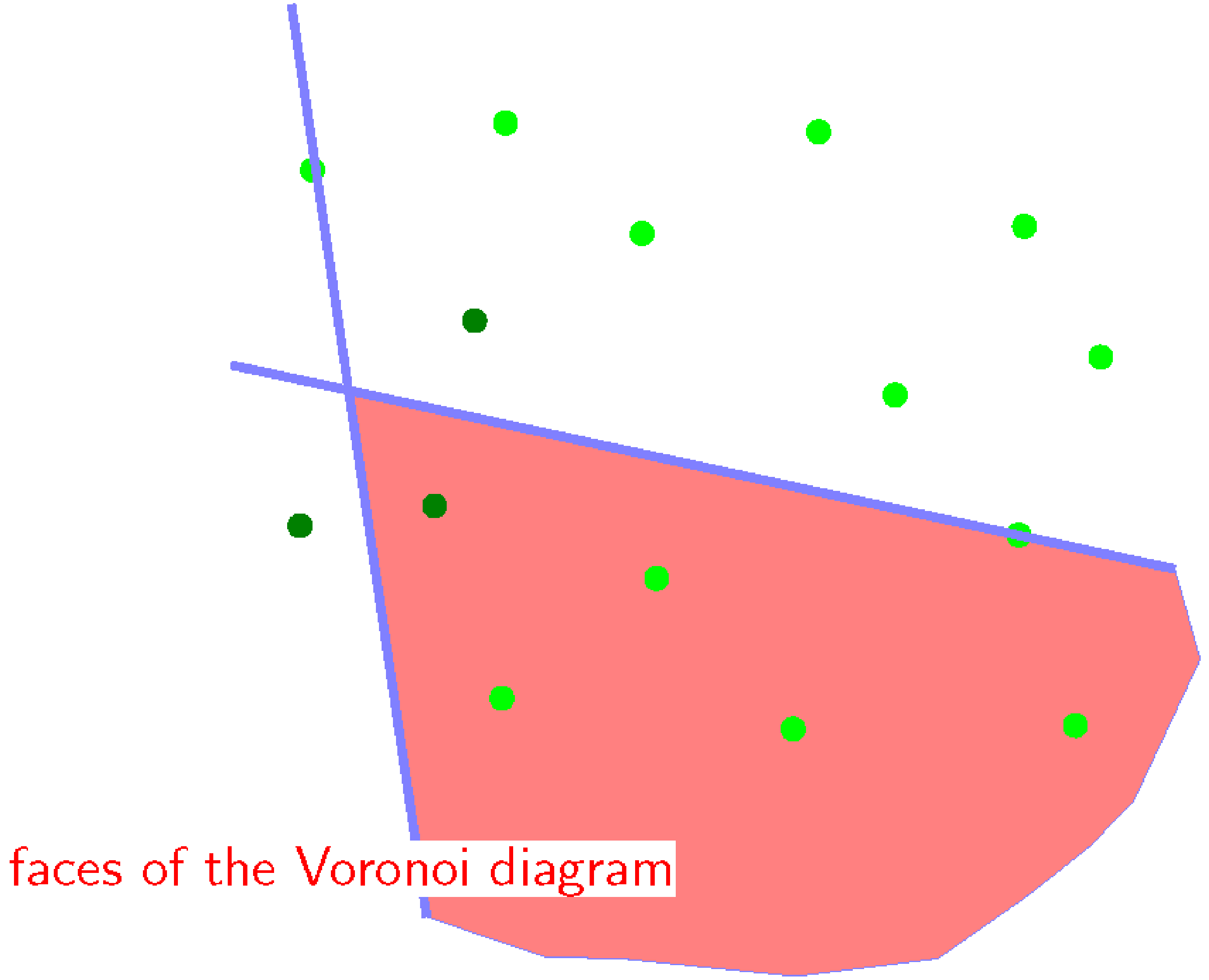
faces of the Voronoi diagram

Voronoi

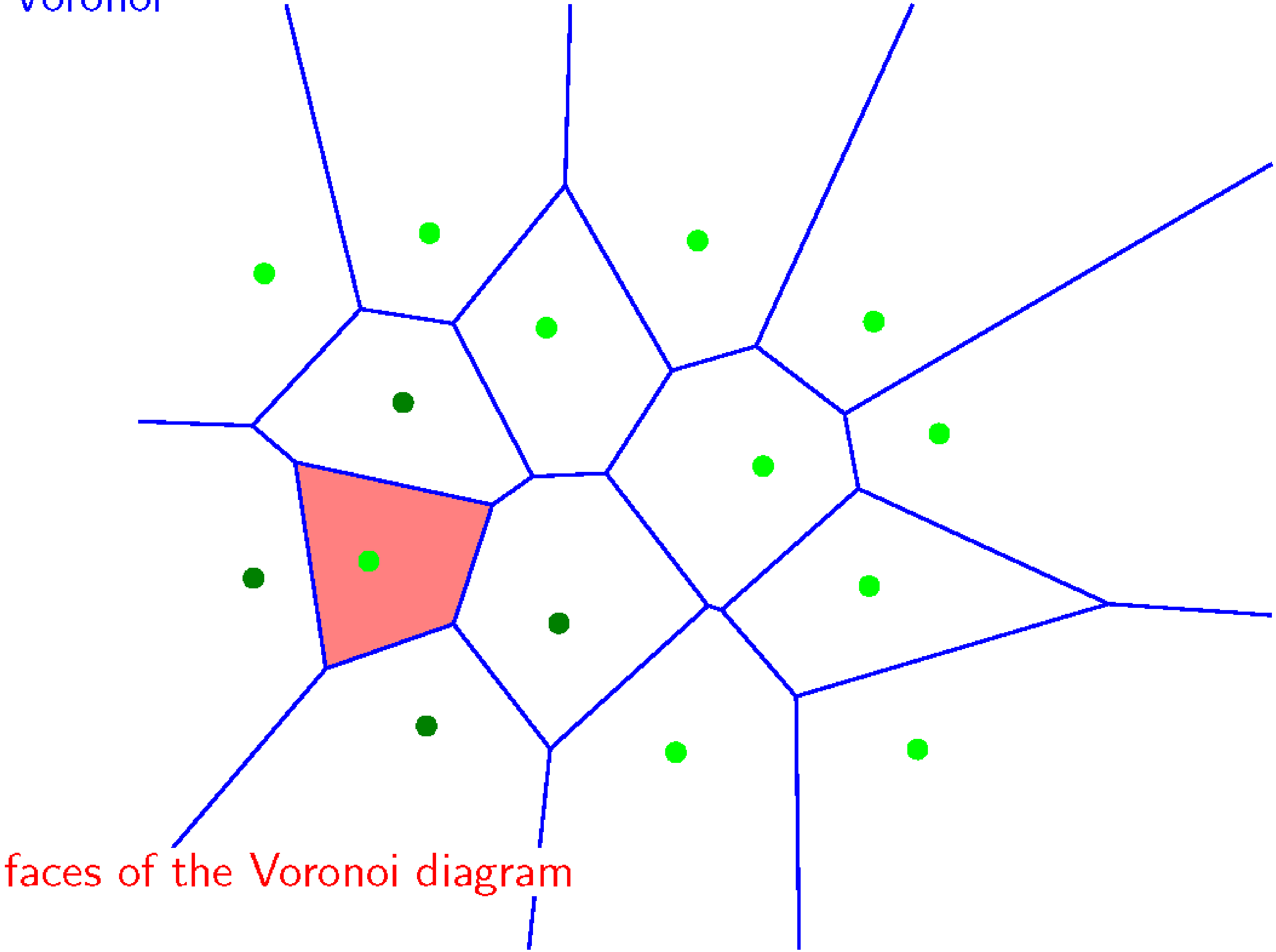


faces of the Voronoi diagram

Voronoi

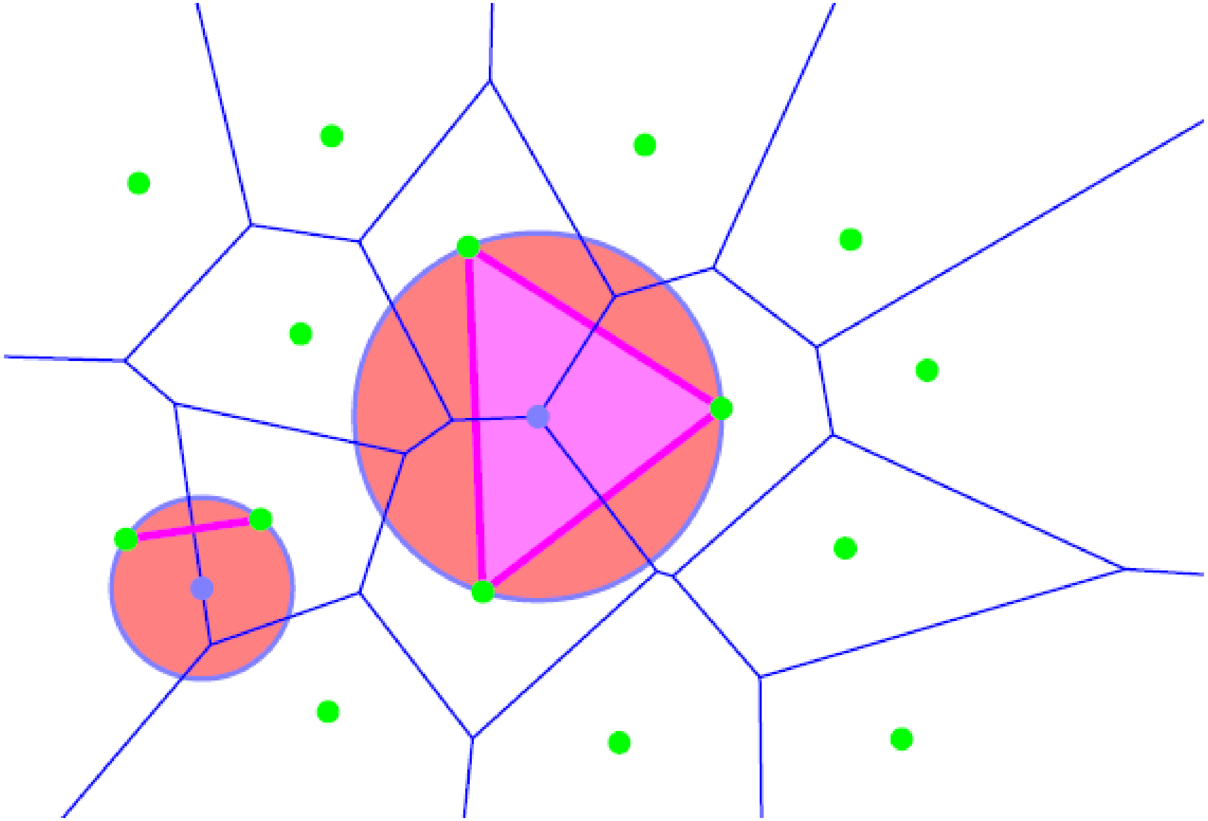


Voronoi



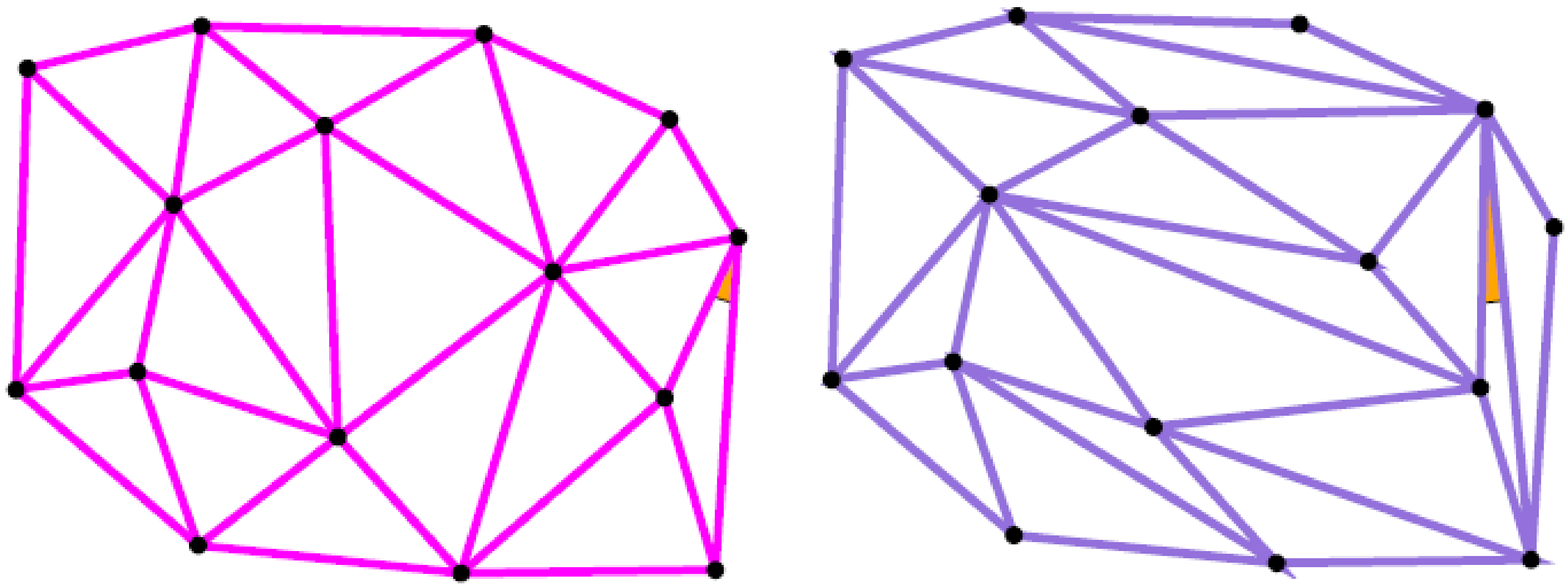
Properties of DT (1)

- Empty sphere property: no points inside the circum-sphere of any simplex
  - Delaunay edge



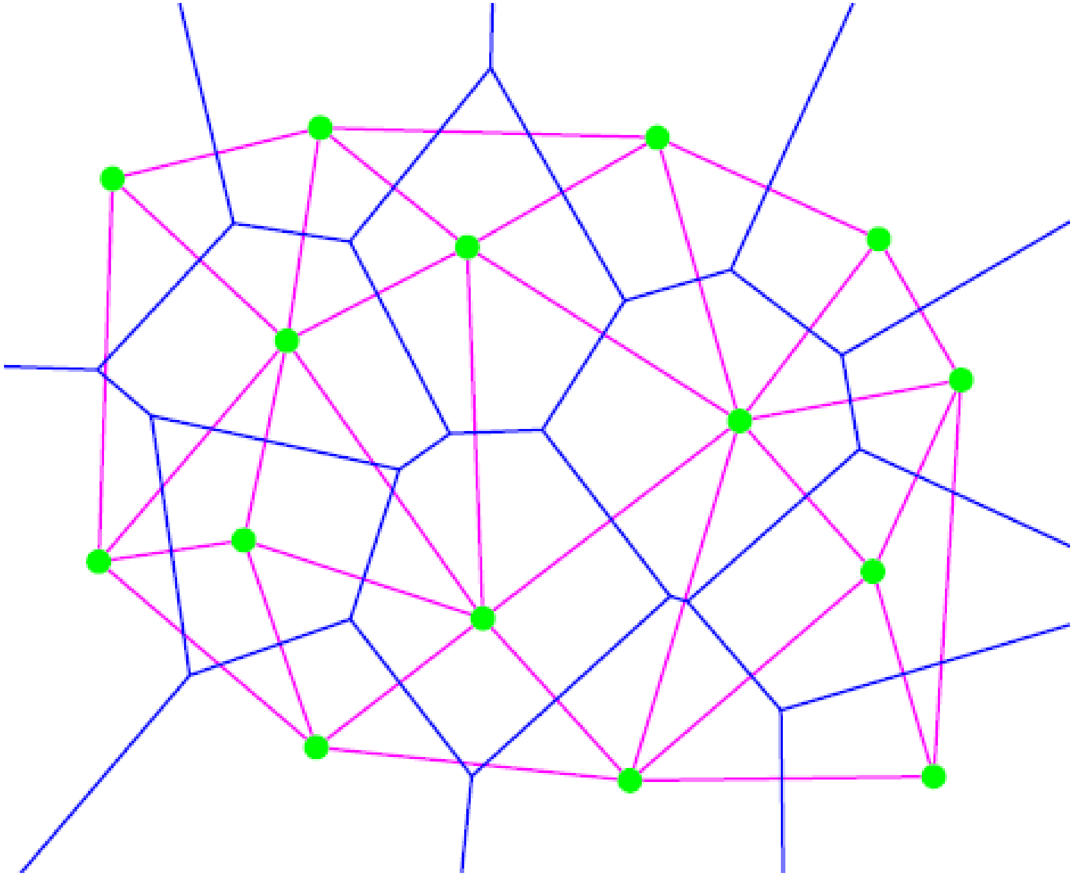
Properties of DT (2)

- DT maximizes the smallest angle
  - [Lawson 1977] and [Sibson 1978]



Properties of DT (3)

- Convex hull: union of all triangles

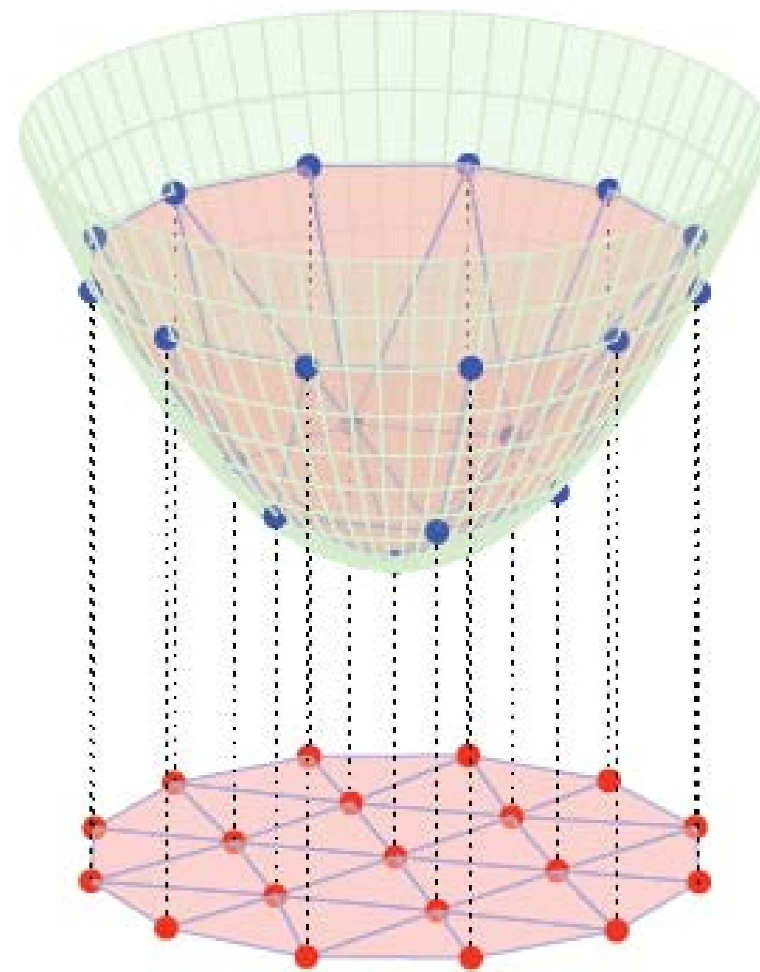


Properties of DT (4)

- DT maximizes the arithmetic mean of the radius of inscribed circles of the triangles.
  - [Lambert 1994]
- DT minimizes roughness (the Dirichlet energy of any piecewise-linear scalar function)
  - [Rippa 1990]
- DT minimizes the maximum containing radius (the radius of the smallest sphere containing the simplex)
  - [Azevedo and Simpson 1989], [Rajan 1991]

Properties of DT (5)

- The DT in d-dimensional spaces is the projection of the points of convex hull onto a (d+1)-dimensional paraboloid.
  - [Brown 1979]



Properties of DT (6)

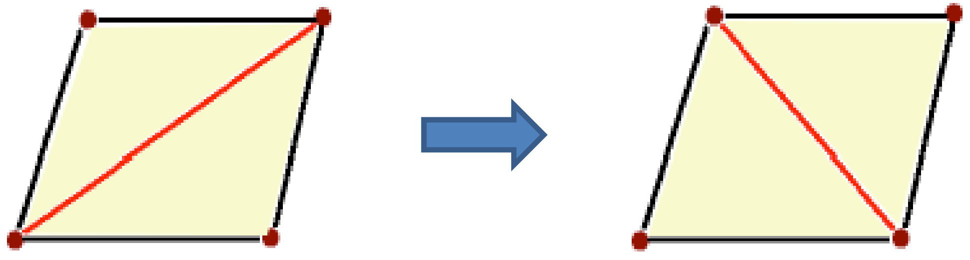
- DT minimizes the spectrum of the geometric Laplacian (spectral characterization)
  - [Chen et al. 2010]

Delaunay Spectral Theorem

The spectrum of the geometric Laplacian obtains its minimum on a Delaunay triangulation. Namely if $\{\lambda_1 = 0, \lambda_2, \ldots, \lambda_n\}$ and $\{\mu_1 = 0, \mu_2, \ldots, \mu_n\}$ are the sequences of non-decreasing eigenvalues of the geometric Laplacian of a Delaunay triangulation and of any other triangulation of the same set of points, respectively, then $\lambda_i \le \mu_i$ for $i = 1, \ldots, n$.

Edge Swapping/Flipping [Sibson 1978]

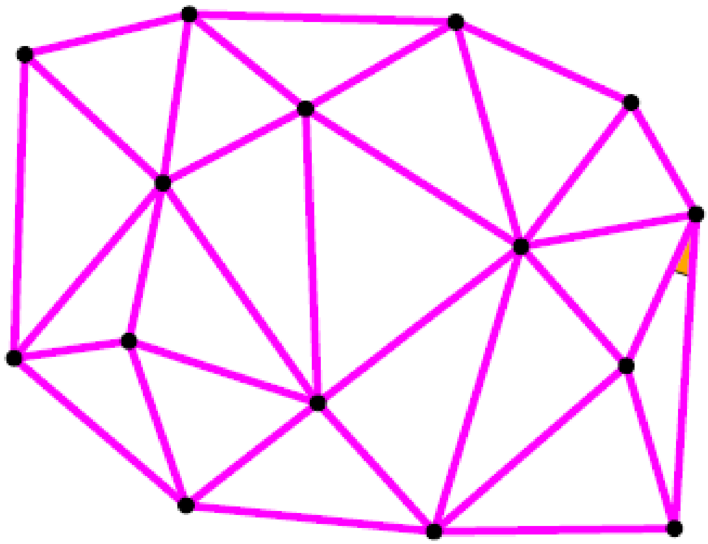
- Start with any triangulation
  - 1. find any two adjacent triangles that form a convex quadrilateral that does not satisfy empty sphere condition
  - 2. swap the diagonal of the quadrilateral to be a Deluany triangulation of that four points
  - 3. repeat step 1,2 until stuck.

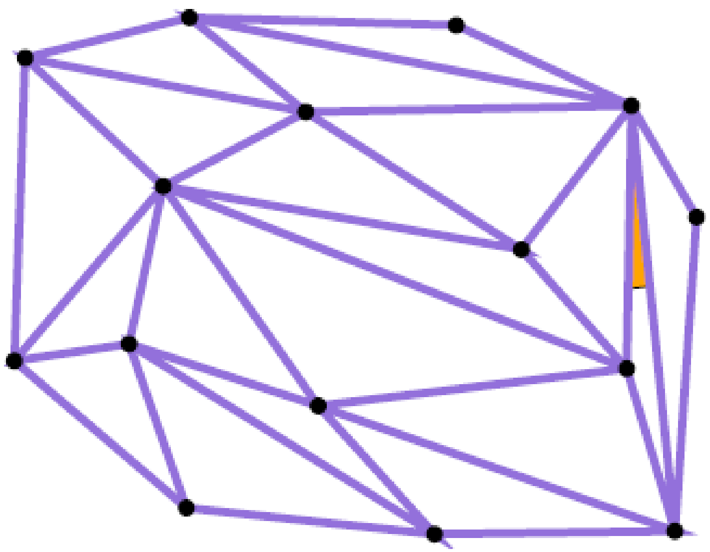


Convergence? Is it possible to end with an infinite loop?

Mesh Generation

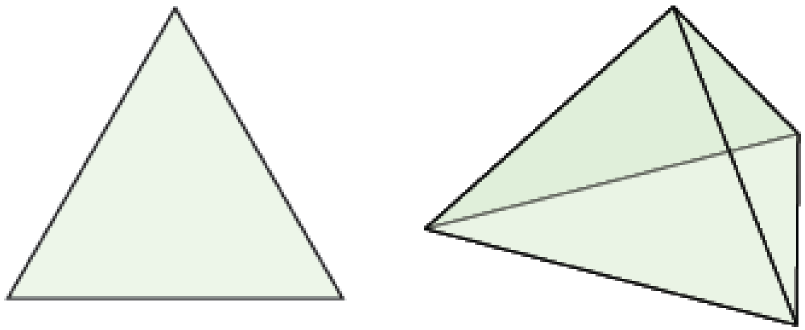
- Given a fixed point set, Delaunay triangulation will try to make the triangulation more shape regular and thus is considered as a “good” unstructured mesh.





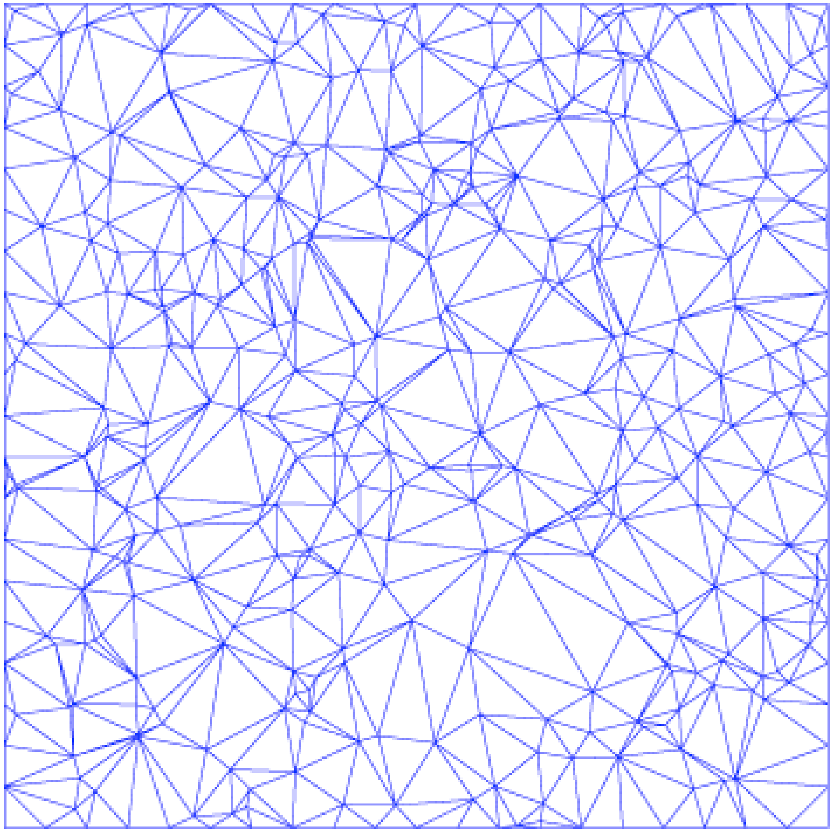
Mesh Quality

- What do we mean a “good” mesh/simplex (triangle)?
  - Minimal angle
  - Mean ratio
  - Aspect/radius ratio
  - …



- It is not easy to define a universal mesh quality acceptable by everyone. But everyone agrees on the "best" simplex: equilateral triangle and tetrahedra.

DT is not necessary a good mesh



DT only optimize the **connectivity** when points are fixed. The **distribution of points** is more important for a good mesh.

What is the usage of a mesh?

- One Answer: Mesh is used to approximate functions.

For a given function $f \in C(\Omega)$ and a triangulation $\mathcal{T}$ of Ω, we define the quality of $\mathcal{T}$ as the interpolation error:

$$Q(\mathcal{T}, f, p) = \|f - f_{I,\mathcal{T}}\|_{L^p,\Omega} = \left(\int_\Omega |f(\mathbf{x}) - f_{I,\mathcal{T}}(\mathbf{x})|^p d\mathbf{x}\right)^{1/p}.$$

Mesh quality is a function depedent concept from the approximation point of view.

On the other hand, by choosing appropriate functions, we can get meshes with good geometric shape.

Optimality of Delaunay Triangulation

Given a point set $\mathbf{P} \subset \mathbb{R}^n$, denote by Ω be the convex hull of $\mathbf{P}$ and $\mathscr{T}_\mathbf{P}$ all possible triangulations of Ω by using the points in $\mathbf{P}$.

Theorem [Chen and Xu 2004]

Delaunay triangulation optimizes the connvectivity for a given point set. Namely

$$Q(DT, \|\mathbf{x}\|^2, p) = \min_{\mathcal{T} \in \mathscr{T}_\mathbf{P}} Q(\mathcal{T}, \|\mathbf{x}\|^2, p), \quad \forall\, 1 \leq p \leq \infty.$$

- $\mathbb{R}^2, p = \infty$ (D'Azevedo and Simpson 1989 [12])
- $\mathbb{R}^n, p = \infty$ (Rajan 1991 [21])
- $\mathbb{R}^2, 1 \leq p < \infty$ (Rippa 1992 [23])
- $\mathbb{R}^n, 1 \leq p \leq \infty$ (C. and Xu 2004 [11])

Graph of the Linear Interpolation

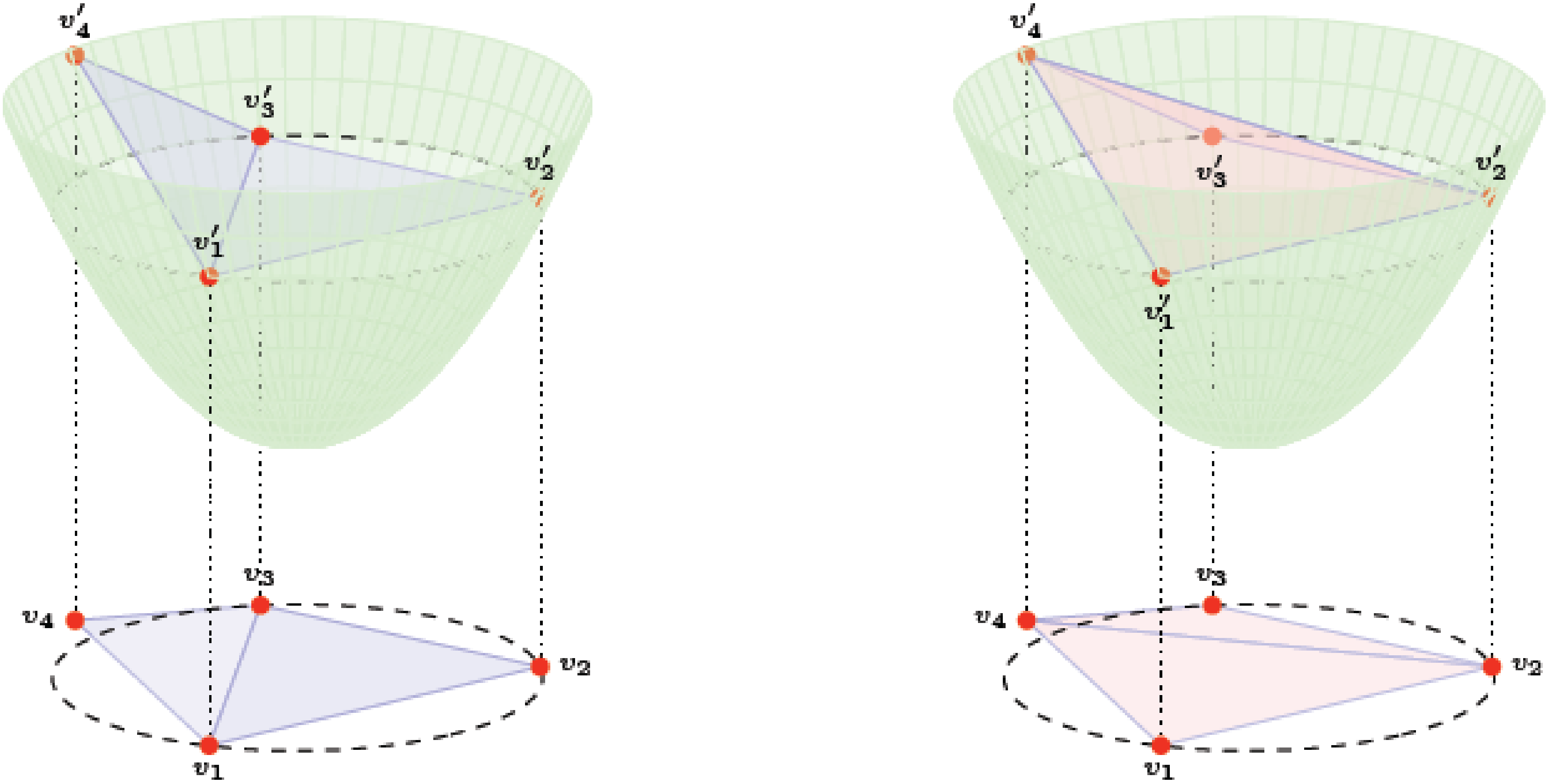
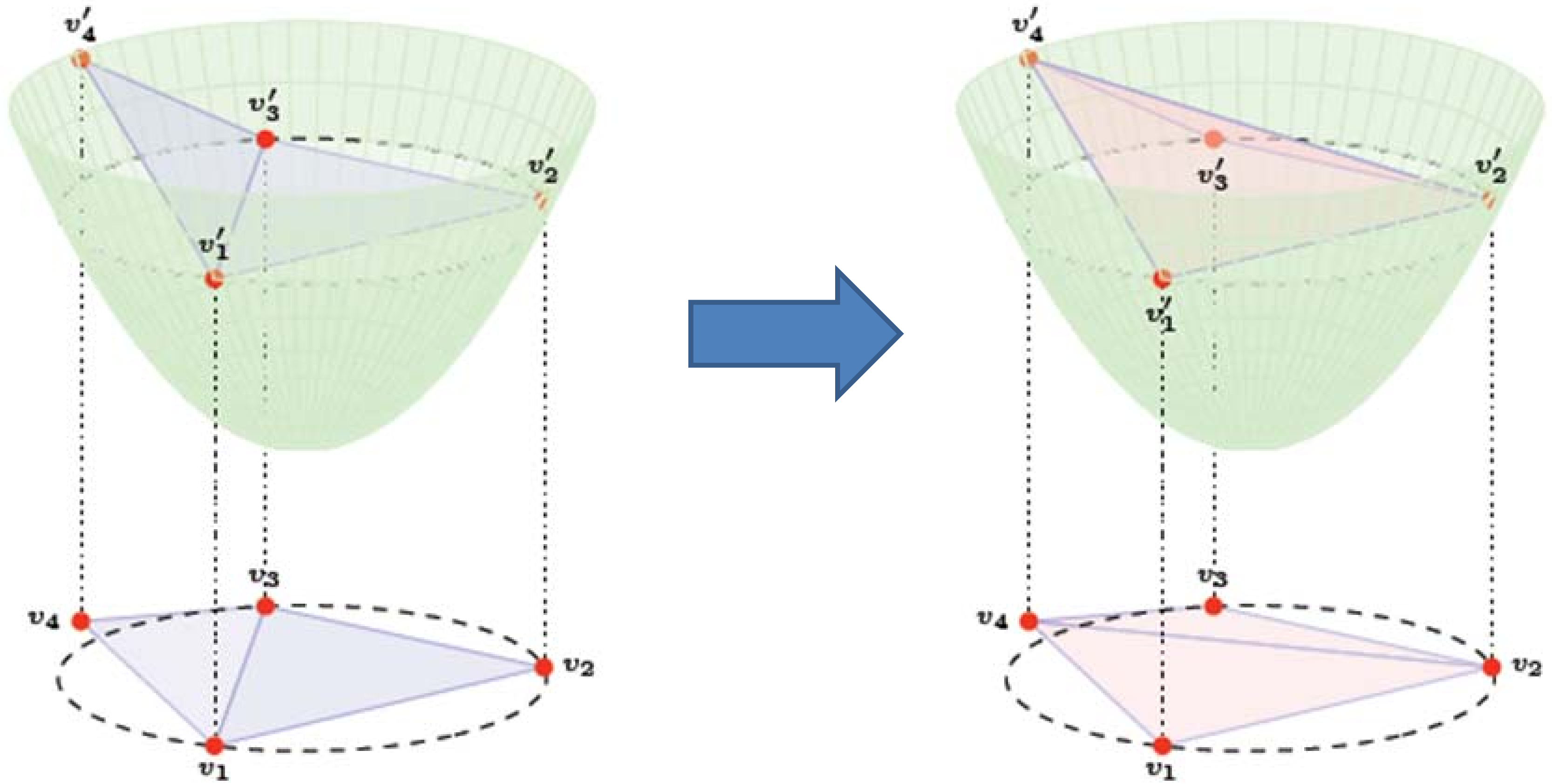


Figure: Different triangulations and graphs of linear interpolations

Convergence of Edge Swapping Algorithm



$$Q(\mathcal{T}_1, \|\boldsymbol{x}\|^2, p) > Q(\mathcal{T}_2, \|\boldsymbol{x}\|^2, p) > \cdots > Q(\mathcal{T}_n, \|\boldsymbol{x}\|^2, p) > \cdots > 0$$

Lifting Trick

We lift $\mathbf{P}$ to the paraboloid $x_{n+1} = \|\mathbf{x}\|^2$ and construct the convex hull of $\mathbf{P}'$. A facet of this convex hull in $\mathbb{R}^{n+1}$ belongs to the lower convex hull if it is supported by a hyper-plane that separates $\mathbf{P}'$ from $(\mathbf{0}, -\infty)$.

The projection of the lower convex hull of $\mathbf{P}'$ to $\mathbb{R}^n$ is the Delaunay triangulation of $\mathbf{P}$.

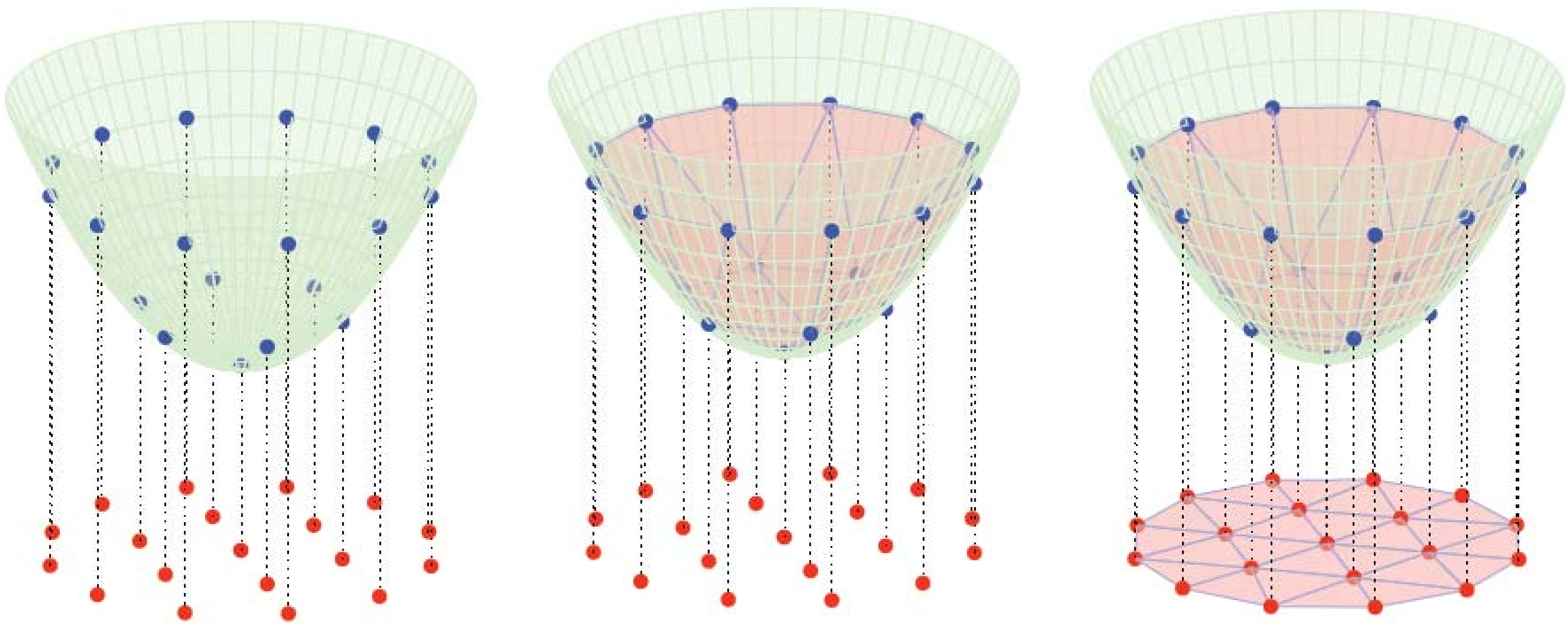
(Brown 1979 [3], Edelsbrunner and Seidel 1986 [13])

Let $f(\boldsymbol{x}) = u(\boldsymbol{x}) := \|\boldsymbol{x}\|^2$. It is equivalent to say

$$u_{I,DT}(\boldsymbol{x}) \leq u_{I,\mathcal{T}}(\boldsymbol{x}), \quad \forall \mathcal{T} \in \mathscr{T}_{\mathbf{P}}.$$

Qhull: a Global Algorithm to Construct DT

- Lift points to the parabolid
- Form the lowest convex hull in R^{n+1}
- Project the lowest convex hull to R^n



Optimal Delaunay Triangulation (ODT)

$$Q(DT, \|\mathbf{x}\|^2, p) = \min_{\mathcal{T} \in \mathscr{T}_{\mathrm{P}}} Q(\mathcal{T}, \|\mathbf{x}\|^2, p), \quad \forall\, 1 \le p \le \infty.$$

$$Q(ODT, \|\mathbf{x}\|^2, p) = \inf_{\mathcal{T} \in \mathscr{T}_{\mathrm{N}}} Q(\mathcal{T}, \|\mathbf{x}\|^2, p), \quad \forall\, 1 \le p \le \infty,$$

where $\mathscr{T}_N$ is the set of all triangulations of Ω with at most N vertices. We allow the move of points inside the domain.

- [Chen and Xu 2004] proved the existence of ODT. In general, it is not unique.

However…

- ODT energy function is only C^0 (piecewise smooth)
  - Both the point positions and its connectivity are the variables in the function

$$E(p,t) = Q(\mathcal{T}, \|\boldsymbol{x}\|^2, 1)$$

- Can not be solved using general numeric optimization methods

Mesh Optimization based on ODT

Write $\mathcal{T} = (p, t)$ with:

- p: the set of points;
- t: the connectivity of points.

We define the energy $E(p, t) = Q(\mathcal{T}, \|\boldsymbol{x}\|^2, 1)$.

Basic Algorithm

WHILE $E(p, t) > \epsilon$ **DO**

1. For a fixed p, find t for the optimization problem

$$\min_{t} E(p, t);$$

2. For a fixed t, find p for the optimization problem

$$\inf_{p} E(p, t).$$

END WHILE

Optimize the Connectivity

Fix points, find the optimal connvectivity using these points

$$\min_{t} E(p, t)$$

We have studied this question in the section of Delaunay triangulations.

- Local algorithm: edge swapping;
- Global algorithm: Qhull (Lifting trick).

Remark

We are not really interested in the optimal mesh. Instead a good mesh will be obtained by performing few steps of optimization methods.

Optimize the Location

Fix connectivity, find the optimal location of points

$$\inf_{p} E(p, t)$$

We will present two algorithms in this section.

- Local algorithm: mesh smoothing;
- Global algorithm: H^{-1} preconditioner.

Generalization: non-uniform density

We can define our energy as

$$E(p, t, \rho) = \int_{\Omega} |u_I - u| \rho(\boldsymbol{x}) d\boldsymbol{x},$$

where $\rho > 0$ is a density function.

Since we only change the measure for the integration, DT is still the optimal triangulation for a fixed point set.

The exact formula for ∇E is not easy. Instead we choose a piecewise constant approximation of ρ on each element. And use the weighted volume $|\tau|_\rho$ to replace $|\tau|$.

Generalization: anisotropic metric

We can choose any convex function in the definition

$$Q(\mathcal{T}, f, p).$$

The error formula holds in the metric given by $\nabla^2 f$. So the optimization will give an equilateral triangulation under this new metric.

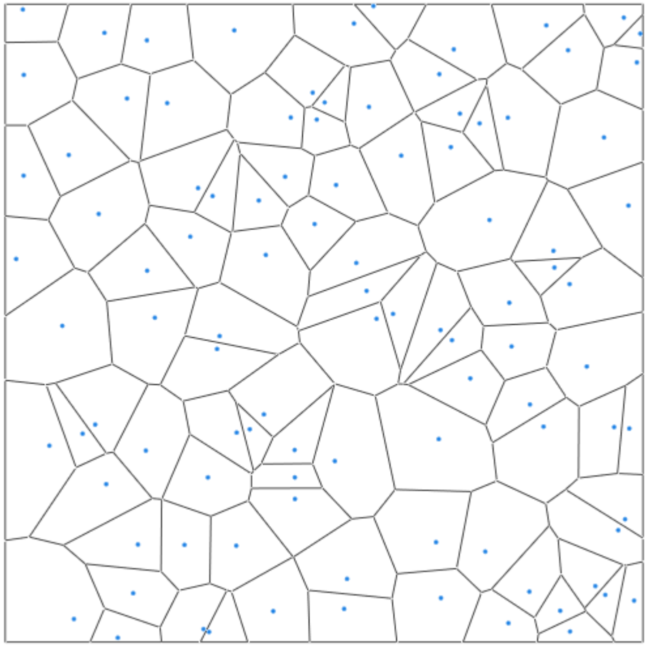
Some References

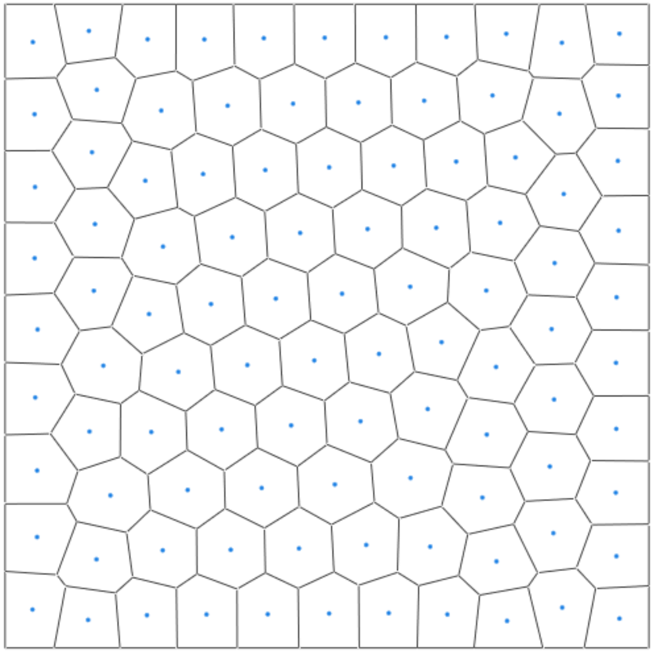
- Huang 2001,2003 Huang and Sun 2003, Huang 2005 [15, 16].
- Shewchuk 2002 [24].
- Cao 2005, 2007, 2008 [4, 5, 6].
- C. 2004, 2005, C. and Xu 2004, C., Sun and Xu 2007 [10, 11, 7, 9].

Centroidal Voronoi Tessellation (CVT)

Centroidal Voronoi Tessellation

- *Definition*: The VT is a centroidal Voronoi tessellation (CVT) , if each seed coincides with the centroid of its Voronoi cell





Centroidal Voronoi Tessellation

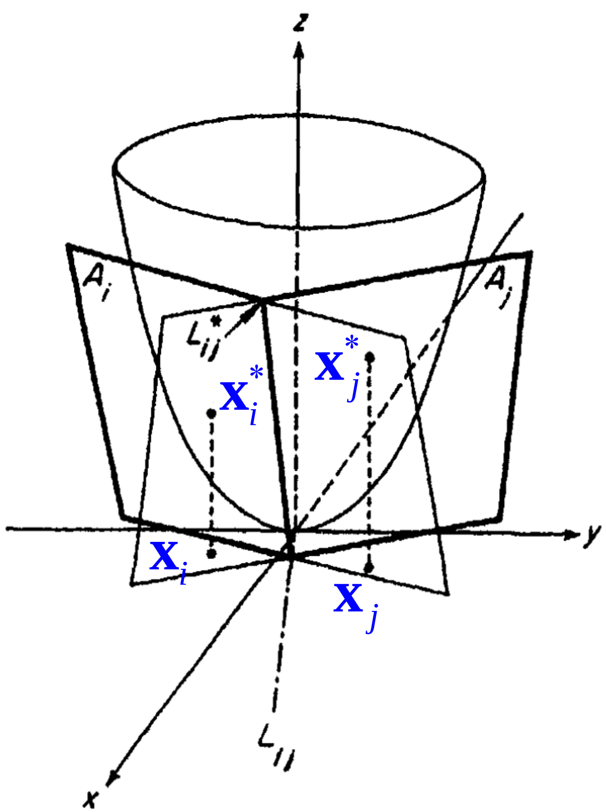
- *Definition*: The VT is a centroidal Voronoi tessellation (CVT) , if each seed coincides with the centroid of its Voronoi cell
- CVT energy function:

$$F(X) = \sum_{i=1}^{N} \int_{V_i} \rho(\mathbf{x}) \|\mathbf{x} - \mathbf{x}_i\|^2 d\mathbf{x}$$

- CVT is a critical point of $F(X)$, an *optimal* CVT is a global minimizer of $F(X)$

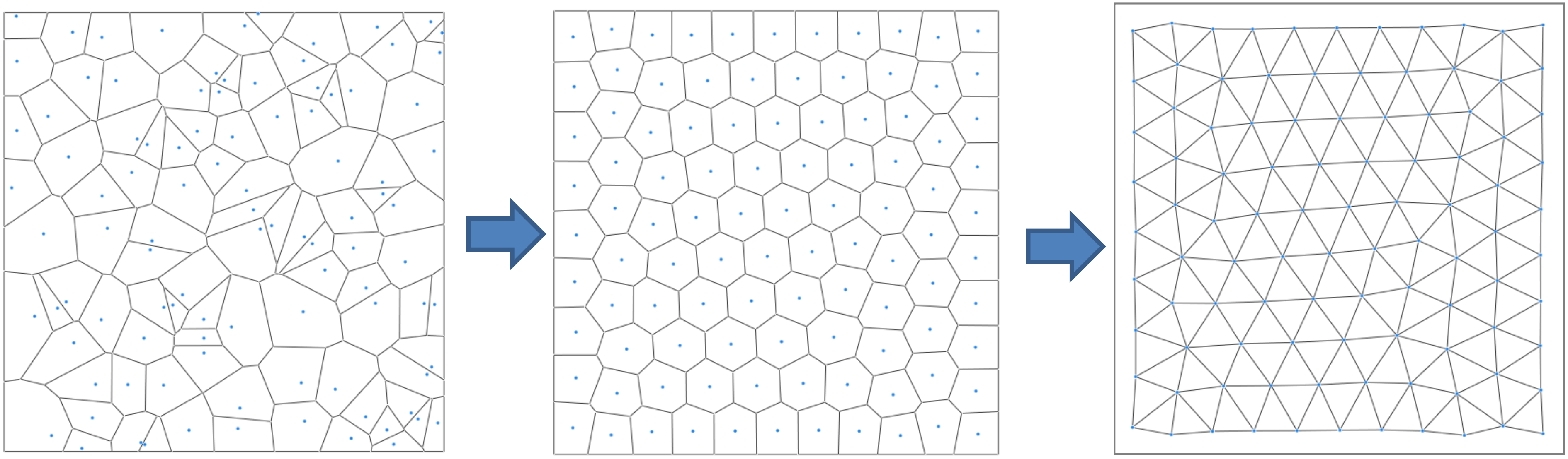
Geometric Interpretation

- *Theorem*: The CVT energy with $\rho(\mathbf{x})$ identical to 1, is the volume difference between the circumscribed polytope and the paraboloid.
  - An optimal CVT is a best circumscribed piecewise linear approximant to the paraboloid



Lloyd Algorithm

- Construct the VT associated with the points
- Compute the centroids of the Voronoi regions
- Move the points to the centroids
- Iterate until convergent

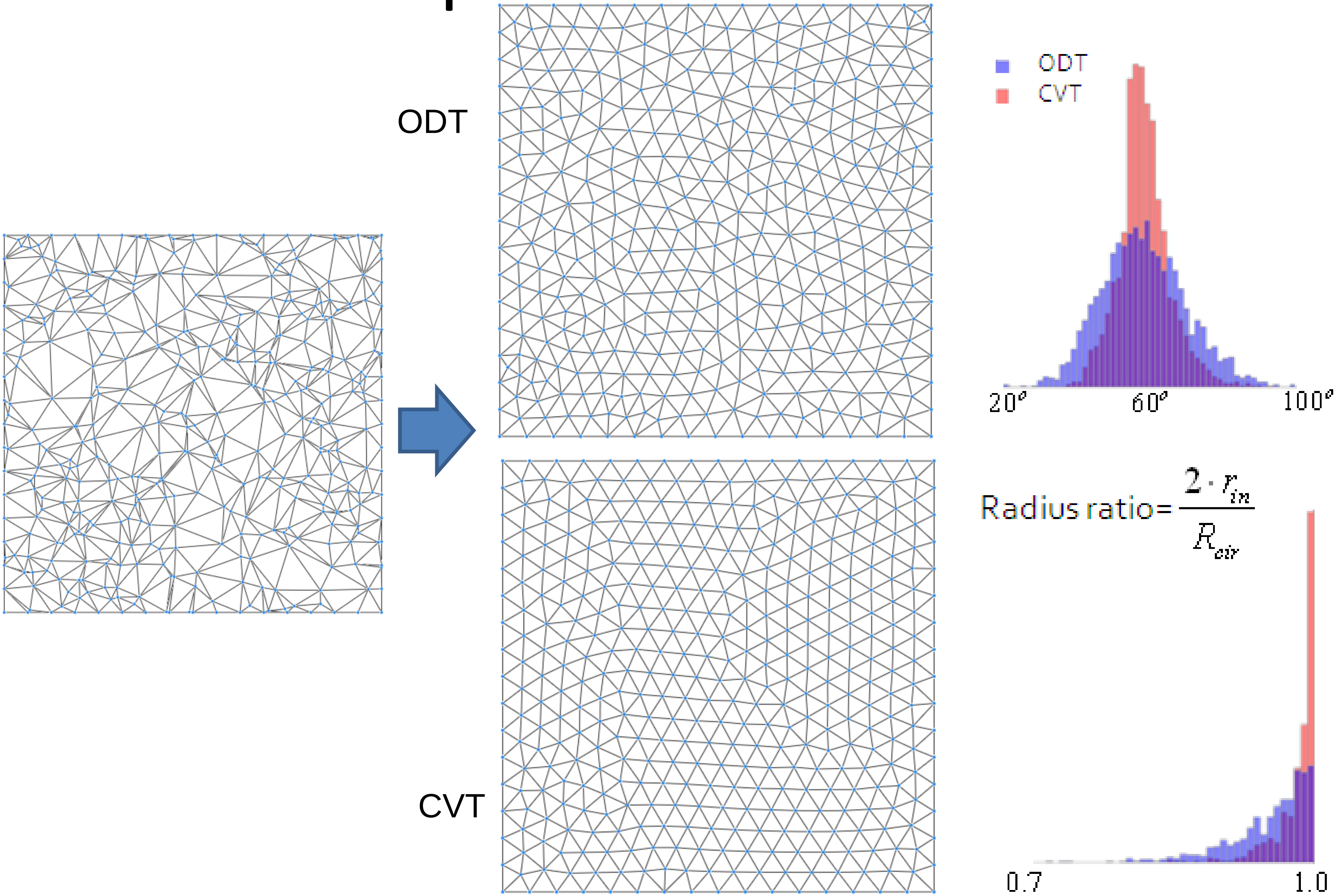


CVT Energy

- CVT energy is C^2 smooth!
  - [Liu et al. 2009]

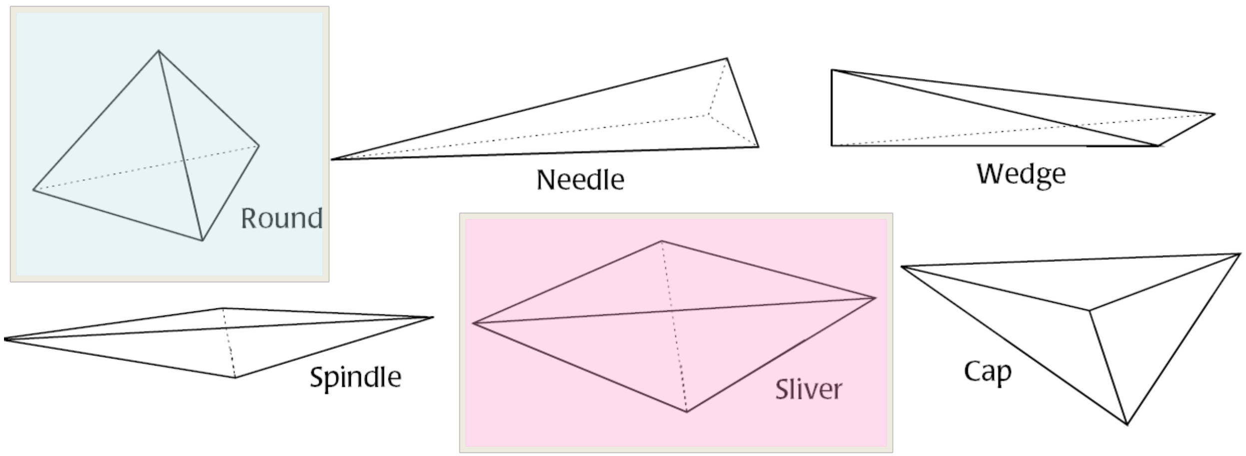
- Could be solved much faster than Lloyd algorithm

Compare ODT and CVT



Mesh Generation — 3D

- Regular tetrahedra can't tile the space
- Tetrahedra classified by bad angles

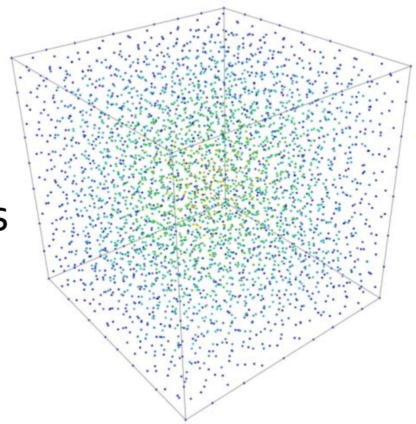


- well-spaced points generate only round or sliver Delaunay tetrahedra *[Eppstein 01]*

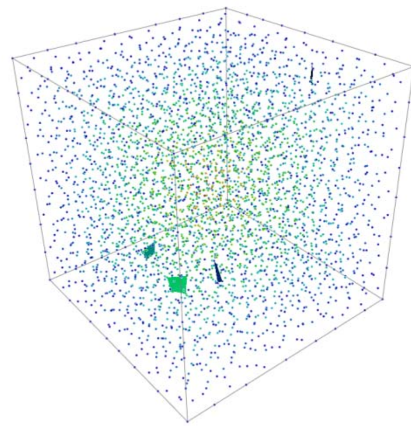
Comparison

ODT:
4000 seeds
21726 tets

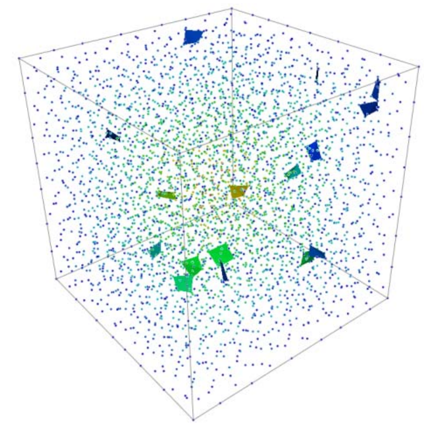
0 sliver < 5 deg



4 slivers < 10 deg

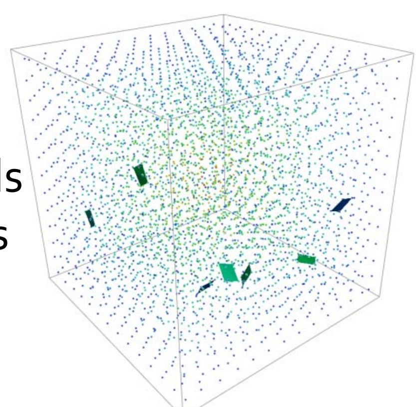


16 slivers < 15 deg

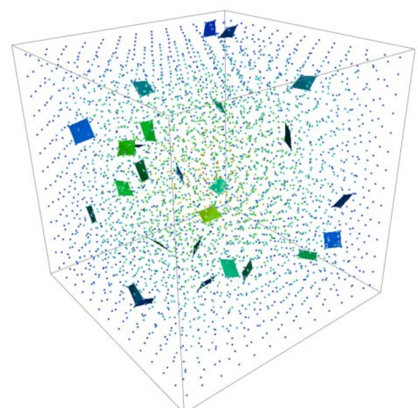


CVT:
4000 seeds
20179 tets

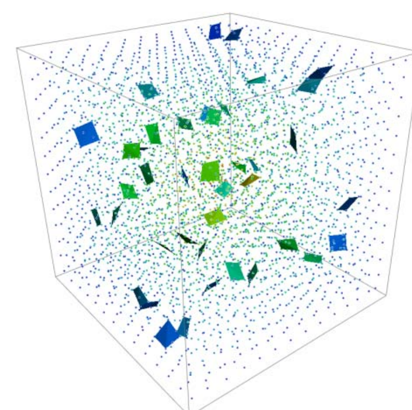
7 slivers < 5 deg



26 slivers < 10 deg

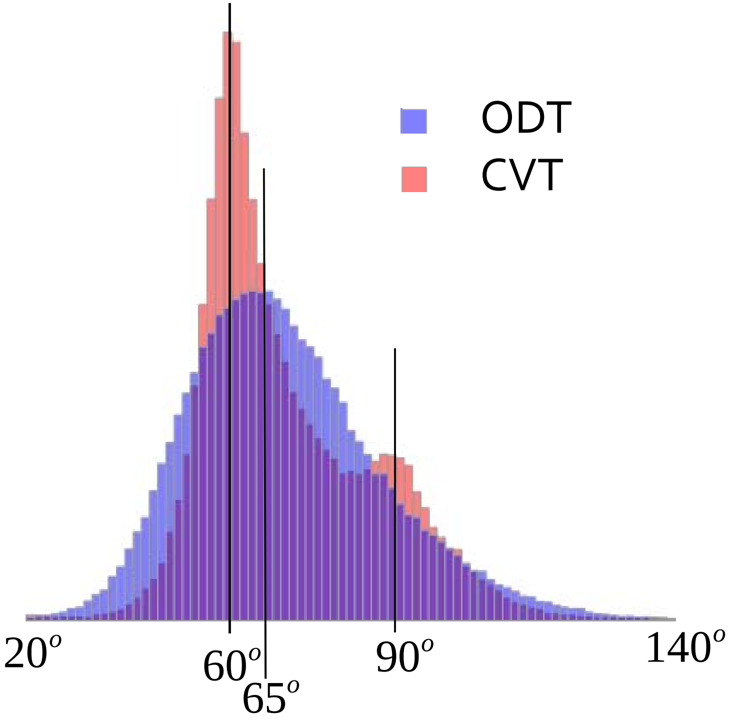


40 slivers < 15 deg



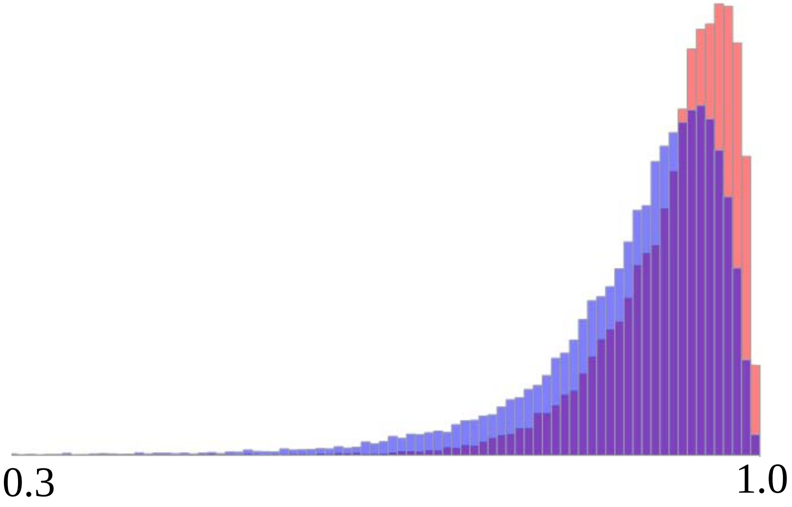
Tetrahedron Qualities

- Dihedral Angle distribution



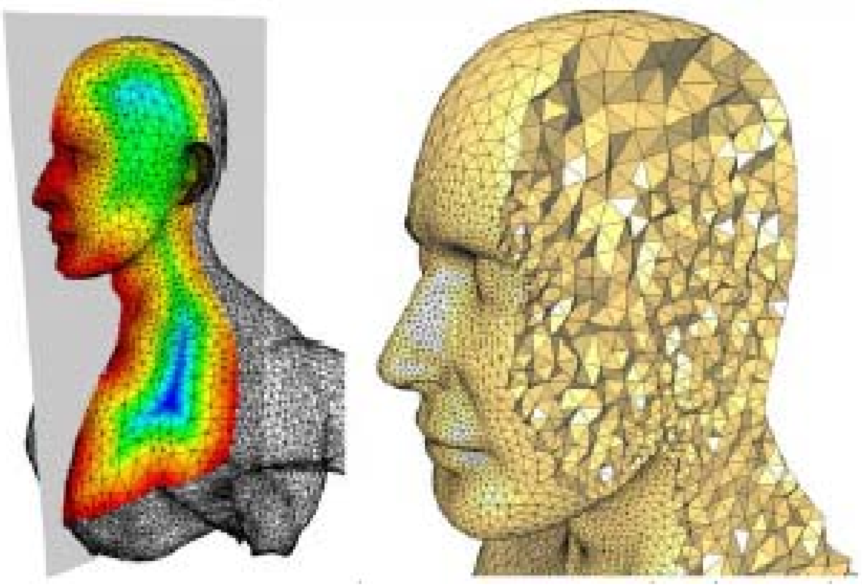
- Radius ratio distribution

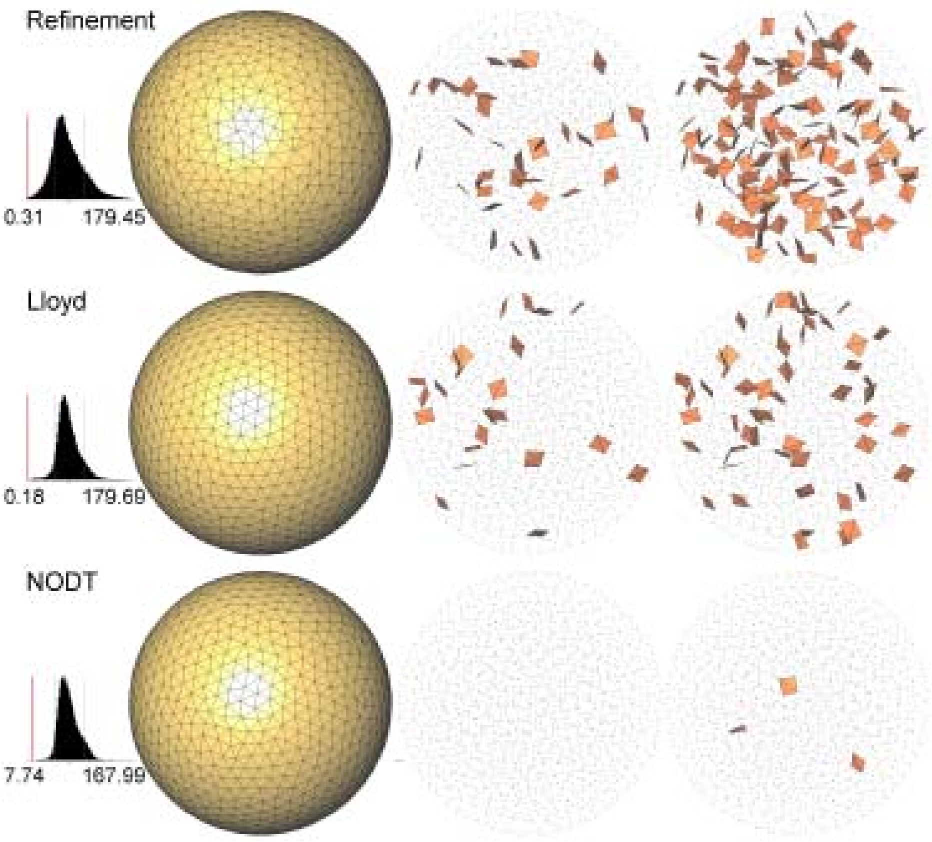
Radius ratio= $\dfrac{3 \cdot r_{in}}{R_{cir}}$



3D Tetrahedral Mesh

- [Alliez et al. 2005, Tournois et al. 2009]





Summary

- **CVT**
  - best circumscribed PL approximant
  - compact Voronoi cells
  - more regular triangles in 2D meshing
- **ODT**
  - best inscribed PL interpolant
  - compact simplices
  - Less slivers in 3D meshing

Q&A