

LECTURE 22: THE EXTERIOR DERIVATIVE

1. THE EXTERIOR DERIVATIVE

¶ The exterior derivative: A local definition.

Now we define the exterior derivative for differential forms. It generalizes the conception of the differential on functions, and it is the most important operation for the rest of this semester. Unlike the wedge product, the interior product and the pull-back operations that we defined last time, the exterior derivative is no longer a pointwise operation, but is a local operation (i.e. depends on the “nearby values”)

We start with $f \in \Omega^0(U) = C^\infty(U)$. In this case we have already seen that $df \in \Omega^1(U)$. So we get a linear map

$$d : \Omega^0(U) \rightarrow \Omega^1(U), \quad f \mapsto df.$$

Locally on each coordinate chart we have

$$df = \sum_i (\partial_i f) dx^i.$$

We also have an “invariant definition” of $df \in \Omega^1(U)$, via

$$df(X) = Xf, \quad \forall X \in \Gamma^\infty(TU).$$

Now suppose ω is a k -form on M , so that locally

$$\omega = \sum_I \omega_{i_1, \dots, i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}.$$

We want to define $d\omega$ as a $(k+1)$ -form. It is natural to define

Definition 1.1. The *exterior derivative* of ω is the $(k+1)$ -form $d\omega$ given by the formula

$$(1) \quad d\omega = \sum_I d\omega_{i_1, \dots, i_k} \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k} = \sum_{I,i} \partial_i(\omega_{i_1, \dots, i_k}) dx^i \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}.$$

¶ The exterior derivative: A coordinate-free definition.

Before we proceed, we need to clarify that $d\omega$ defined above is well-defined. In other words, the $(k+1)$ -form $d\omega$ defined above should be independent of the choices of coordinate patches.

Usually one has two ways to prove the well-definedness of a conception on manifolds. One is to check that the definition is unchanged if one use another coordinate chart (for example, the definition of smoothness of a function), the other is to give an equivalent but coordinate-free definition (usually called *the invariant formulation*). We will take the second approach here since the coordinate-free expression of $d\omega$ is also very useful.

Let's start with small k 's to find out the invariant formula of $d\omega$.

- For $k = 0$, i.e. $\omega = f \in C^\infty(U)$, we can regard df as a $C^\infty(U)$ -linear map

$$df : \Gamma^\infty(TU) \rightarrow C^\infty(U)$$

such that

$$df(X) = Xf.$$

- For $k = 1$, i.e. $\omega \in \Omega^1(U)$, we want to regard $d\omega$ as a $C^\infty(U)$ -bilinear map

$$d\omega : \Gamma^\infty(TU) \times \Gamma^\infty(TU) \rightarrow C^\infty(U).$$

We write $\omega = \sum_i \omega_i dx^i$, $X = \sum_k X^k \partial_k$ and $Y = \sum_l Y^l \partial_l$. Then

$$\begin{aligned} d\omega(X, Y) &= \sum_{i,j,k,l} (\partial_j \omega_i) dx^j \wedge dx^i (X^k \partial_k, Y^l \partial_l) \\ &= \sum_{i,j} ((\partial_j \omega_i) X^j Y^i - (\partial_j \omega_i) X^i Y^j) \\ &= \sum_{i,j} (X^j \partial_j (\omega_i Y^i) - \omega_i X^j \partial_j (Y^i) - Y^j \partial_j (\omega_i X^i) + \omega_i Y^j \partial_j (X^i)) \\ &= X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y]). \end{aligned}$$

So we arrive at

$$d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y]).$$

- For $k = 2$, i.e. $\omega \in \Omega^2(U)$, by tedious computations one gets: as a $C^\infty(U)$ -trilinear map

$$d\omega : \Gamma^\infty(TU) \times \Gamma^\infty(TU) \times \Gamma^\infty(TU) \rightarrow C^\infty(U),$$

one has

$$d\omega(X, Y, Z) = X(\omega(Y, Z)) - Y(\omega(X, Z)) + Z(\omega(X, Y)) - \omega([X, Y], Z) + \omega([X, Z], Y) - \omega([Y, Z], X).$$

So we are naturally led to the following *the invariant formula* for $d\omega$:

Theorem 1.2. For any $\omega \in \Omega^k(U)$, the $(k+1)$ -form $d\omega$, viewed as a $C^\infty(U)$ -multilinear map

$$d\omega : \Gamma^\infty(TU) \times \cdots \times \Gamma^\infty(TU) \rightarrow C^\infty(U),$$

is given by the formula

$$(2) \quad \begin{aligned} d\omega(X_1, \dots, X_{k+1}) &:= \sum_i (-1)^{i-1} X_i(\omega(X_1, \dots, \widehat{X}_i, \dots, X_{k+1})) \\ &+ \sum_{i < j} (-1)^{i+j} \omega([X_i, X_j], X_1, \dots, \widehat{X}_i, \dots, \widehat{X}_j, \dots, X_{k+1}). \end{aligned}$$

Sketch of proof. Define $(k+1)$ -form $\widetilde{d\omega}$ via the formula (2). We need to show that

- (1) $\widetilde{d\omega}$ is anti-symmetric, i.e. for any $r < s$, a simple but messy computation yields

$$\widetilde{d\omega}(X_1, \dots, X_r, \dots, X_s, \dots, X_{k+1}) = -\widetilde{d\omega}(X_1, \dots, X_s, \dots, X_r, \dots, X_{k+1}).$$

- (2) $\widetilde{d}\omega$ is multi-linear at each point, i.e. $\widetilde{d}\omega$ is $C^\infty(U)$ -linear on U . Note that $\widetilde{d}\omega$ is obviously \mathbb{R} -linear. So in view of (1), it is enough to prove for any $f \in C^\infty(U)$,

$$\widetilde{d}\omega(fX_1, X_2, \dots, X_{k+1}) = f\widetilde{d}\omega(X_1, \dots, X_{k+1}).$$

This can be checked by a direct computation:

$$\begin{aligned} \widetilde{d}\omega(fX_1, X_2, \dots, X_{k+1}) &= fX_1(\omega(X_2, \dots, X_{k+1})) \\ &\quad + \sum_{i>1} (-1)^{i-1} X_i(\omega(fX_1, \dots, \widehat{X}_i, \dots, X_{k+1})) \\ &\quad + \sum_{i>1} (-1)^{i+1} \omega([fX_1, X_i], X_2, \dots, \widehat{X}_i, \dots, X_{k+1}) \\ &\quad + \sum_{1<i<j} (-1)^{i+j} \omega([X_i, X_j], fX_1, \dots, \widehat{X}_i, \dots, \widehat{X}_j, \dots, X_{k+1}) \\ &= f\widetilde{d}\omega(X_1, \dots, X_{k+1}) \\ &\quad + \sum_{i>1} (-1)^{i-1} (X_i f) \omega(X_1, \dots, \widehat{X}_i, \dots, X_{k+1}) \\ &\quad - \sum_{i>1} (-1)^{i+1} (X_i f) \omega(X_1, \dots, \widehat{X}_i, \dots, X_{k+1}) \\ &= f\widetilde{d}\omega(X_1, \dots, X_{k+1}). \end{aligned}$$

- (3) It remains to check that $\widetilde{d}\omega$ has the local expression (1) as desired. Obviously the map

$$d : \Omega^k(U) \rightarrow \Omega^{k+1}(U)$$

is linear. So without loss of generality, we may assume

$$\omega = f dx^1 \wedge \dots \wedge dx^k$$

in a local chart U . Note that $[\partial_i, \partial_j] = 0$. It follows that for any increasing indices $j_1 < \dots < j_{k+1}$, the right hand side of

$$\widetilde{d}\omega(\partial_{j_1}, \dots, \partial_{j_{k+1}}) = \sum_i (-1)^{i-1} \partial_{j_i}(\omega(\partial_{j_1}, \dots, \widehat{\partial}_{j_i}, \dots, \partial_{j_{k+1}}))$$

vanishes except for the case $j_1 = 1, \dots, j_k = k$ and $i = k+1$ (and thus $j_i \geq k+1$). In other words, the only non-zero terms in all possible expressions $\widetilde{d}\omega(\partial_{j_1}, \dots, \partial_{j_{k+1}})$ are

$$\widetilde{d}\omega(\partial_1, \dots, \partial_k, \partial_r) = (-1)^k \partial_r(f).$$

It follows that

$$\widetilde{d}\omega = \sum_{r>k} (-1)^k \partial_r(f) dx^1 \wedge \dots \wedge dx^k \wedge dx^r = \sum \partial_r(f) dx^r \wedge dx^1 \wedge \dots \wedge dx^k,$$

which is exactly the local expression in Definition 1.1. \square

¶ Properties of the exterior derivative.

The following properties will be used frequently later in this course:

Proposition 1.3. *Suppose $\omega \in \Omega^k(U)$, $\eta \in \Omega^l(U)$, $X \in \Gamma^\infty(TU)$ and $\varphi \in C^\infty(U', U)$. Then*

- (1) $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$.
- (2) $d \circ d = 0$.
- (3) $\varphi^* \circ d = d \circ \varphi^*$.

Proof. (1): Since d is linear, it is enough to assume $\omega = f dx^{i_1} \wedge \cdots \wedge dx^{i_k}$ and $\eta = g dx^{j_1} \wedge \cdots \wedge dx^{j_l}$, with indices set $I \cap J = \emptyset$. Then the formula follows from a direct computation:

$$\begin{aligned} d(\omega \wedge \eta) &= d(fg dx^{i_1} \wedge \cdots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_l}) \\ &= \sum_i \partial_i (fg) dx^i \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_l} \\ &= \sum_i (\partial_i f) dx^i \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} \wedge \eta + (-1)^k \omega \wedge \sum_i (\partial_i g) dx^i \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_l} \\ &\quad - d\omega \wedge \eta + (-1)^k \omega \wedge d\eta. \end{aligned}$$

(2): We first check this for $k = 0$:

$$d(df)(X, Y) = X(df(Y)) - Y(df(X)) - df([X, Y]) = X(Y(f)) - Y(X(f)) - [X, Y]f = 0.$$

For $k > 0$, by linearity we may assume $\omega = f dx^1 \wedge \cdots \wedge dx^k$. Since $ddf = 0$ and $ddx^i = 0$, we get

$$\begin{aligned} d(d\omega) &= d(df \wedge dx^1 \wedge \cdots \wedge dx^k) \\ &= d(df) \wedge dx^1 \wedge \cdots \wedge dx^k + \sum_i (-1)^i df \wedge dx^1 \wedge \cdots \wedge d(dx^i) \wedge \cdots \wedge dx^k \\ &= 0. \end{aligned}$$

(3): Again we first check this for $k = 0$:

$$(\varphi^* df)_p(X_p) = df_{\varphi(p)}(d\varphi_p(X_p)) = d(\varphi^* f)_p(X_p).$$

In general, assume $\omega = f dx^1 \wedge \cdots \wedge dx^k$. Then by (1), (2) and Proposition 3.5(2) in Lecture 21,

$$\begin{aligned} \varphi^* d\omega &= \varphi^*(df \wedge dx^1 \wedge \cdots \wedge dx^k) \\ &= \varphi^*(df) \wedge \varphi^*(dx^1) \wedge \cdots \wedge \varphi^*(dx^k) \\ &= d(\varphi^* f) \wedge d(\varphi^* x^1) \wedge \cdots \wedge d(\varphi^* x^k) \\ &= d(\varphi^* f d(\varphi^* x^1) \wedge \cdots \wedge d(\varphi^* x^k)) \\ &= d(\varphi^* \omega). \end{aligned}$$

□

2. **READING MATERIALS:** THE LIE DERIVATIVES (CONTINUED)¶ **The Lie derivative of differential forms along a vector field.**

Recall that in Lecture 15, we defined the Lie derivative of functions:

The *Lie derivative* of a $f \in C^\infty(M)$ with respect to $X \in \Gamma^\infty(TM)$ is

$$\mathcal{L}_X(f) := \left. \frac{d}{dt} \right|_{t=0} \phi_t^* f \quad \left(= \lim_{t \rightarrow 0} \frac{\phi_t^* f - f}{t} \right),$$

where ϕ_t is the (local) flow generated by the vector field X .

Since we have defined pull-back ϕ^* on differential forms, one can easily define the Lie derivative of differential forms:

Definition 2.1. The *Lie derivative* of a $\omega \in \Omega^k(M)$ with respect to $X \in \Gamma^\infty(TM)$ is

$$\mathcal{L}_X(\omega) := \left. \frac{d}{dt} \right|_{t=0} \phi_t^* \omega \quad \left(= \lim_{t \rightarrow 0} \frac{\phi_t^* \omega - \omega}{t} \right),$$

where ϕ_t is the (local) flow generated by the vector field X .

Note that $\mathcal{L}_X f$ that we defined in Lecture 5 is just a special case of this definition, since $C^\infty(M) = \Omega^0(M)$.

We have seen in PSet 5-1-4 that

$$\mathcal{L}_X f = Xf \quad \text{and} \quad \mathcal{L}_{[X,Y]} f = \mathcal{L}_X \mathcal{L}_Y(f) - \mathcal{L}_Y \mathcal{L}_X(f).$$

For the Lie derivative defined on differential forms, we have

Proposition 2.2. Suppose $\omega \in \Omega^k(M)$, $\eta \in \Omega^l(M)$ and $X, X_1, X_2 \in \Gamma^\infty(TM)$. Then

- (1) $d\mathcal{L}_X \omega = \mathcal{L}_X d\omega$.
- (2) $\mathcal{L}_X(\omega \wedge \eta) = \mathcal{L}_X \omega \wedge \eta + \omega \wedge \mathcal{L}_X \eta$.
- (3) [**Cartan's Magic Formula**] $\mathcal{L}_X \omega = d\iota_X \omega + \iota_X d\omega$.
- (4) $\mathcal{L}_{[X_1, X_2]} \omega = \mathcal{L}_{X_1} \mathcal{L}_{X_2} \omega - \mathcal{L}_{X_2} \mathcal{L}_{X_1} \omega$.
- (5) $(\mathcal{L}_X \omega)(X_1, \dots, X_k) = \mathcal{L}_X(\omega(X_1, \dots, X_k)) - \sum_i \omega(X_1, \dots, \mathcal{L}_X X_i, \dots, X_k)$.

Sketch of proof (Details left as an exercise.)

(1) follows from $\varphi^* d = d\varphi^*$, and (2) follows from $\varphi^*(\omega \wedge \eta) = \varphi^* \omega \wedge \varphi^* \eta$.

To prove (3), one proceed by induction on k . First it is not hard to prove Cartan's magic formula for $k = 0$ (where we set $\iota_X f = 0$). For a general k -form ω , by linearity, we may assume that locally $\omega = f dx^1 \wedge \dots \wedge dx^k = dx^1 \wedge \omega_1$, where $\omega_1 = f dx^2 \wedge \dots \wedge dx^k$. Then it is not hard to check Cartan's magic formula for ω by induction. (Do we need to prove Cartan's magic formula for $k = 1$ for the induction process to work?)

(4) also follows from a similar induction argument.

To prove (5), one can first check it for simple 1-forms like $\omega = dx^1$. Then use (2) and Proposition 2.6 in Lecture 21. \square

¶ **The Lie derivative of differential forms along a vector field.**

In Lecture 15 we also defined the Lie derivatives of vector fields:

The *Lie derivative* of $Y \in \Gamma^\infty(TM)$ with respect to $X \in \Gamma^\infty(TM)$ is

$$\mathcal{L}_X(Y) := \left. \frac{d}{dt} \right|_{t=0} (\phi_{-t})_* Y \quad \left(= \lim_{t \rightarrow 0} \frac{(\phi_{-t})_* Y_{\phi_t(p)} - Y_p}{t} \right).$$

where the push-forward $\varphi_* X$ on N is defined by

$$(\varphi_* X)_{\varphi(p)} = d\varphi_p(X_p), \quad \forall p \in M.$$

and we have seen in PSet 5-1-4 that

$$\mathcal{L}_X Y = [X, Y] \quad \text{and} \quad \mathcal{L}_{[X_1, X_2]} Y = \mathcal{L}_{X_1} \mathcal{L}_{X_2}(Y) - \mathcal{L}_{X_2} \mathcal{L}_{X_1}(Y).$$

It is then straightforward to extend the Lie derivatives to tensor fields by using

$$\mathcal{L}_X(T \otimes S) := (\mathcal{L}_X T) \otimes S + T \otimes (\mathcal{L}_X S).$$

One can show that for a general (l, k) -tensor T , the Lie derivative $\mathcal{L}_X T$ is the (l, k) -tensor defined via

$$\begin{aligned} (\mathcal{L}_X T)(\omega_1, \dots, \omega_l, X_1, \dots, X_k) &= X(T(\omega_1, \dots, \omega_l, X_1, \dots, X_k)) \\ &\quad - \sum_i T(\omega_1, \dots, \mathcal{L}_X \omega_i, \dots, \omega_l, X_1, \dots, X_k) \\ &\quad - \sum_j T(\omega_1, \dots, \omega_l, X_1, \dots, \mathcal{L}_X X_j, \dots, X_k) \end{aligned}$$