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Stochastic modeling of multidimensional diffusion in the radiation belts

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[1] A new code for solving radiation belt diffusion equations has been developed and applied to the 2-D bounce-averaged energy pitch angle quasi-linear diffusion equation. The code uses Monte Carlo methods to solve Itô stochastic differential equations (SDEs) which are mathematically equivalent to radiation belt diffusion equations. We show that our SDE code solves the diffusion equation with off-diagonal diffusion coefficients in contrast to standard finite difference codes which are generally unstable when off-diagonal diffusion coefficients are included. Our results are in excellent agreement with previous results. We have also investigated effects of assuming purely parallel propagating electromagnetic waves when calculating the diffusion coefficients and find that this assumption leads to errors of more than an order of magnitude in flux at some equatorial pitch angles for the specific chorus wave model we use. Further work is needed to investigate the sensitivity of our results to the wave model parameters. Generalization of the method to 3-D is straightforward, thus making this method a very promising new way to investigate the relative roles of pitch angle, energy, and radial diffusion in radiation belt dynamics.

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1. Introduction

[2] The Earth's outer radiation belt is very dynamic, and electron fluxes can vary by several orders of magnitude during storm times, which makes it very hazardous to spacecrafts and astronauts [e.g., *Baker et al.*, 1997]. Quasi-linear diffusion theory has been used to evaluate dynamic changes of particle fluxes in the radiation belts [*Albert*, 2004; *Albert and Young*, 2005; *Horne and Thorne*, 2003; *Horne et al.*, 2003]. Using the quasi-linear diffusion theory to model radiation belt dynamics requires at least two kinds of computations: numerical solution of a diffusion equation, which is a one-dimensional or multidimensional Fokker-Planck equation, depending on diffusion processes we are interested in, and calculation of diffusion coefficients.

[3] *Albert* [2004] has shown that numerical problems arise when applying standard finite difference methods to pitch angle and energy diffusion equations because of rapidly varying off-diagonal diffusion coefficients. *Albert and Young* [2005] developed a method for the 2-D diffusion equation which diagonalizes the diffusion tensor by transforming to a new set of coordinates and solves the transformed equation by simple finite difference methods. In this work we introduce another method which uses probabilistic representations of solutions of Fokker-Planck equations [*Freidlin*, 1985; *Costantini et al.*, 1998] via stochastic differential equations (SDEs), and we develop a 2-D code for solving pitch angle and energy diffusion equations. Compared with finite difference methods, the SDE method has three main advantages. First, the SDE method is very efficient when solutions on only a small number of points are desired, particularly when applied to high-dimensional problems, and it is easy to code and parallelize, with parallelization efficiency close to one. Second, with the SDE method, we are able to handle complicated boundary geometry other than constant-coordinate boundaries (see section 2.2). Third, generalization of the SDE method to higher dimensions is straightforward, and we expect the method to be applicable to general 3-D radiation belt diffusion equations. For more applications of similar methods using relations between Fokker-Planck equations and SDEs, see, e.g., *Zhang* [1999], *Albright et al.* [2003], *Alanko-Huotari et al.* [2007], *Qin et al.* [2005], and *Yamada et al.* [1998].

[4] Besides solving diffusion equations, correctly calculating quasi-linear diffusion coefficients is also important for numerical modeling of the radiation belt dynamics using quasi-linear theory. *Albert* [2005] and *Glauert and Horne* [2005] have shown full calculations of diffusion coefficients for cyclotron resonant wave-particle interactions, where up to $n = \pm 5$ resonances are included. However, the full calculation of diffusion coefficients is very time consuming. *Summers* [2005] derived simplified formulae for coefficients with a parallel propagation approximation (and hence

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only the $n = -1$ resonance is included [*Albert*, 2007]), and the computation becomes much faster. *Shprits et al.* [2006] calculated bounce-averaged pitch angle and energy diffusion coefficients $D_{\alpha 0 \alpha 0}$ and D_{pp} with the parallel propagation approximation for $E \leq 1$ MeV particles and compared them with fully calculated coefficients from the PADIE code of *Glauert and Horne* [2005]. They concluded that coefficients for field-aligned waves are close to coefficients for waves with mildly oblique wave normal angle distribution from the PADIE code. However, using the wave model from *Horne et al.* [2005], we compute particle fluxes and we show that for $E = 2$ MeV electrons, $D_{\alpha 0 \alpha 0}$ and D_{pp} calculated with the parallel propagation approximation produce flux differences of about 1 order of magnitude at some pitch angles, compared to using fully calculated coefficients. Furthermore, we show that by including off-diagonal terms in the calculation, the parallel propagation approximation also produces large errors in fluxes for both $E = 0.5$ MeV and 2 MeV electrons at small pitch angles.

[5] The remainder of this paper is organized as follows. The SDE method and its numerical implementation are introduced in section 2. In section 3 we present the application of the SDE method to a bounce-averaged radiation belt pitch angle and energy diffusion equation. After describing the implementation of the SDE method for the pitch angle energy equation (section 3.1), we show comparisons between results from the SDE method and the Albert and Young [2005] transformation method (section 3.2). Then fluxes calculated from diffusion coefficients with the parallel propagation approximation [*Summers*, 2005] are compared with fluxes computed with coefficients from full quasi-linear theory [*Albert*, 2005] (section 3.3). We summarize our work and discuss future work in section 4.

2. SDE Method

[6] Our SDE code is based on mathematical results which show that solutions of diffusion equations can be obtained using an equivalent stochastic process. Thus, we first give a description of a stochastic process using Itô stochastic differential equations in section 2.1. Then we show how these lead to probabilistic representations of solutions of diffusion equations in section 2.2.

2.1. Itô Stochastic Differential Equations

[7] Stochastic differential equations (SDEs) are used to describe stochastic processes. They differ from ordinary differential equations by having terms involving random variables [*Gardiner*, 1985; *Freidlin*, 1985]. A general m-dimensional SDE with an n-dimensional Wiener process is written as

$$d\mathbf{X}(t) = \mathbf{b}(\mathbf{X}, t)\, dt + \boldsymbol{\sigma}(\mathbf{X}, \mathbf{t})\, d\mathbf{W}(\mathbf{t}), \tag{1}$$

where the m vector $\mathbf{X}$ represents an m-dimensional stochastic process $(X_1, X_2, \ldots, X_m)$. Throughout this work, stochastic processes are indicated by uppercase characters, and their values at a given time are represented by corresponding lowercase characters. The n vector $\mathbf{W}$ is an n-dimensional Wiener process $(W_1, W_2, \ldots, W_n)$ and $d\mathbf{W}(t) = \mathbf{W}(t + dt) - \mathbf{W}(t)$ [*Gardiner*, 1985]; an increment of a one-dimensional Wiener process is proportional to a Gaussian random number. The m vector $\mathbf{b}$ and the $m \times n$ matrix $\boldsymbol{\sigma}$ are coefficients that determine the values of $\mathbf{X}(t)$, they will be directly related to the coefficients of a corresponding diffusion equation in section 2.2. Stepping equation (1) in time generates a random walk trajectory through $\mathbf{X}$ space.

[8] Note that SDEs may be formulated using two main mathematical methods: the Itô method and the Stratonovich method [*Gardiner*, 1985]. In this work we use Itô SDEs because they are directly related to diffusion equations of interest for the radiation belts and they are mathematically more convenient [*Oksendal*, 1992; *Freidlin*, 1985; *Costantini et al.*, 1998].

2.2. Probabilistic Representation of Solutions of Diffusion Equations

[9] To solve a diffusion equation using SDEs, we can first write the diffusion equation in Fokker-Planck form and then obtain equivalent "time-forward" SDEs from the diffusion equation. These time-forward SDEs can then be used to simulate particle trajectories using a Monte Carlo technique, and the distribution of particles at any given time can be obtained by binning particles in phase space. This time-forward SDE method is presented in Appendix A to show local effects of off-diagonal terms on the distribution of particles. Alternatively, in this section we present a "time-backward" SDE method, where solutions of diffusion equations are represented by the mean value of a functional of trajectories of a stochastic process [*Freidlin*, 1985]. This is the method used in our current SDE code. Compared with the time-forward method, the time-backward method is more efficient when solutions on fewer points are of interest, and it is better for handling a variety of boundary conditions.

[10] To introduce the time-backward SDE method, let us first consider a d-dimensional diffusion equation written as

$$\frac{\partial f}{\partial t} = \sum_{i,j=1}^{d} \frac{1}{2}\, a_{ij}(t, \mathbf{x}) \frac{\partial^2 f}{\partial x_i \partial x_j}(t, \mathbf{x}) + \sum_{i=1}^{d} b_i(t, \mathbf{x}) \frac{\partial f}{\partial x_i}(t, \mathbf{x}) + c(t, \mathbf{x}) f(t, \mathbf{x}), \tag{2}$$

with initial and Dirichlet boundary conditions

$$f(0, \mathbf{x}) = g_0(\mathbf{x}), \quad \mathbf{x} \in D, \tag{3}$$

$$f(t, \mathbf{x}) = g_1(t, \mathbf{x}), \quad \mathbf{x} \in \partial D. \tag{4}$$

Here D is the domain of the problem with boundary ∂D, and $g_1(0, \mathbf{x}) = g_0(\mathbf{x})$ on ∂D. Note that ∂D is not restricted to constant coordinate surfaces in the SDE method [*Freidlin*, 1985].

[11] The solution $f(\mathbf{x}, t)$ of equation (2) is related to the following d-dimensional stochastic process:

$$d\mathbf{X}(s) = \mathbf{b}(t - s, \mathbf{X})\, ds + \boldsymbol{\sigma}(t - s, \mathbf{X})\, d\mathbf{W}(s), \; 0 \leq s \leq t, \tag{5}$$

where $\mathbf{X}(s = 0) = \mathbf{x}$ and $\mathbf{W}(s)$ is a d-dimensional Wiener process. Here the $d \times d$ matrix $\boldsymbol{\sigma}$ is defined by $\boldsymbol{\sigma}\boldsymbol{\sigma}^T = \mathbf{a}$. Note that $\boldsymbol{\sigma}$ is not uniquely determined by this equation, but

according to Levy's theorem [*Zhang*, 1999; *Freidlin*, 1985], different choices of $\boldsymbol{\sigma}$ generate equivalent stochastic processes that yield the same solution of the diffusion equation (2). Also, note that equation (5) is a time-backward SDE: at $s = 0$, we evaluate $\mathbf{b}$ and $\boldsymbol{\sigma}$ at time t, while at $s = t$, we evaluate $\mathbf{b}$ and $\boldsymbol{\sigma}$ at time zero. The solution $f(\mathbf{x}, t)$ is then represented by the stochastic process defined in equation (5) as

$$f(\mathbf{x}, t) = \mathbb{E}(F_{\mathbf{x}}), \tag{6}$$

where $\mathbb{E}$ denotes the expectation value and $F_{\mathbf{x}}$ is defined by

$$F_{\mathbf{x}} = \begin{cases} g_0\left(\mathbf{X}|_{s=t}\right)\exp\left(Y|_{s=t}\right), & \tau \geq t; \\ g_1\left(t-\tau, \mathbf{X}|_{s=\tau}\right)\exp\left(Y|_{s=\tau}\right), & \tau < t, \end{cases} \tag{7}$$

where τ has the value of s when the stochastic process $\mathbf{X}(s)$ exits from the boundary ∂D for the first time and $Y(s)$ is defined by

$$Y(s) = \int_0^s c(t-r, \mathbf{X}(r))dr. \tag{8}$$

[12] Numerical calculation of f can be constructed easily from equations (6)–(8). To obtain $f(\mathbf{x}, t)$, we sample a number of trajectories of the stochastic process defined by equation (5) starting from $\mathbf{x}$ and $s = 0$, using a Monte Carlo technique. The simulation of a trajectory will stop either by reaching the initial condition at $s = t$ (where time = 0) or by reaching the boundary of the domain D at $s = \tau$, whichever comes first, and returns a value defined by equation (7). Then we use the average of values returned by all trajectories to approximate $f(\mathbf{x}, t)$. This process is repeated if we want to calculate f at other points.

[13] Now let us also consider a particular type of Neumann boundary condition that is commonly encountered in radiation belt diffusion equations:

$$\nabla f \cdot \mathbf{n} = 0, \quad \mathbf{x} \in \partial_1 D, \tag{9}$$

where $\nabla f = (\partial f/\partial x^1, \partial f/\partial x^2, \ldots, \partial f/\partial x^d)$, the boundary $\partial_1 D$ is the part of ∂D with the Neumann condition, and $\mathbf{n}$ is the inward unit normal vector on $\partial_1 D$. General methods for implementing Neumann boundary conditions in SDE solutions are given by *Freidlin* [1985] and *Costantini et al.* [1998]; here we simply note that condition (9) can be enforced in our numerical calculation of $f(\mathbf{x}, t)$ as follows: Every time a trajectory reaches the Neumann boundary $\partial_1 D$, we immediately reflect it about the normal vector $\mathbf{n}$ [*Bossy et al.*, 2004]. This trajectory will later be stopped by either reaching the initial condition or a Dirichlet boundary, and at that time the trajectory returns a value defined by equation (7).

3. Application

[14] In this section, we apply the above (see section 2) SDE method to a bounce-averaged pitch angle and energy diffusion equation [*Albert*, 2004]. In section 3.1 we derive the stochastic process used to solve the diffusion equation. In section 3.2 fluxes calculated using the SDE code are compared with results from *Albert and Young* [2005] to show that the SDE code is capable of solving the diffusion equation with off-diagonal diffusion coefficients. To show the effect of diffusion coefficients with the parallel propagation approximation [*Summers*, 2005] on particle fluxes, we solve the diffusion equation using these diffusion coefficients and in section 3.3 the results are compared with those obtained from fully calculated coefficients.

3.1. Application to Pitch Angle and Energy Diffusion Equations

[15] We apply the above SDE method to the bounce-averaged pitch angle and energy diffusion equation written in equatorial pitch angle and momentum (α_0, p)

$$\frac{\partial f}{\partial t} = \frac{1}{Gp}\frac{\partial}{\partial \alpha_0} G\left(D_{\alpha_0\alpha_0}\frac{1}{p}\frac{\partial f}{\partial \alpha_0} + D_{\alpha_0 p}\frac{\partial f}{\partial p}\right) + \frac{1}{G}\frac{\partial}{\partial p} G\left(D_{\alpha_0 p}\frac{1}{p}\frac{\partial f}{\partial \alpha_0} + D_{pp}\frac{\partial f}{\partial p}\right), \tag{10}$$

where $D_{\alpha 0 \alpha 0}$, $D_{\alpha 0 p}$, and D_{pp} are bounce-averaged pitch angle, mixed, and momentum diffusion coefficients [*Albert*, 2004]. Here G is a Jacobian factor, $G = p^2 T(\alpha_0)\sin(\alpha_0)\cos(\alpha_0)$, and $T(\alpha_0) \approx 1.30 - 0.56\sin(\alpha_0)$ is the normalized bounce period. Initial and boundary conditions are chosen to be the same as from *Albert and Young* [2005]. Thus, the initial flux is $j(t = 0) = \exp[-(E - 0.2)/0.1][\sin(\alpha_0) - \sin(\alpha_{0L})]$, where the loss cone angle $\alpha_{0L} = 5^\circ$ and flux j is related to phase space density f by $j = f/p^2$. Boundary conditions are

$$f|_{\alpha_0 = \alpha_{0L}} = 0, \tag{11}$$

$$\left.\frac{\partial f}{\partial \alpha_0}\right|_{\alpha_0 = 90^\circ} = 0, \tag{12}$$

$$f|_{E = E_{\max}} = 0, \tag{13}$$

$$f|_{E = E_{\min}} = j(t = 0)|_{E = E_{\min}}/p_{\min}^2, \tag{14}$$

where $E_{\min} = 0.2$ MeV, $E_{\max} = 5$ MeV, and $p_{\min}$ is the momentum corresponding to $E_{\min}$ [*Albert and Young*, 2005].

[16] To solve the equation using the time-backward SDE method, we first write equation (10) in the form of (2):

$$\frac{\partial f}{\partial t} = \frac{D_{\alpha_0\alpha_0}}{p^2}\frac{\partial^2 f}{\partial \alpha_0^2} + 2\frac{D_{\alpha_0 p}}{p}\frac{\partial^2 f}{\partial \alpha_0 \partial p} + D_{pp}\frac{\partial^2 f}{\partial p^2} + b_{\alpha_0}\frac{\partial f}{\partial \alpha_0} + b_p\frac{\partial f}{\partial p}, \tag{15}$$

with

$$b_{\alpha_0}(t, \alpha_0, p) = \frac{1}{Gp}\frac{\partial}{\partial \alpha_0}\left(\frac{GD_{\alpha_0\alpha_0}}{p}\right) + \frac{1}{G}\frac{\partial}{\partial p}\left(\frac{GD_{\alpha_0 p}}{p}\right), \tag{16}$$

$$b_p(t, \alpha_0, p) = \frac{1}{Gp}\frac{\partial}{\partial \alpha_0}\left(GD_{\alpha_0 p}\right) + \frac{1}{G}\frac{\partial}{\partial p}\left(GD_{pp}\right). \tag{17}$$

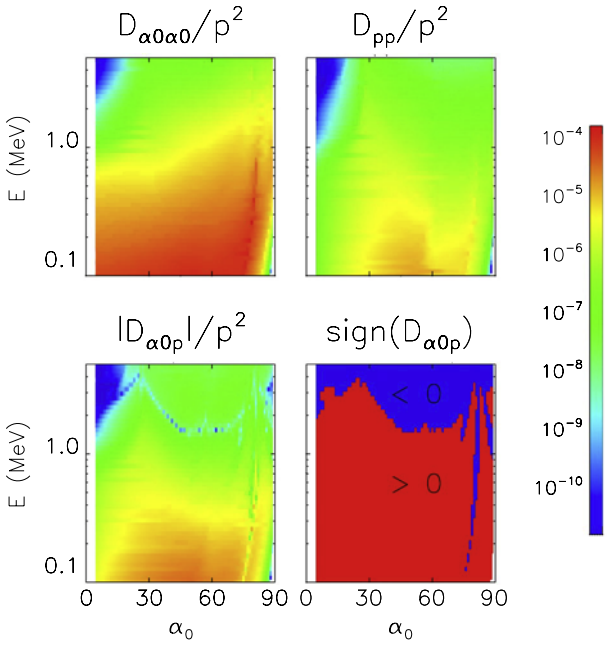


Figure 1. $D_{\alpha 0\alpha 0}/p^2$, D_{pp}/p^2, and $|D_{\alpha 0p}|/p^2$: inverse time scales in units of s^{-1} from diffusion coefficients of *Albert and Young* [2005]. Also shown is the sign of the cross diffusion coefficients.(Reprinted from *Albert and Young* [2005].)

Thus, the two-dimensional stochastic process defined in equation (5) becomes

$$dA_0(s) = b_{\alpha_0}(t-s, A_0, P)\,ds + \sigma_{11}\,dW_1 + \sigma_{12}\,dW_2, \tag{18}$$

$$dP(s) = b_p(t-s, A_0, P)\,ds + \sigma_{21}\,dW_1 + \sigma_{22}\,dW_2, \tag{19}$$

with $A_0(s{=}0) = \alpha_0$ and $P(s{=}0) = p$. Then, because of the Neumann boundary condition at $\alpha_0 = 90°$, we numerically reflect A_0 with respect to $\alpha_0 = 90°$ if it is larger than 90°. Here components of the matrix $\boldsymbol{\sigma}$ are defined by

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{21} \\ \sigma_{12} & \sigma_{22} \end{bmatrix} = \begin{bmatrix} 2D_{\alpha_0\alpha_0}/p^2 & 2D_{\alpha_0 p}/p \\ 2D_{\alpha_0 p}/p & 2D_{pp} \end{bmatrix}. \tag{20}$$

[17] In this work, we choose $\sigma_{12} = 0$ for simplicity and then the other components are

$$\sigma_{11} = \sqrt{2D_{\alpha_0\alpha_0}}/p, \tag{21}$$

$$\sigma_{21} = \sqrt{2}D_{\alpha_0 p}/\sqrt{D_{\alpha_0\alpha_0}}, \tag{22}$$

$$\sigma_{22} = \sqrt{2D_{pp} - \sigma_{21}^2}, \tag{23}$$

where we have used the fact that $D_{\alpha 0\alpha 0}$ is never zero in equation (22).

[18] We have developed a 2-D SDE code to solve the diffusion equation (10) where SDEs (18) and (19) are integrated using the Euler-Maruyama method [*Kloeden and Platen*, 1992]:

$$A_0(s_{n+1}) = A_0(s_n) + b_{\alpha_0}[t - s_n, A_0(s_n), P(s_n)]\,\Delta s + \sigma_{11}(s_n)\,\Delta W_1 + \sigma_{12}(s_n)\,\Delta W_2, \tag{24}$$

$$P(s_{n+1}) = P(s_n) + b_p[t - s_n, A_0(s_n), P(s_n)]\,\Delta s + \sigma_{21}(s_n)\,\Delta W_1 + \sigma_{22}(s_n)\,\Delta W_2. \tag{25}$$

Here $\Delta W = \sqrt{s_{n+1} - s_n}\,N(0, 1)$, where $N(0, 1)$ is a standard Gaussian random number with zero mean and unit variance generated using the Box-Muller algorithm [*Press et al.*, 2002]. Because the original time-backward SDE method requires fresh samples of trajectories for every different (α_0, p) and traces trajectories back to the initial condition or to a boundary every time, the current SDE code is less efficient when solutions on many grid points for long times are needed. Improving the efficiency of the SDE code is one of tasks in our future work. In this work, we mainly want to show that the method can be used to solve multidimensional diffusion equations. Results from the SDE code are compared with those of *Albert and Young* [2005] in section 3.2.

3.2. Comparisons With Results of *Albert and Young* [2005]

[19] *Albert and Young* [2005] solve the diffusion equation (10) by first transforming to new coordinates which diagonalize the diffusion tensor and then applying standard finite difference methods to the transformed diffusion equation. The bounce-averaged diffusion coefficients $D_{\alpha 0\alpha 0}$, $D_{\alpha 0p}$, and D_{pp} for storm time chorus waves were calculated at $L = 4.5$, with computational methods of *Albert* [2005]. The wave model used to calculate diffusion coefficients is described by *Horne et al.* [2005] and *Albert and Young* [2005]; the wave magnetic field is given by $B_\omega^2 = B^2(\omega)g_\omega(\tan\theta)$, where the wave power spectral density $B^2(\omega)$ and the wave normal angle ($\tan\theta$) distribution function $g_\omega(\tan\theta)$ are truncated Gaussian functions defined between lower and upper frequency cutoffs ($\omega_{LC} < \omega < \omega_{UC}$) and wave normal angle cutoffs ($\theta_{LC} < \theta < \theta_{UC}$). The latitudinal distribution of the waves and the ratio of electron plasma frequency (f_{pe}) to electron cyclotron frequency (f_{ce}) are the same as those used by *Horne et al.* [2005] and *Albert and Young* [2005] and are shown in Table 1. Similar models were used by *Li et al.* [2007]. Up to $n = \pm 5$ resonance harmonics were included in the calculation. The calculated diffusion coefficients $D_{\alpha 0\alpha 0}$ are proportional to $(p\Delta\alpha_0)^2$ / Δt, as from *Lyons* [1974a, 1974b] and are divided by p^2 to give the inverse time scales plotted in Figure 1.

[20] Using the above diffusion coefficients in equation (10), we obtain fluxes for $E = 0.5$ MeV and 2.0 MeV electrons with α_0 ranging from 6° to 88° with 1° spacing at $t = 0.1$ and 1 day. We have sampled $N = 9000$ trajectories at each α_0 for $E = 0.5$ MeV and $N = 18000$ trajectories for $E = 2.0$ MeV with $dt = 0.0004$ day. The chosen dt gives small relative change in α_0 and E per step, compared with scales of the diffusion coefficients and initial phase space density. Our choices of N and dt might not be optimal, and choosing N adaptively is probably better (G. Cunningham, personal communication, 2007). Results from the SDE code are

Table 1. Latitudinal Distribution of the Waves and f_{pe}/f_{ce} of the Wave Model [*Horne et al.*, 2005] Used to Calculate Diffusion Coefficients

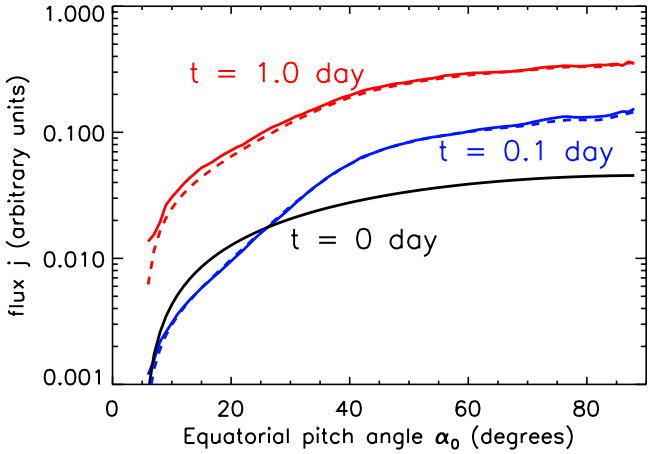
	Local Time Sector		
	2300–0600 MLT	0600–1200 MLT	1200–1500 MLT
f_{pe}/f_{ce}	~3.4 to 2.5	~3.0 to 0.9	~5.9 to 1.4
Latitudinal range	0° to 15°	15° to 35°	10° to 35°

compared with those of *Albert and Young* [2005]. Figure 2 shows the comparisons for $E = 0.5$ MeV electrons (Figure 2, top) and $E = 2.0$ MeV electrons (Figure 2, bottom), with results from the SDE method smoothed using a six-point moving window average in α_0 with $\Delta\alpha_0 = 1°$. Within small numerical errors associated with each of the methods, the two sets of results are in excellent agreement, and they demonstrate that our SDE code is able to successfully solve the bounce-averaged pitch angle and energy diffusion equation.

[21] To show the effects of ignoring off-diagonal terms on change of flux, we rerun the SDE code, setting off-diagonal diffusion coefficients to zero. Results are shown in Figure 3 for 0.5 MeV (Figure 3, top) and 2 MeV (Figure 3, bottom) electrons. From Figure 3 we see that for 0.5 MeV electrons, while there is a relatively small effect at large pitch angles, ignoring off-diagonal terms overestimates electron fluxes at small pitch angles by a factor of 2 to ~5 at $t = 1$ day. For 2 MeV electrons, ignoring off-diagonal terms overestimates fluxes by a factor of 5 to ~10 at $t = 1$ day, with larger errors at smaller pitch angles. Thus, off-diagonal terms are more important for 2 MeV electrons. We emphasize that these results are for the *Horne et al.* [2005] wave model, and we note that the peak in flux of 2 MeV electrons near 30° may be related to the cutoff in wave power at 35° latitude in the *Horne et al.* [2005] model (see discussion in section 4).

3.3. Effects of Parallel Propagation Approximation

[22] *Summers* [2005] and *Summers et al.* [2007a, 2007b] have derived cyclotron resonance diffusion coefficients for field-aligned waves, where only the $n = -1$ resonance is included (henceforth denoted by $D^{\|}$). This assumption of parallel propagation greatly improves the computation efficiency. Bounce-averaged $D^{\|}$ are given and compared with diffusion coefficients obtained from the PADIE code



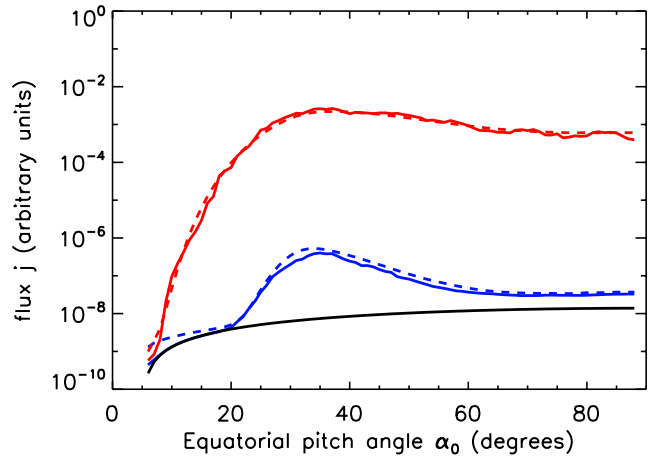
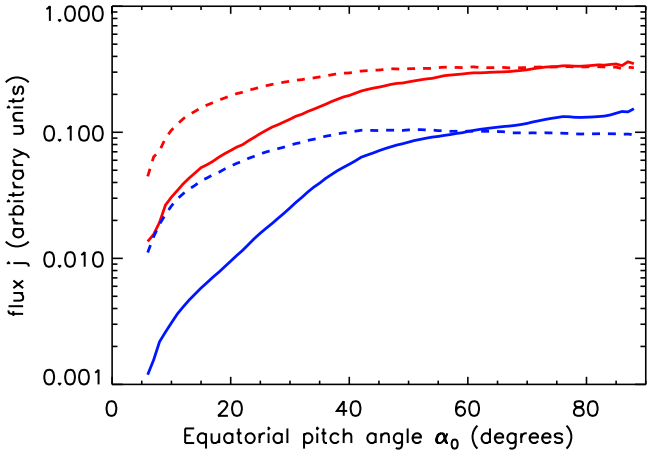


Figure 2. Comparisons between results obtained from the SDE method (solid lines) and the *Albert and Young* [2005] method (dashed lines) for (top) $E = 0.5$ MeV and (bottom) $E = 2.0$ MeV at $t = 0.1$ day (blue lines) and $t = 1$ day (red lines). Here black lines show the initial condition.



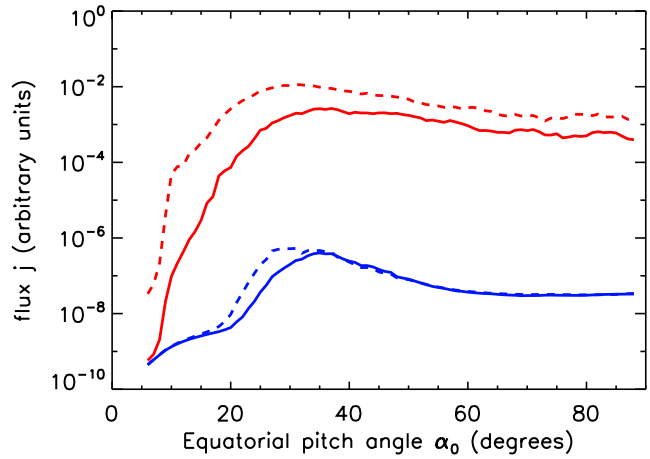


Figure 3. Fluxes for (top) $E = 0.5$ MeV and (bottom) $E = 2.0$ MeV at $t = 0.1$ day (blue lines) and $t = 1$ day (red lines) with and without off-diagonal diffusion terms. Dashed lines are results without off-diagonal diffusion coefficients, and solid lines are results with off-diagonal terms.

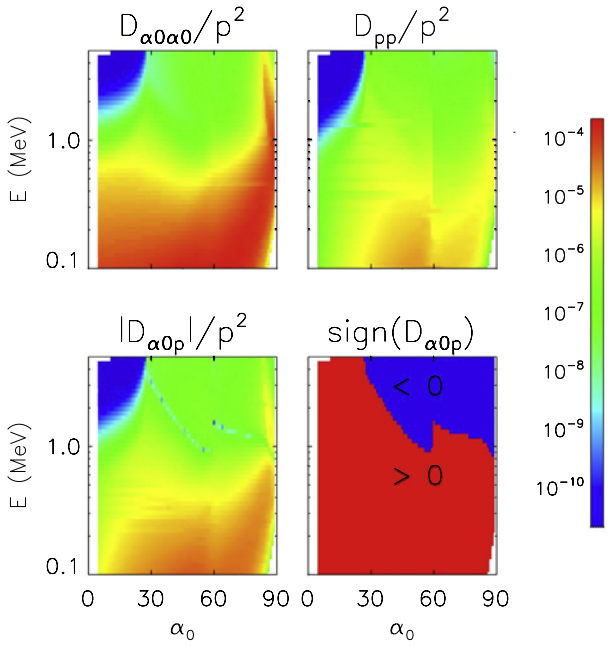


Figure 4. Same as Figure 1, except that diffusion coefficients are calculated with the parallel propagation approximation.

[*Glauert and Horne*, 2005] by *Shprits et al.* [2006]. In the present work, we also calculate $D^{\|}$ using the methods of *Albert* [2005] with the same wave parameters as the wave model described in section 3.2, except that $\theta_{LC} = \theta_{UC} = 0$. The resulting diffusion coefficients are the same as those obtained from the PADIE code and are half of those given by *Summers et al.* [2007a] (this factor of 2 difference is discussed by *Albert* [2007]).

[23] Figure 4 shows inverse time scales from diffusion coefficients with the parallel wave approximation. Compared with Figure 1, we see that the general behavior of $D^{\|}$ is quite good, with larger differences for $E > 1$ MeV electrons. The off-diagonal terms of $D^{\|}$ are worse approximations than the diagonal terms, with details discussed by *Albert* [2007].

[24] To compare effects of $D^{\|}$ with fully calculated diffusion coefficients D, we solve equation (10) for 0.5 MeV and 2 MeV electrons using the following four sets of diffusion coefficients: (1) $D^{\|}$, (2) diagonal terms of $D^{\|}$ (hereinafter referred to as $D_d^{\|}$), (3) D, and (4) diagonal terms of D (hereinafter referred to as D_d). Results are shown in Figures 5–7.

[25] Figure 5 (top) shows the comparison between fluxes calculated using $D_d^{\|}$ and D_d for 0.5 MeV electrons. We see that results from $D_d^{\|}$ agree very well with D_d, with slight differences for α_0 greater than about 40°. Figure 5 (bottom) shows the same comparison for 2.0 MeV electrons from which we see that the flux from $D_d^{\|}$ is smaller than that from D_d by up to ~5 orders of magnitude at low α_0 (<15°) at t = 1 day. This behavior occurs because $D_d^{\|}$ underestimates energy diffusion coefficients for high-energy particles at small pitch angles, where $n \neq -1$ resonances also make a significant contribution. Thus, $D_d^{\|}$ produces larger differences in fluxes for 2 MeV electrons than 0.5 MeV at small α_0 compared with D_d.

[26] Figure 6 shows comparisons between fluxes calculated using $D_d^{\|}$ and D for 0.5 MeV electrons (Figure 6, top) and 2 MeV electrons (Figure 6, bottom). Figure 6 (top) shows that $D_d^{\|}$ overestimates increase of flux at small pitch angles for 0.5 MeV electrons, which is expected, because $D_d^{\|}$ yields very similar flux increases as D_d for 0.5 MeV electrons. For 2.0 MeV electrons, fluxes from $D_d^{\|}$ are smaller than that from D for $\alpha_0 \lesssim 18°$ and larger for $\alpha_0 \gtrsim 18°$ at t = 1 day (where the difference can be about 1–2 orders of magnitude).

[27] Fluxes calculated from $D^{\|}$ and D (i.e., with off-diagonal terms included) for 0.5 MeV and 2 MeV electrons are shown in Figure 7 (top) and Figure 7 (bottom), respectively. Reasonable agreement between $D^{\|}$ and D fluxes is obtained for $\alpha_0 \gtrsim 50°$, but significant differences occur at smaller pitch angles. For 0.5 MeV electrons, $D^{\|}$ underestimates increases of flux at t = 1 day by approximately an order of magnitude for $\alpha_0 < 20°$. For 2.0 MeV electrons, behavior of $D^{\|}$ is worse at t = 1 day. We see from Figure 7 (bottom) that $D^{\|}$ underestimates increases of flux

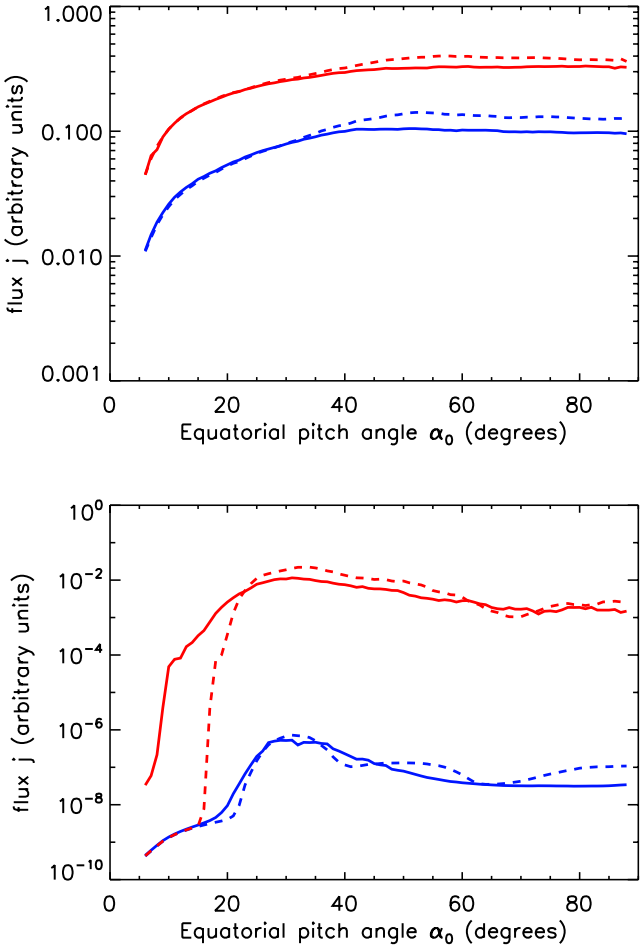


Figure 5. Comparisons between results obtained from diffusion coefficients $D_d^{\|}$ (dashed lines) and D_d (solid lines) for (top) E = 0.5 MeV and (bottom) E = 2.0 MeV at t = 0.1 day (blue lines) and t = 1 day (red lines).

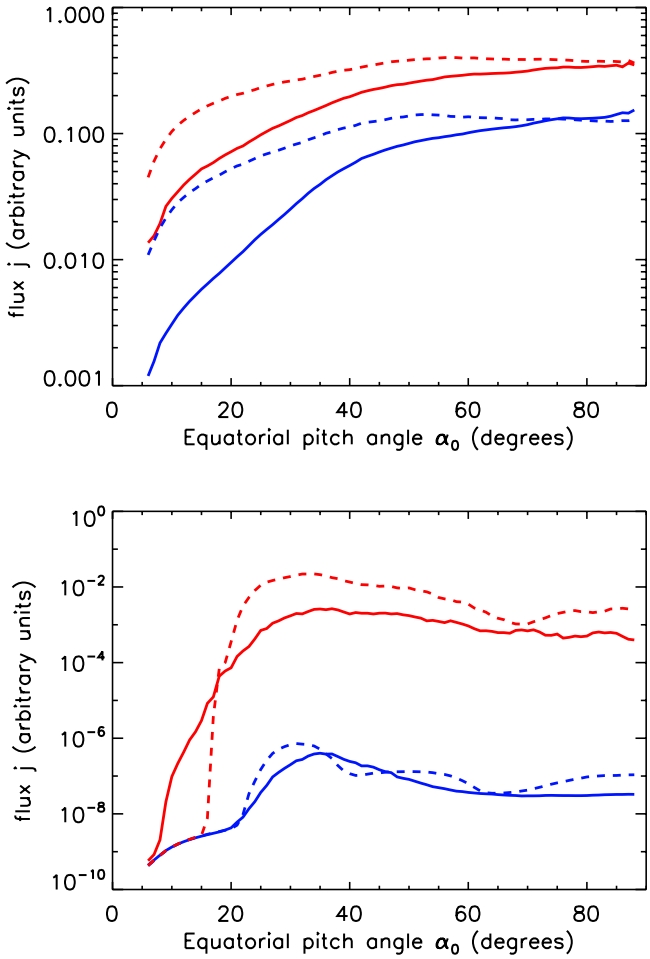


Figure 6. Comparisons between results obtained from diffusion coefficients $D_d^{\|}$ (dashed lines) and D (solid lines) for (top) E = 0.5 MeV and (bottom) E = 2.0 MeV at t = 0.1 day (blue lines) and t = 1 day (red lines).

by $\sim$ 1–4 orders of magnitude for $10^\circ \lesssim \alpha_0 \lesssim 35^\circ$. Thus, the approximation of parallel propagation produces larger differences at small pitch angles for higher-energy particles, especially when off-diagonal terms are included.

4. Summary and Discussion

[28] In this work a new code, based on the mathematical theory of expressing solutions of diffusion equations in terms of related stochastic processes, has been developed for solving multidimensional radiation belt diffusion equations. Two examples are used to show its applications.

[29] First, we apply the SDE code to a bounce-averaged pitch angle and energy diffusion equation and obtain excellent agreement with a previously developed method [*Albert and Young*, 2005]. We also confirm that ignoring off-diagonal terms in the diffusion equation overestimates increase of flux, especially at small pitch angles, at t = 1 day (by a factor of 2 to $\sim$5 for 0.5 MeV, and 5 to $\sim$10 for 2 MeV electrons) using the *Albert and Young* [2005] diffusion coefficients.

[30] Second, by solving the bounce-averaged pitch angle and energy diffusion equation using fully calculated diffusion coefficients D [*Albert and Young*, 2005] and coefficients with the parallel propagation approximation $D^{\|}$ [*Summers*, 2005; *Summers et al.*, 2007a, 2007b], both calculated using the chorus wave model of *Horne et al.* [2005], we show that diagonal diffusion coefficients of $D^{\|}$ agree well with those of D only for low-energy particles (e.g., E=0.5 MeV). For high-energy electrons, the difference between the diagonal terms of $D^{\|}$ and D produces large differences in fluxes at some pitch angles (difference of up to 5 orders of magnitude for 2 MeV electrons at $\alpha_0 \lesssim 15^\circ$, at t = 1 day). By including off-diagonal diffusion coefficients in our calculation, we show that the off-diagonal terms of $D^{\|}$ can produce differences in fluxes of 4 orders of magnitude for 2 MeV electrons at t=1 day. A discussion of the details of different diffusion coefficients and another approximation for a full calculation of diffusion coefficients are presented by *Albert* [2007].

[31] Note that the above conclusions on the magnitude and location of differences that occur by omitting off-diagonal terms and assuming parallel propagating waves are very likely to be dependent on the wave model used. For example, a different latitudinal distribution of wave power may result in different diffusion coefficients and thus

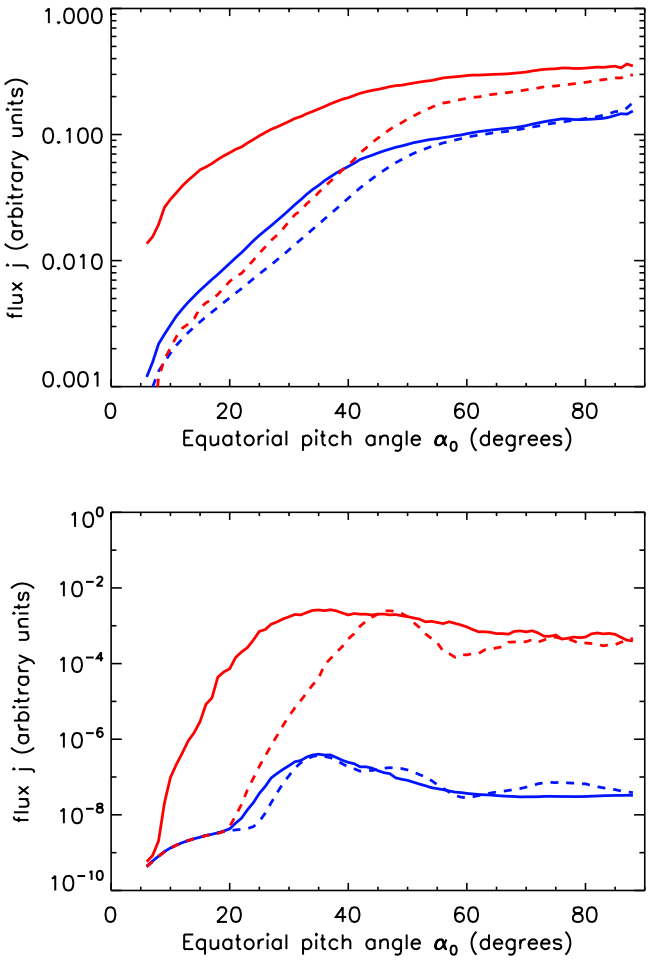


Figure 7. Comparisons between results obtained from diffusion coefficients $D^{\|}$ (dashed lines) and D (solid lines) for (top) E = 0.5 MeV and (bottom) E = 2.0 MeV at t = 0.1 day (blue lines) and t =1 day (red lines).

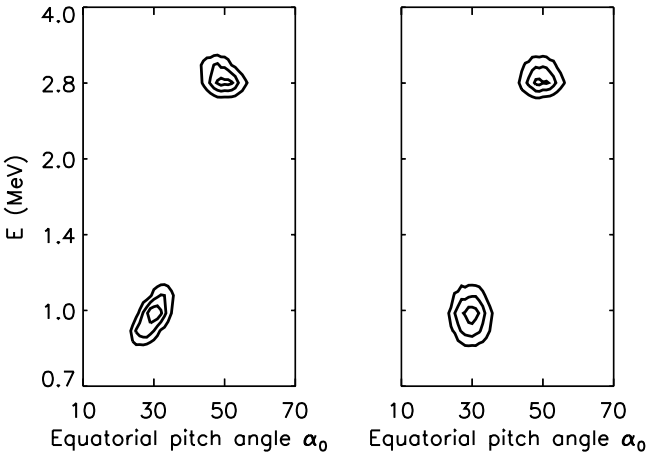


Figure A1. Local effects of ignoring off-diagonal terms. Lines are contours of particle numbers. Particles are released from $\alpha_0 = 30°$ for $E = 1.0$ MeV and $\alpha_0 = 50°$ for $E = 3$ MeV. (left) Off-diagonal terms are kept. $D_{\alpha 0p}$ is positive at $\alpha_0 = 30°$, $E = 1.0$ MeV and negative at $\alpha_0 = 50°$, $E = 3$ MeV. (right) Off-diagonal terms are set to zero.

different conclusions. The sensitivity of our results to wave models needs further study. However, before such work is done, it is safer to include both off-diagonal terms and oblique waves in calculations of electron flux.

[32] The SDE method is less efficient when solutions on many grid points are desired. However, when parallel computers are available, computation time can be greatly reduced because of high parallelization efficiency. Generalization to 3-D including pitch angle, energy, and radial diffusion is straightforward. The SDE method is very promising for providing new insights into the relative roles of local acceleration and radial diffusion as acceleration mechanisms and the importance of pitch angle diffusion as a loss process.

Appendix A: Time-Forward SDE Method

[33] To use the time-forward SDE method, we first set $F = Gf$ and write the bounce-averaged pitch angle and energy diffusion equation (10) in the following form:

$$\frac{\partial F}{\partial t} = \frac{\partial^2}{\partial \alpha_0^2}\left(\frac{D_{\alpha_0\alpha_0}}{p^2}F\right) + 2\,\frac{\partial^2}{\partial \alpha_0 \partial p}\left(\frac{D_{\alpha_0 p}}{p}F\right) + \frac{\partial^2}{\partial p^2}\left(D_{pp}F\right) - \frac{\partial}{\partial \alpha_0}\left(b_{\alpha_0}F\right) - \frac{\partial}{\partial p}\left(b_pF\right), \tag{A1}$$

where $b_{\alpha 0}$ and b_p are defined in equations (16) and (17). Thus, the time-forward stochastic differential equations corresponding to equation (A1) are [*Alanko-Huotari et al.*, 2007; *Yamada et al.*, 1998; *Qin et al.*, 2005]

$$dA_0(t) = b_{\alpha_0}(t, A_0, P)\,dt + \sigma_{11}\,dW_1 + \sigma_{12}\,dW_2, \tag{A2}$$

$$dP(t) = b_p(t, A_0, P)\,dt + \sigma_{21}\,dW_1 + \sigma_{22}\,dW_2, \tag{A3}$$

where components of the matrix $\boldsymbol{\sigma}$ are also defined by equations (21)–(23).

[34] Equations (A2) and (A3) are solved to give changes of particle coordinates (α_0, p). Thus, after a given time period, the distribution of electrons can be obtained. Here we choose a time period short enough to ignore boundary effects. To explore local effects of off-diagonal diffusion coefficients on distributions of particles, we release 9000 particles from $\alpha_0 = 30°$, $E = 1$ MeV, where $D_{\alpha 0p}$ is positive, and $\alpha_0 = 50°$, $E = 3$ MeV, where $D_{\alpha 0p}$ is negative. We obtain the distribution of particles shown in Figure A1 after $t = 0.06$ day for $E = 3$ MeV and $t = 0.01$ day for $E = 1$ MeV. We also turned off-diagonal diffusion coefficients on and off to show local effects of ignoring off-diagonal terms. Figure A1 (left) has $D_{\alpha 0p} \neq 0$, and Figure A1 (right) has $D_{\alpha 0p} = 0$. We see from Figure A1 that without $D_{\alpha 0p}$, the local distribution of particles has a shape of an ellipse, while with $D_{\alpha 0p}$ this ellipse is tilted, and the tilt direction is determined by the sign of $D_{\alpha 0p}$. With $D_{\alpha 0p}$ positive (as for the $\alpha_0 = 30°$, $E = 1$ MeV case) the ellipse tilts clockwise, and with $D_{\alpha 0p}$ negative ($\alpha_0 = 50°$, $E = 3$ MeV), the ellipse tilts counterclockwise. These results are consistent with previous analytical results using Green functions [*Albert and Young*, 2005].

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