## LECTURE 6: THE EXPONENTIAL MAP

### 1. One-parameter Subgroups

Let $G$ be a Lie group, $X_e \in T_eG$ be a tangent vector at the identity element and $X \in \mathfrak{g}$ the left invariant vector field generated by $X_e$. One can show that (exercise) any left invariant vector field on $G$ is complete. So for any $g \in G$ there is a unique integral curve of $X$ defined on the whole real line $\mathbb{R}$,

$$\gamma_g : \mathbb{R} \to G,$$

so that $\gamma_g(0) = g$. We are interested in the special map $\phi := \gamma_e$, i.e. the integral curve of $X$ that starts at $e$.

**Lemma 1.1.** *The map $\phi = \gamma_e$ is a Lie group homomorphism from $\mathbb{R}$ to $G$, i.e.*

$$\phi(s+t) = \phi(s)\phi(t)$$

*holds for all $s, t \in \mathbb{R}$.*

*Proof.* For any $s \in \mathbb{R}$ fixed, consider the curves

$$\gamma_1 : \mathbb{R} \to G, \quad t \mapsto \gamma_e(s)\gamma_e(t)$$

and

$$\gamma_2 : \mathbb{R} \to G, \quad t \mapsto \gamma_e(t+s).$$

We claim that both $\gamma_1$ and $\gamma_2$ are integral curves of the vector field $X$ with identical initial condition $\gamma_1(0) = \gamma_e(s) = \gamma_2(0)$. In fact, $\gamma_2$ is an integral curve of $X$ since it is the translation-reparametrization of the integral curve $\gamma_e$. For $\gamma_1$, use the left-invariance of $X$ we have

$$\dot{\gamma}_1(t) = (dL_{\gamma_e(s)})_{\gamma_e(t)}\dot{\gamma}_e(t) = (dL_{\gamma_e(s)})_{\gamma_e(t)}X_{\gamma_e(t)} = X_{\gamma_e(s)\gamma_e(t)} = X_{\gamma_1(t)}.$$

It follows that $\gamma_1 \equiv \gamma_2$. $\square$

**Definition 1.2.** A *one-parameter subgroup* of a Lie group $G$ is a Lie group homomorphism $\phi : \mathbb{R} \to G$, i.e. $\phi$ is smooth such that $\phi(s+t) = \phi(s)\phi(t)$ for all $s, t \in \mathbb{R}$.

So the arguments above shows that for any $X \in \mathfrak{g}$ (or any for any $X_e \in T_eG$), one can construct a one-parameter subgroup $\phi$ of $G$. Conversely, for any one-parameter subgroup $\phi : \mathbb{R} \to G$, we must have $\phi(0) = e$, and thus construct a left-invariant vector field $X$ on $G$ via the vector

$$X_e = \dot{\phi}(0) = (d\phi)_0(\frac{d}{dt}) \in T_eG.$$

It is not hard to see that different vectors in $T_eG$ give rise to different one-parameter subgroups, and different one-parameter subgroups give rise to different vectors in $T_eG$.

As a consequence, we get one-to-one correspondences between

- One-parameter subgroups of $G$.
- Left invariant vector fields on $G$.
- Tangent vectors at $e \in G$.

So we have three different descriptions of the Lie algebra $\mathfrak{g}$.

## 2. The Exponential Map

For any $X \in \mathfrak{g}$, let $\phi_X$ be the one-parameter subgroup of $G$ corresponding to $X$.

**Definition 2.1.** The *exponential map* of $G$ is the map

$$\exp : \mathfrak{g} \to G, \quad X \mapsto \phi_X(1).$$

Since $\widetilde{\phi}(s) = \phi_X(ts)$ is the one parameter subgroup corresponding to $tX$, we have

$$\exp(tX) = \phi_X(t).$$

*Example.* (1) For $G = \mathbb{R}^*$, we can identify $T_1G = \mathbb{R}$. For any $x \in T_1G = \mathbb{R}$, the map

$$\phi : \mathbb{R} \to G, \quad t \mapsto e^{tx}$$

is the one-parameter subgroup of $G$ with $\dot{\phi}(0) = x$. It follows $\exp(x) = e^x$.

(2) For $G = S^1$, we can identify $T_1S^1 = i\mathbb{R}$. The one-parameter subgroup corresponding to $ix \in T_1S^1 = i\mathbb{R}$ is

$$\phi : \mathbb{R} \to S^1, \quad t \mapsto e^{itx}.$$

So the exponential map is given by $\exp(ix) = e^{ix}$.

(3) For $G = \mathbb{R}$, we identify $T_0G = \mathbb{R}$. The one-parameter subgroup for $x \in \mathbb{R}$ is

$$\phi : \mathbb{R} \to \mathbb{R}, \quad t \mapsto tx.$$

So the exponential map is $\exp(x) = x$.

Note that the zero vector $0 \in T_eG$ generates the zero vector field on $G$, whose integral curve through $e$ is the constant curve. So $\exp(0) = e$.

**Lemma 2.2.** *The exponential map* $\exp : \mathfrak{g} \to G$ *is smooth, and if we identify both* $T_0\mathfrak{g}$ *and* $T_eG$ *with* $\mathfrak{g}$*,*

$$(d\exp)_0 = \mathrm{Id}.$$

*Proof.* Consider the map

$$\Phi : \mathbb{R} \times G \times \mathfrak{g} \to G \times \mathfrak{g}, \quad (t, g, X) \mapsto (g \cdot \exp(tX), X).$$

One can check that this is the flow on $G \times \mathfrak{g}$ corresponding to the (left invariant) vector field $(g, X) \mapsto (X_g, 0)$ on $G \times \mathfrak{g}$, thus it is smooth. It follows that $\exp = \pi_1 \circ \Phi|_{\{1\}\times\{e\}\times\mathfrak{g}}$ is smooth.

Since $\exp(tX) = \phi_X(t)$, $\frac{d}{dt}|_{t=0} \exp(tX) = X$. On the other hand,

$$\left.\frac{d}{dt}\right|_{t=0} \exp \circ tX = (d\exp)_0 \frac{d(Xt)}{dt} = (d\exp)_0 X.$$

We conclude that $(d\exp)_0$ equals to the identity map. □

Since $(d\exp)_0$ is bijective, we have

**Corollary 2.3.** $\exp$ *is a local diffeomorphism near* $0$*, i.e. it is a diffeomorphism from a neighborhood of* $0 \in T_eG$ *to a neighborhood of* $e \in G$*.*

Recall that for any Lie group homomorphism $\varphi : G \to H$, its differential at $e$,

$$d\varphi : \mathfrak{g} \to \mathfrak{h},$$

is a Lie algebra homomorphism.

**Proposition 2.4** (exp is Natural)**.** *Given any Lie group homomorphism* $\varphi : G \to H$*, the diagram*

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{d\varphi} & \mathfrak{h} \\ \Big\downarrow{\scriptstyle \exp_{\mathfrak{g}}} & & \Big\downarrow{\scriptstyle \exp_{\mathfrak{h}}} \\ G & \xrightarrow{\varphi} & H \end{array}$$

*is commutative, i.e.* $\varphi \circ \exp_{\mathfrak{g}} = \exp_{\mathfrak{h}} \circ d\varphi$*.*

*Proof.* Let $X \in \mathfrak{g}$, then

$$\varphi \circ \exp_{\mathfrak{g}} : \mathbb{R} \to H, \quad t \mapsto \varphi \circ \exp_{\mathfrak{g}}(tX)$$

is the one-parameter subgroup of $H$ associated to the vector

$$\left.\frac{d}{dt}\right|_{t=0} \varphi \circ \exp_{\mathfrak{g}}(tX) = d\varphi(X).$$

So $\varphi \circ \exp_{\mathfrak{g}}(tX) = \exp_{\mathfrak{h}} \circ t d\varphi(X)$. □

As an application, one can show that if $G$ is connected, any Lie group homomorphism $\varphi : G \to H$ is determined by the induced Lie algebra homomorphism $d\varphi : \mathfrak{g} \to \mathfrak{h}$.

## 3. Different Descriptions of Lie Bracket

Now we have three different descriptions of the Lie algebra $\mathfrak{g}$ of $G$. Consequently we should also have three different descriptions of the Lie bracket operation $[\cdot, \cdot]$:

(a) $\mathfrak{g}$ = the set of left invariant vector fields on $G$: For left invariant vector fields $X$ and $Y$ on $G$,

$$[X, Y] := XY - YX.$$

(b) $\mathfrak{g} = T_eG$: For $X, Y \in T_eG$,

$$[X, Y] := \mathrm{ad}(X)Y,$$

where $\mathrm{ad} : T_eG \to \mathrm{End}(T_eG)$ is defined as follows. Each element $g \in G$ gives rise to an automorphism

$$c(g) : G \to G, \quad x \mapsto gxg^{-1}.$$

Notice that $c(g)$ maps $e$ to $e$, its differential at $e$ gives us a linear map

$$\mathrm{Ad}_g = (dc(g))_e : T_eG \to T_eG.$$

In other words, we get a map (the *adjoint representation* of the Lie group $G$)

$$\mathrm{Ad} : G \to \mathrm{End}(T_eG), \quad g \mapsto \mathrm{Ad}_g.$$

Note that $\mathrm{Ad}(e)$ is the identity map in $\mathrm{End}(T_eG)$. Moreover, since $\mathrm{End}(T_eG)$ is a linear space, its tangent space at Id can be identified with $\mathrm{End}(T_eG)$ itself in a natural way. Taking derivative again at $e$, we get (the *adjoint representation* of the Lie algebra $\mathfrak{g}$)

$$\mathrm{ad} : T_eG \to \mathrm{End}(T_eG).$$

Applying the naturality of exp to the Lie group homomorphism $\mathrm{Ad} : G \to \mathrm{End}(\mathfrak{g})$ and to the conjugation map $c(g) : G \to G$, we have

**Proposition 3.1.** (1) $\mathrm{Ad}(\exp(tX)) = \exp(t\mathrm{ad}(X))$.
(2) $g(\exp tX)g^{-1} = \exp(t\mathrm{Ad}_gX)$.

(c) $\mathfrak{g} =$ the set of one-parameter subgroups: The one-parameter subgroups generated by $X, Y \in T_eG$ are $\phi_X$ and $\phi_Y$. Define

$$a(t, s) = \phi_X(t)\phi_Y(s)\phi_X(-t).$$

Then

$$[\phi_X, \phi_Y] := \text{the one-parameter subgroup generated by } \left.\frac{\partial}{\partial t}\right|_{t=0} \left.\frac{\partial}{\partial s}\right|_{s=0} a(t, s),$$

Now we show that the three different Lie bracket described above are equivalent:

**Theorem 3.2.** *The three different Lie brackets defined in (a), (b), (c) are equivalent.*

*Proof.* First let's compute $(\mathrm{ad}X)Y$. According to proposition 3.1,

$$(\mathrm{ad}X)Y = \left.\frac{d}{dt}\right|_{t=0} (\mathrm{Ad}(\exp tX)Y).$$

On the other hand, since $\mathrm{Ad}_g$ is the differential of $c(g)$, we have

$$\mathrm{Ad}(\exp tX)Y = \left.\frac{d}{ds}\right|_{s=0} c(\exp tX)\exp sY = \left.\frac{d}{ds}\right|_{s=0} \exp(tX)\exp(sY)\exp(-tX).$$

This shows that (b) is equivalent to (c). To show that they are also equivalent to (a), we compute for any $f \in C^\infty(G)$,

$$\begin{aligned}(\mathrm{ad}(X)Y)f &= (\frac{d}{dt}\Big|_{t=0}(\mathrm{Ad}(\exp tX)Y))f \\ &= \frac{\partial^2}{\partial s\partial t}\Big|_{s=t=0} f(\exp(tX)\exp(sY)\exp(-tX)) \\ &= \frac{\partial^2}{\partial s\partial t}\Big|_{s=t=0} f(\exp(tX)\exp(sY)) + \frac{\partial^2}{\partial s\partial t}\Big|_{s=t=0} f(\exp(sY)\exp(-tX)) \\ &= XYf(e) - YXf(e).\end{aligned}$$

$\square$