Lectures on p-adic zeta functions and (φ, Γ)-modules

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Abstract

This paper is based on the course notes given in the 13-th National Graduate Student Summer School in Pure Mathematics at University of Science and Technology of China in July 2008. We first discuss the spaces of continuous functions, locally analytic functions, $\mathcal{C}^r$-continuous functions over $\mathbb{Z}_p$, and the dual spaces of measures, distributions and tempered distributions of order r. we prove Kummer's congruences and use Leopoldt's Γ-transform to construct the p-adic zeta function of Kubota-Leopoldt. The theory of (φ, Γ)-modules of Fontaine is then introduced and its connection to Iwasawa theory is explained. At last we compute the (φ, Γ)-module of the p-adic representation $\mathbb{Z}_p(1)$ and obtain its connection to Kubota-Leopoldt zeta function.

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1 Introduction

This paper is based on the course notes given in the 13-th National Graduate Student Summer School in Pure Mathematics at University of Science and Technology of China in July 2008. It follows heavily the course notes "Fontaine rings and p-adic L-functions"([C]) of Pierre Colmez at Tsinghua University in 2004 (available at http://staff.ustc.edu.cn/~yiouyang/).

This paper is divided into four sections. In §1, we first discuss the p-adic Banach space $\mathcal{C}^0(\mathbb{Z}_p, \mathbb{Q}_p)$, the space of continuous functions over $\mathbb{Z}_p$ and prove

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the classical Mahler's theorem. We then study its dual space $\mathcal{D}_0(\mathbb{Z}_p, \mathbb{Q}_p)$, the space of measures over $\mathbb{Z}_p$. The Amice transform A_μ of a measure μ is given, as well as the φ, ψ and Galois actions of μ. We then show the map $\mu \mapsto A_\mu$ gives an isometry between Banach algebras $\mathcal{D}_0(\mathbb{Z}_p, \mathbb{Q}_p)$ and $B^+_{\mathbb{Q}_p}$, a space with deep root in the theory of (φ, Γ)-modules. In the same manner, we study the Frechet space of locally analytic functions $LA(\mathbb{Z}_p, \mathbb{Q}_p)$, and the dual spaces $\mathcal{D}(\mathbb{Z}_p, \mathbb{Q}_p)$ of distributions over $\mathbb{Z}_p$, the Amice transform of a distribution and the actions of φ, ψ and Galois on a distribution, and the isometry of $\mathcal{D}(\mathbb{Z}_p, \mathbb{Q}_p)$ and the Robba ring $\mathcal{R}^+$. At last, we study the space $\mathcal{C}^r(\mathbb{Z}_p, \mathbb{Q}_p)$ of $\mathcal{C}^r$-functions over $\mathbb{Z}_p$, and its dual space $\mathcal{D}_r(\mathbb{Z}_p, \mathbb{Q}_p)$ of tempered distributions of order r.

In §2, we prove Kummer's congruences and use Leopoldt's Γ-transform to construct the p-adic zeta function of Kubota-Leopoldt. In §3, we first review Fontaine's theory of (φ, Γ)-modules of p-adic Galois representations, then use the (φ, Γ)-module $D(V)$ of a p-adic representation V to compute the Galois cohomology of V and obtain its Euler-Poincaré formula (the theory of Herr).

In §4, we define the Iwasawa module of V, and describe it in terms of $D(V)$. When $V = \mathbb{Z}_p(1)$, we are able to do explicit computation and obtain Coleman's power series.

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2 Continuous functions, measures and distributions over $\mathbb{Z}_p$

2.1 Continuous functions on $\mathbb{Z}_p$

2.1.1 p-adic Banach spaces.

We first recall properties about p-adic Banach spaces.

Definition 2.1. A *p-adic Banach space* B is a $\mathbb{Q}_p$-vector space which contains a (full) $\mathbb{Z}_p$-lattice B^0 separated and complete for the p-adic topology, i.e.,

$$B = B^0 \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \quad \text{and} \quad B^0 = \varprojlim_{n\in\mathbb{N}} B^0/p^n B^0.$$

For every $x \in B$, there exists $n \in \mathbb{Z}$, such that $x \in p^n B^0$. We define the associated *valuation* $v_B : B \to \mathbb{Z} \cup \{+\infty\}$ by

$$v_B(x) = \sup_{n\in\mathbb{Z}\cup\{+\infty\}} \{n \mid x \in p^n B^0\}. \tag{2.1}$$

Then v_B satisfies the following properties:

$$v_B(x+y) \geq \min\{v_B(x), v_B(y)\}, \text{ if } x, y \in C; \tag{2.2}$$

$$v_B(\lambda x) = v_p(\lambda) + v_B(x), \text{ if } \lambda \in \mathbb{Q}_p \text{ and } x \in C. \tag{2.3}$$