

## LECTURE 17: DIFFERENTIAL FORMS

### 1. DIFFERENTIAL FORMS

Now let  $M$  be a smooth manifold of dimension  $n$ . We start by defining the tensor product of vector bundles over  $M$ . Let  $(\pi_1, E_1, M)$  and  $(\pi_2, E_2, M)$  be two vector bundles over  $M$  of rank  $k_1, k_2$  respectively. Then the *tensor product bundle*  $(\pi_1 \otimes \pi_2, E_1 \otimes E_2, M)$  is the rank  $k_1 k_2$  vector bundle over  $M$  with fiber

$$(\pi_1 \otimes \pi_2)^{-1}(p) = (E_1)_p \otimes (E_2)_p = \text{span}\{e_i^1 \otimes e_j^2\},$$

where  $e_1^1, \dots, e_{k_1}^1$  is a basis of  $(E_1)_p$ , while  $e_1^2, \dots, e_{k_2}^2$  is a basis of  $(E_2)_p$ . The local trivialization map is

$$\Phi_U^1 \otimes \Phi_U^2 : (\pi_1 \otimes \pi_2)^{-1}(U) \rightarrow U \times (\mathbb{R}^{k_1} \otimes \mathbb{R}^{k_2}), \quad a_{ij}(p)e_i^1 \otimes e_j^2 \mapsto (p, a_{ij}(p)\Phi_U^1(e_i^1) \otimes \Phi_U^2(e_j^2)),$$

where  $\Phi_U^1$  and  $\Phi_U^2$  are local trivializations of  $E_1$  and  $E_2$  respectively. One can check that this does defines a vector bundle.

By using this way iteratively, one can construct the  $(k, l)$ -*tensor bundle* of  $M$

$$\otimes^{k,l} TM := (TM)^{\otimes k} \otimes (T^*M)^{\otimes l}.$$

**Definition 1.1.** Any smooth section of this bundle is called a *smooth  $(k, l)$ -tensor field* on  $M$ .

Note that locally any smooth  $(k, l)$ -tensor field can be expressed as

$$T = \sum T_{j_1 \dots j_l}^{i_1 \dots i_k}(x) \frac{\partial}{\partial x^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_k}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_l},$$

where  $T_{j_1 \dots j_l}^{i_1 \dots i_k}$ 's are smooth functions on a coordinate patch.

*Example.* A symmetric positive  $(0, 2)$ -tensor field  $g$  on  $M$  is called a Riemannian metric on  $M$ . Locally  $g = \sum g_{ij}(x) dx^i \otimes dx^j$ , where  $(g_{ij}(x))$  is a positive definite symmetric matrix depending smoothly on  $x$ .

Similarly one can define the *exterior power bundle*  $\Lambda^k T^*M$ , whose fiber at point  $p \in M$  is the linear space  $\Lambda^k T_p^*M$ . A local trivialization  $\Lambda^k \Phi_U : (\Lambda^k \pi)^{-1}(U) \rightarrow U \times \Lambda^k \mathbb{R}^n$  is given by

$$\sum \omega_{i_1, \dots, i_k}(p) dx^{i_1} \wedge \dots \wedge dx^{i_k} \mapsto (p, \sum \omega_{i_1, \dots, i_k}(p) \Phi_U(dx^{i_1}) \wedge \dots \wedge \Phi_U(dx^{i_k})).$$

**Definition 1.2.** A *smooth  $k$ -form* (or simply a  $k$ -*form*)  $\omega$  on  $M$  is a smooth section of  $\Lambda^k T^*M$ . The set of  $k$ -forms is denoted by  $\Omega^k(M)$ .

Note that  $\Omega^0(M) = C^\infty(M)$ , and  $\Omega^k(M) = 0$  for  $k > n$ .

Locally in a chart  $U$  any smooth  $k$ -form  $\omega$  can be represented as

$$\omega = \sum_I \omega_I dx^I = \sum_I \omega_{i_1, \dots, i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k},$$

where  $I = \{i_1 < \dots < i_k\}$  is taking as increasing  $k$ -tuples in  $\{1, \dots, n\}$ , and  $\omega_I$ 's are smooth functions on  $U$ . Observe that if  $X_1, \dots, X_k$  are smooth vector fields on  $M$ , and if we denote  $X_j = \sum X_j^i \partial_i$ , then

$$dx^{i_1} \wedge \dots \wedge dx^{i_k}(X_1, \dots, X_k) = \det(X_s^{i_r})_{1 \leq r, s \leq k}.$$

As a consequence, if  $\omega$  is a smooth  $k$ -form on  $M$ , then for any smooth vector fields  $X_1, \dots, X_k$  on  $M$ ,  $\omega(X_1, \dots, X_k)$  is a smooth function on  $M$ .

One can define the *wedge product* of two differential forms as before. For example,

$$(dx^1 + 2dx^2) \wedge (dx^1 \wedge dx^2 - dx^2 \wedge dx^3 + 3dx^1 \wedge dx^3) = -7dx^1 \wedge dx^2 \wedge dx^3.$$

Note that the wedge product is still anti-commutative: if  $\omega \in \Omega^k(M), \eta \in \Omega^l(M)$ , then

$$\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega \in \Omega^{k+l}(M).$$

Similarly one can contract a differential  $k$ -form  $\omega$  with a smooth vector field  $X$ , which gives a  $(k-1)$ -form  $\iota_X \omega$  on  $M$  defined as

$$\iota_X \omega(X_1, \dots, X_{k-1}) = \omega(X, X_1, \dots, X_{k-1}).$$

$\iota_X \omega$  is called the *contraction* of  $\omega$  with respect to  $X$ . Obviously the map  $\iota_X : \Omega^k(M) \rightarrow \Omega^{k-1}(M)$  is linear and satisfies  $\iota_X \circ \iota_X = 0$ . Moreover, a direct computation yields

$$\iota_X dx^{i_1} \wedge \dots \wedge dx^{i_k} = \sum_r (-1)^{r-1} dx^{i_r}(X) dx^{i_1} \wedge \dots \wedge \widehat{dx^{i_r}} \wedge \dots \wedge dx^{i_k}.$$

Using this, one can easily see  $\iota_X$  is an *anti-derivative*:

**Lemma 1.3.** *Let  $X$  be a smooth vector field,  $\omega \in \Omega^k(M)$  and  $\eta \in \Omega^l(M)$ . Then*

$$\iota_X(\omega \wedge \eta) = (\iota_X \omega) \wedge \eta + (-1)^k \omega \wedge \iota_X \eta.$$

We can also pull back differential forms. In fact, if  $\varphi : M \rightarrow N$  is a smooth map, then for any  $p \in M$ ,  $d\varphi_p$  is a linear map from  $T_p M$  to  $T_{f(p)} N$ . So one can define the *pull-back* of  $\omega \in \Omega^k(N)$  to be the  $k$ -form  $\varphi^* \omega \in \Omega^k(M)$  defined at any  $p \in M$  by

$$(\varphi^* \omega)_p(X_1, \dots, X_k) = \omega_{f(p)}(d\varphi_p(X_1), \dots, d\varphi_p(X_k)).$$

From the linear theory one can easily see that  $\varphi^* : \Omega^k(N) \rightarrow \Omega^k(M)$  is linear and

$$\varphi^*(\omega \wedge \eta) = \varphi^* \omega \wedge \varphi^* \eta.$$

Moreover, by chain rule one can prove  $(\psi \circ \varphi)^* = \psi^* \circ \varphi^*$ .

*Remark.* If  $k = 0$ , then the pullback  $\varphi^*$  is exactly the pullback  $\varphi^* : C^\infty(N) \rightarrow C^\infty(M)$  on functions that we learned earlier.

Let  $\{y^1, \dots, y^m\}$  be local coordinates on  $N$ . Then we get

$$\varphi^*\left(\sum_I \omega_I dy^{i_1} \wedge \dots \wedge dy^{i_k}\right) = \sum_I (\varphi^* \omega_I) \varphi^*(dy^{i_1}) \wedge \dots \wedge \varphi^*(dy^{i_k}).$$

Note that by definition,

$$(\varphi^* dy^i)_p(X_p) = dy_{\varphi(p)}(d\varphi_p(X_p)) = d(y \circ \varphi)_p(X_p) = d(\varphi^* y^i)_p(X_p).$$

As a consequence, we get

**Corollary 1.4.** *Let  $\varphi : M \rightarrow N$  be a smooth map between  $n$ -manifolds, and  $(x^i), (y^i)$  be smooth coordinates on  $U \subset M$  and  $V \subset N$  respectively. Then*

$$\varphi^*(g dy^1 \wedge \dots \wedge dy^n) = (\varphi^* g)(\det d\varphi) dx^1 \wedge \dots \wedge dx^n.$$

*Proof.* If we denote  $\varphi = (\varphi^1, \dots, \varphi^n)$  in the coordinate system  $(y^i)$ , then we have get

$$\varphi^*(g dy^1 \wedge \dots \wedge dy^n) = (\varphi^* g) d\varphi^1 \wedge \dots \wedge d\varphi^n.$$

So the formula to be proved follows from

$$d\varphi^1 \wedge \dots \wedge d\varphi^n(\partial_1, \dots, \partial_n) = \det(d\varphi^i(\partial_j)) = \det\left(\frac{\partial \varphi^i}{\partial x_j}\right) = \det(d\varphi).$$

□

## 2. THE EXTERIOR DERIVATIVE

Another very important operation on differential forms is the exterior derivative. It is not a pointwise operation, but a local operation. Suppose  $\omega$  is a  $k$ -form on  $M$ , so that locally

$$\omega = \sum_I \omega_{i_1, \dots, i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}.$$

**Definition 2.1.** The *exterior derivative* of  $\omega$  is the  $(k+1)$ -form  $d\omega$  given by the formula

$$(1) \quad d\omega = \sum_I d\omega_{i_1, \dots, i_k} \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k} = \sum_{I, i} \partial_i(\omega_{i_1, \dots, i_k}) dx^i \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}.$$

Before we proceed, we need to clarify that  $d\omega$  defined above is well-defined. In other words, if we choose different coordinate patches (in which  $\omega$  has different local expressions),  $d\omega$  calculated in different ways should match. To prove this, we will give a new expression (the invariant formula) of  $d\omega$  and check that locally we have the previous expression.

**Theorem 2.2.** *For any  $\omega \in \Omega^k(M)$ , the formula*

$$\begin{aligned} d\omega(X_1, \dots, X_{k+1}) &:= \sum_i (-1)^{i-1} X_i(\omega(X_1, \dots, \widehat{X}_i, \dots, X_{k+1})) \\ &\quad + \sum_{i < j} (-1)^{i+j} \omega([X_i, X_j], X_1, \dots, \widehat{X}_i, \dots, \widehat{X}_j, \dots, X_{k+1}). \end{aligned}$$

*defines a  $(k+1)$ -form  $d\omega \in \Omega^{k+1}(M)$  which is given by (1) locally.*

*Remark.* Special cases of the invariant formula

- If  $f \in C^\infty(M) = \Omega^0(M)$ , then  $df(X) = Xf$ .
- if  $\omega \in \Omega^1(M)$ , then for any smooth vector fields  $X, Y$  on  $M$ ,

$$d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y]).$$

- If  $\omega \in \Omega^2(M)$ , then for any smooth vector fields  $X, Y, Z$  on  $M$ ,

$$\begin{aligned} d\omega(X, Y, Z) = & X(\omega(Y, Z)) - Y(\omega(X, Z)) + Z(\omega(X, Y)) \\ & - \omega([X, Y], Z) + \omega([X, Z], Y) - \omega([Y, Z], X). \end{aligned}$$

*Proof of theorem 2.2.* To show that  $d\omega$  is a  $(k+1)$ -form, we need to show that

- (1)  $d\omega$  is anti-symmetric, i.e. for any  $r < s$ ,

$$d\omega(X_1, \dots, X_r, \dots, X_s, \dots, X_{k+1}) = -d\omega(X_1, \dots, X_s, \dots, X_r, \dots, X_{k+1}).$$

This follows from a simple but very long computation. But you can get the idea by staring at the formulae for  $d\omega$  with  $\omega \in \Omega^1(M)$  or  $\omega \in \Omega^2(M)$  above.

- (2)  $d\omega$  is multi-linear at each point, i.e.  $d\omega$  is  $C^\infty(M)$ -linear on  $M$ . Note that  $d\omega$  is obviously  $\mathbb{R}$ -linear. So in view of (1), it is enough to prove for any  $f \in C^\infty(M)$ ,

$$d\omega(fX_1, X_2, \dots, X_{k+1}) = fd\omega(X_1, \dots, X_{k+1}).$$

This can be checked by a direct computation:

$$\begin{aligned} d\omega(fX_1, X_2, \dots, X_{k+1}) &= fX_1(\omega(X_2, \dots, X_{k+1})) \\ &\quad + \sum_{i>1} (-1)^{i-1} X_i(\omega(fX_1, \dots, \widehat{X}_i, \dots, X_{k+1})) \\ &\quad + \sum_{i>1} (-1)^{i+1} \omega([fX_1, X_i], X_2, \dots, \widehat{X}_i, \dots, X_{k+1}) \\ &\quad + \sum_{1<i<j} (-1)^{i+j} \omega([X_i, X_j], fX_1, \dots, \widehat{X}_i, \dots, \widehat{X}_j, \dots, X_{k+1}) \\ &= fd\omega(X_1, \dots, X_{k+1}) \\ &\quad + \sum_{i>1} (-1)^{i-1} (X_i f) \omega(X_1, \dots, \widehat{X}_i, \dots, X_{k+1}) \\ &\quad - \sum_{i>1} (-1)^{i+1} (X_i f) \omega(X_1, \dots, \widehat{X}_i, \dots, X_{k+1}) \\ &= fd\omega(X_1, \dots, X_{k+1}). \end{aligned}$$

It remains to check that  $d\omega$  has the local expression (1) as we anticipated. Obviously the map

$$d : \Omega^k(M) \rightarrow \Omega^{k+1}(M)$$

is linear. So without loss of generality, we may assume

$$\omega = f dx^1 \wedge \dots \wedge dx^k$$

in a local chart  $U$ . Note that  $[\partial_i, \partial_j] = 0$ . It follows that for any increasing indices  $j_1 < \cdots < j_{k+1}$ , the right hand side of

$$d\omega(\partial_{j_1}, \dots, \partial_{j_{k+1}}) = \sum_i (-1)^{i-1} \partial_{j_i}(\omega(\partial_{j_1}, \dots, \hat{\partial}_{j_i}, \dots, \partial_{j_{k+1}}))$$

vanishes except for the case  $j_1 = 1, \dots, j_k = k$  and  $i = k + 1$ . In other words, the only non-zero terms in all possible expressions  $d\omega(\partial_{j_1}, \dots, \partial_{j_{k+1}})$  are

$$d\omega(\partial_1, \dots, \partial_k, \partial_r) = (-1)^k \partial_r(f).$$

It follows that

$$d\omega = \sum_{r>k} (-1)^k \partial_r(f) dx^1 \wedge \cdots \wedge dx^k \wedge dx^r = \sum \partial_r(f) dx^r \wedge dx^1 \wedge \cdots \wedge dx^k,$$

which is exactly the local expression we want.  $\square$

Here are some important properties of the operator  $d : \Omega^k(M) \rightarrow \Omega^{k+1}(M)$ .

**Lemma 2.3.** *For any  $\omega \in \Omega^k(M), \eta \in \Omega^l(M)$ , we have  $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$ .*

*Proof.* For simplicity one may assume  $\omega = f dx^{i_1} \wedge \cdots \wedge dx^{i_k}$  and  $\eta = g dx^{j_1} \wedge \cdots \wedge dx^{j_l}$ , with indices set  $I \cap J = \emptyset$ . Then the formula follows from a direct computation.  $\square$

**Corollary 2.4.** *For any  $\omega \in \Omega^k(M)$ ,  $d(d\omega) = 0$ .*

*Proof.* We first check this for  $k = 0$ :

$$d(df)(X, Y) = X(df(Y)) - Y(df(X)) - df([X, Y]) = X(Y(f)) - Y(X(f)) - [X, Y]f = 0.$$

For  $k > 0$ , we may assume  $\omega = f dx^1 \wedge \cdots \wedge dx^k$  locally, so that

$$d(d\omega) = d(df \wedge dx^1 \wedge \cdots \wedge dx^k) = d(df) \wedge dx^1 \wedge \cdots \wedge dx^k + \sum_i (-1)^i df \wedge dx^1 \wedge \cdots \wedge d(dx^i) \wedge \cdots \wedge dx^k = 0$$

since  $d(df) = 0$  and  $d(dx^i) = 0$ .  $\square$

**Lemma 2.5.** *If  $\varphi : M \rightarrow N$  is smooth, then for any  $\omega \in \Omega^k(N)$ ,  $\varphi^*d\omega = d(\varphi^*\omega)$ .*

*Proof.* We have already checked this for 1-forms. In general, It is enough to check this for  $\omega = f dx^1 \wedge \cdots \wedge dx^k$ , in which case

$$\varphi^*d\omega = \varphi^*(df \wedge dx^1 \wedge \cdots \wedge dx^k) = d(f \circ \varphi) \wedge d(x^1 \circ \varphi) \wedge \cdots \wedge d(x^k \circ \varphi)$$

while

$$d\varphi^*\omega = d(f \circ \varphi d(x^1 \circ \varphi) \wedge \cdots \wedge d(x^k \circ \varphi)) = d(f \circ \varphi) \wedge d(x^1 \circ \varphi) \wedge \cdots \wedge d(x^k \circ \varphi).$$

$\square$