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Newton polygons of L functions of polynomials $x^d + ax$

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ABSTRACT

Let p be a prime number and $q = p^h$. For $f(x) = x^d + ax \in \mathbb{F}_q[x]$ $(a \neq 0)$, we obtain the slopes of the Newton polygons of the L-functions of the exponential sums associated to $f(x)$ for any nontrivial finite character χ. For χ of order p, our result recovers Zhu's genericity result [10] by giving p an explicit bound. The general χ case is based on improvement of results of Davis–Wan–Xiao [2].

1. Introduction and main results

Let p be a fixed prime number, h a positive integer and $q = p^h$. For any positive integer m, denote by $\mathbb{F}_{p^m}$ the finite field of p^m elements, and by $\mathbb{Q}_{p^m}$ the unramified extension of $\mathbb{Q}_p$ of degree m in a fixed algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$. Let $\mathbb{C}_p$ be the p-adic completion of $\overline{\mathbb{Q}}_p$. Denote by ord the additive valuation on $\mathbb{C}_p$ normalized by $\mathrm{ord}p = 1$.

For a Laurent polynomial $f(x_1, x_2, \cdots, x_n) \in \mathbb{F}_q[x_1^{\pm 1}, x_2^{\pm 1}, \cdots, x_n^{\pm 1}]$, denote by $\widehat{f}(x)$ the Teichmüller lifting of $f(x)$ in $\mathbb{Q}_q[x_1^{\pm 1}, x_2^{\pm 1}, \cdots, x_n^{\pm 1}]$. Let $\chi : \mathbb{Z}_p \to \mathbb{C}_p^\times$ be a nontrivial

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additive finite character. We suppose that its order is p^{m_χ} from now on, that is, $m_\chi = \log_p(\#\chi(\mathbb{Z}_p))$. The L-function

$$L^*(f,\chi,t) = \exp\left(\sum_{m=1}^{\infty} S_m^*(f,\chi)\frac{t^m}{m}\right), \tag{1.1}$$

where $S_m^*(f,\chi)$ is the exponential sum

$$S_m^*(f,\chi) = \sum_{(x_1,x_2,\cdots,x_n)\in(\mu_{q^m-1})^n} \chi(\mathrm{Tr}_{\mathbb{Q}_{q^m}/\mathbb{Q}_p}(\widehat{f}(x_1,x_2,\cdots,x_n))), \tag{1.2}$$

is a rational function of t over $\mathbb{Q}_p(\zeta_{p^{m_\chi}})$ by well-known theorems of Dwork–Bombieri–Grothendieck. Furthermore, if f is non-degenerate, $L^*(f,\chi,t)^{(-1)^{n-1}}$ is shown to be a polynomial for χ of order p by Adolphson–Sperber [1] and by Liu–Wei [5] for general χ.

From now on we suppose $f(x) \in \mathbb{F}_q[x]$ monic of degree d. Then $L^*(f,\chi,t)$ is a polynomial of degree $p^{m_\chi-1}d$. We fix Ψ a character of order p and write

$$L^*(f,t) = L^*(f,\Psi,t). \tag{1.3}$$

For any $i = 0,1,2,\cdots,d-1$, we can write ip uniquely in the form k_id+r_i with $k_i \in \mathbb{Z}$ and $0 \le r_i < d$. Denote

$$w_i = \frac{k_i+r_i-i}{p-1} = \frac{i}{d} + \frac{d-1}{d(p-1)}(r_i-i). \tag{1.4}$$

The following theorem is the main result of this paper.

Theorem 1.1. *Let $q = p^h$ and let*

$$N(d) = \begin{cases} \frac{d^2(d-1)}{4}+1, & \text{if } q=p; \\ \frac{d^2(d-1)}{2}+1, & \text{if } q>p. \end{cases} \tag{1.5}$$

Suppose $f(x) = x^d + ax \in \mathbb{F}_q[x]$, $a \neq 0$. For any non-trivial finite character χ of order p^{m_χ}, if

$$p > \begin{cases} N(d), & \text{if } m_\chi = 1, \\ \max\{N(d),\ \frac{hd(d-1)}{4}+1\}, & \text{if } m_\chi > 1, \end{cases}$$

the q-adic Newton polygon of $L^*(f,\chi,t)$ has slopes

$$\bigcup_{i=0}^{p^{m_\chi-1}-1}\left\{\frac{i+w_0}{p^{m_\chi-1}}, \frac{i+w_1}{p^{m_\chi-1}}, \cdots, \frac{i+w_{d-1}}{p^{m_\chi-1}}\right\}.$$

Remark. (1) The case $m_\chi = 1$ (i.e., $\chi = \Psi$) was first obtained (albeit in a slightly different form) by H.J. Zhu [10] if $q = p \geq (d-1)^3+2$. Through this she proved D. Wan's Conjecture (see [8]) in this case. Earlier R. Yang [9] obtained the first slope w_1, and other slopes in the case $p \equiv -1 \bmod d$. To obtain our result in this case, we need Zhu's Rigid Transformation Theorem [11, Theorem 5.3] to study the slopes of Fredholm determinants of nuclear matrices when q is general.

(2) For the case $m_\chi > 1$, we need an improvement of results in [2] about the Newton polygons of L-functions of Artin–Shreier–Witt towers associated to a monic polynomial $f(x) \in \mathbb{F}_q[x]$, especially [2, Theorems 1.2 and 3.8]. Our results are stated as Theorem 4.1 and Theorem 4.2.

2. Preliminaries

2.1. Dwork's trace formula

Let $E(t)$ be the Artin–Hasse exponential series:

$$E(t) = \exp\left(\sum_{m=0}^{\infty} \frac{t^{p^m}}{p^m}\right) \in (\mathbb{Z}_p \cap \mathbb{Q})[[t]]. \tag{2.1}$$

Let $\gamma \in \mathbb{Q}_p(\zeta_p)$ be a root of $\sum\limits_{m=0}^{\infty} \frac{t^{p^m}}{p^m} = 0$ satisfying $\mathrm{ord}\gamma = \mathrm{ord}(\zeta_p - 1) = \frac{1}{p-1}$. Fix a system of elements $\{\gamma^{1/1}, \gamma^{1/2}, \gamma^{1/3}, \cdots\} \subset \overline{\mathbb{Q}}_p$ such that

$$\left(\gamma^{1/(m_1 m_2)}\right)^{m_1} = \gamma^{1/m_2}, \text{ for all } m_1, m_2 \geq 1.$$

Denote $\gamma^{n/m} = \left(\gamma^{1/m}\right)^n$ for any $n \in \mathbb{Z}$ and any positive integer m. The Frobenius automorphism $x \mapsto x^p$ of $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ lifts to a generator φ of $\mathbb{Q}_p^{ur}/\mathbb{Q}_p$ which is extended to $\mathbb{Q}_p^{ur}(\gamma^{1/1}, \gamma^{1/2}, \gamma^{1/3}, \cdots)$ by requiring that $\varphi(\gamma^{1/m}) = \gamma^{1/m}$ for all $m \geq 1$. Dwork's splitting function

$$\theta(t) = E(\gamma t) = \sum_{m=0}^{\infty} \gamma_m t^m \tag{2.2}$$

has coefficients $\gamma_m \in \mathbb{Q}_p(\zeta_p)$ satisfying

$$\mathrm{ord}\gamma_m \geq \frac{m}{p-1}, \text{ and } \gamma_m = \frac{\gamma^m}{m!} \text{ for } 0 \leq m \leq p-1. \tag{2.3}$$

Let $f(x) \in \mathbb{F}_q[x]$ of degree d and I be the finite set of all $i \in \mathbb{N}$ such that the coefficient of f at x^i is not 0. Then one can write

$$f(x) = \sum_{i \in I} \bar{a}_i x^i, \;\; \bar{a}_i \neq 0.$$

Let $\widehat{a}_i$ be the Teichmüler lifting of $\bar{a}_i$ in $\mathbb{Q}_q$. Set

$$F(f,x)=\prod_{i\in I}\theta(\widehat{a}_i x^i). \tag{2.4}$$

Write $F(f,x)=\sum_{r=0}^{\infty}F_r(f)x^r$. Then

$$F_r(f)=\sum_{\tau}\left(\prod_{i\in I}\gamma_{\tau_i}\widehat{a}_i^{\tau_i}\right), \tag{2.5}$$

where $\tau=(\tau_i)\in\mathbb{N}^I$ is over all solutions of the linear system $\sum_{i\in I} i\tau_i=r$. By (2.3), $\operatorname{ord}(\prod_{i\in I}\gamma_{\tau_i}\widehat{a}_i^{\tau_i})\geq\sum_{i\in I}\frac{\tau_i}{p-1}\geq\frac{r}{d(p-1)}$. Thus

$$\operatorname{ord}(F_r(f))\geq\frac{r}{d(p-1)}. \tag{2.6}$$

Let $A_1(f)$ be the nuclear matrix

$$A_1(f)=(a_{s,r}(f))=\left(F_{ps-r}(f)\gamma^{(r-s)/d}\right)_{s,r\geq 0} \tag{2.7}$$

over $\mathbb{Q}_q(\gamma^{1/d})$ indexed by $(s,r)\in\mathbb{N}^2$. We have

$$\operatorname{ord}a_{s,r}(f)=\operatorname{ord}F_{ps-r}(f)\gamma^{(r-s)/d}\geq\frac{ps-r}{d(p-1)}+\frac{r-s}{d(p-1)}=\frac{s}{d}. \tag{2.8}$$

Let $A_h(f)$ be the nuclear matrix

$$A_h(f)=A_1(f)A_1(f)^{\varphi}\cdots A_1(f)^{\varphi^{h-1}}. \tag{2.9}$$

Theorem 2.1 *(Dwork's trace formula). For $f(x)\in\mathbb{F}_q[x]$, we have*

$$S_m^*(f)=(q^m-1)\operatorname{Tr}^{\varphi^{-1}}(A_h(f)^m). \tag{2.10}$$

Equivalently,

$$L^*(f,t)=\frac{\det^{\varphi^{-1}}(I-tA_h(f))}{\det^{\varphi^{-1}}(I-tqA_h(f))}, \tag{2.11}$$

where det *is the Fredholm determinant.*

Remark. Note that all objects above can be defined for any Laurent polynomial $f(x_1,x_2,\cdots,x_n)\in\mathbb{F}_q[x_1^{\pm 1},x_2^{\pm 1},\cdots,x_n^{\pm 1}]$, and Dwork's trace formula also holds after a slight modification. See [7,9] for details.

2.2. Zhu's Rigid Transformation Theorem

Let $U = (u_{sr})_{s,r\in\mathbb{N}}$ be a nuclear matrix over $\mathbb{Q}_q(\gamma^{1/d})$. Then the Fredholm determinants $\det(I - tU)$ is well defined and p-adic entire (see [6]). Write

$$\det(I - tU) = c_0 + c_1 t + c_2 t^2 + \cdots. \tag{2.12}$$

For $0 \le i_1 < i_2 < \cdots < i_s$, denote by $U(i_1, \cdots, i_s)$ the principal sub-matrix of U formed by removing all the rows and columns except the i_k-th $(1 \le k \le s)$ ones. In particular, denote $U[s] = U(0, 1, \cdots, s-1)$. Then we have $c_0 = 1$ and for $k \ge 1$,

$$c_k = (-1)^k \sum_{0\le i_1<i_2<\cdots<i_k} \det U(i_1, i_2, \cdots, i_k). \tag{2.13}$$

Denote

$$U_h = N_{\mathbb{Q}_q/\mathbb{Q}_p}(U) = U \cdot U^{\varphi} \cdots U^{\varphi^{h-1}}. \tag{2.14}$$

Write

$$\det(I - tU_h) = C_0 + C_1 t + C_2 t^2 + \cdots. \tag{2.15}$$

Zhu [11, Theorem 5.3] proved the following result.

Theorem 2.2 *(Rigid Transformation Theorem). Suppose $(\beta_s)_{s\ge 0}$ is a strictly increasing sequence such that*

$$\lim_{s\to+\infty} \beta_s = \infty, \ \text{ and } \beta_s \le \inf_{r\ge 0} \operatorname{ord}(u_{sr}).$$

Suppose the inequalities

$$\sum_{s<i} \beta_s \le \operatorname{ord} \det U[i] \le \sum_{s<i} \beta_s + \frac{\beta_{i+1} - \beta_i}{2}$$

hold for every $1 \le i \le k$. Then $\operatorname{ord}_q(C_i) = \operatorname{ord}_p \det U[i]$ for $1 \le i \le k$ and

$$\mathrm{NP}_q(\det(1 - tU_h[k])) = \mathrm{NP}_p(\det(1 - tU[k])).$$

3. Slopes of the Newton polygon of $L^*(f, t)$

In this section we shall use Dwork's trace formula and Zhu's Rigid Transformation Theorem to compute the slopes of the Newton polygon of $L^*(f, t)$ where $f(x) = x^d + ax \in \mathbb{F}_q[x]$ and $a \ne 0$. We denote $A_1 = A_1(f)$ and $A_h = A_h(f)$. Recall that $ip = k_i d + r_i$, $0 \le r_i < d$.

Lemma 3.1. *We have*

$$F_{ip-j}(f) \equiv \gamma_{k_i}\gamma_{r_i-j}\widehat{a}^{r_i-j} \mod \gamma^{k_i+r_i-j+1}, \text{ for } 0 \le j \le r_i;$$

and

$$F_{ip-j}(f) \equiv 0 \mod \gamma^{k_i+r_i-j+1}, \text{ for } j > r_i.$$

Proof. For $m \in \mathbb{Z}_+$, write $m = kd + r$ for unique integers k, r such that $0 \le r < d$. By definition,

$$\begin{aligned} F_m(f) &= \gamma_k \cdot \gamma_r \cdot \widehat{a}^r + \gamma_{k-1} \cdot \gamma_{r+d} \cdot \widehat{a}^{r+d} + \gamma_{k-2} \cdot \gamma_{r+2d} \cdot \widehat{a}^{r+2d} + \cdots + \gamma_0\gamma_m\widehat{a}^m \\ &\equiv \gamma_k \cdot \gamma_r \cdot \widehat{a}^r \mod \gamma^{k+r+1}. \end{aligned}$$

The lemma follows from this fact. □

By Lemma 3.1, if $0 \le j \le r_i$, we have

$$\begin{aligned} a_{ij}(f) &\equiv \gamma^{\frac{j-i}{d}}\gamma_{k_i}\gamma_{r_i-j}\widehat{a}^{r_i-j} \\ &= \left(\gamma_{k_i}\gamma^{r_i-\frac{i}{d}}\widehat{a}^{r_i}\right) \cdot \left(\gamma^{\frac{j}{d}-j}\widehat{a}^{-j}\right) \cdot \frac{1}{(r_i-j)!} \mod \gamma^{\frac{j-i}{d}+k_i+r_i-j+1}. \end{aligned} \tag{3.1}$$

If $j > r_i$, we have

$$a_{ij}(f) = \gamma^{\frac{j-i}{d}} F_{ip-j}(f) \equiv 0 \mod \gamma^{\frac{j-i}{d}+k_i+r_i-j+1}. \tag{3.2}$$

Hence we get the following result.

Lemma 3.2. *For any $0 < s \le d$, we have*

$$T_1 A_1[s] T_2 \equiv \begin{pmatrix} 1 & r_0 & r_0(r_0-1) & \cdots \\ 1 & r_1 & r_1(r_1-1) & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 1 & r_{s-1} & r_{s-1}(r_{s-1}-1) & \cdots \end{pmatrix} \mod \gamma \tag{3.3}$$

where

$$T_1 = \operatorname{diag}\left(\frac{1}{\gamma_{k_0}\gamma^{r_0-\frac{0}{d}}\widehat{a}^{r_0}r_0!}, \frac{1}{\gamma_{k_1}\gamma^{r_1-\frac{1}{d}}\widehat{a}^{r_1}r_1!}, \cdots, \frac{1}{\gamma_{k_i}\gamma^{r_s-\frac{s-1}{d}}\widehat{a}^{r_{s-1}}r_{s-1}!}\right)$$

and

$$T_2 = \operatorname{diag}\left(\gamma^{0-\frac{0}{d}}\widehat{a}^0, \gamma^{1-\frac{1}{d}}\widehat{a}^1, \cdots, \gamma^{(s-1)-\frac{s-1}{d}}\widehat{a}^{s-1}\right).$$

Proposition 3.3. *If $p \geq d$, then for any $s = 1, \cdots, d$,*

$$\operatorname{ord}(\det A_1[s]) = \sum_{i=0}^{s-1} w_i \leq \frac{s^2 - s}{2d} + \frac{d(d-1)}{4(p-1)}. \tag{3.4}$$

Proof. As $s \leq d$, $r_0, r_1, \cdots, r_{s-1}$ are distinct. The determinant of the matrix of the right hand side of (3.3) equals to $\prod_{0 \leq i < j \leq s-1} (r_j - r_i) \neq 0$, of which the prime factors are less than d. Therefore the determinant is invertible in $\mathbb{F}_p$ for $p \geq d$. In this case, one has

$$\operatorname{ord} \det A_1[s] = -\operatorname{ord} \det T_1 - \operatorname{ord} \det T_2.$$

Recall that $w_i = \frac{k_i + r_i - i}{p-1} = \frac{i}{d} + \frac{d-1}{d(p-1)}(r_i - i)$, we have

$$\operatorname{ord} \det A_1[s] = \sum_{i=0}^{s-1} w_i = \frac{s^2 - s}{2d} + \frac{d-1}{d(p-1)} \sum_{i=0}^{s-1} (r_i - i).$$

However

$$\sum_{i=0}^{s-1} (r_i - i) \leq \sum_{i=0}^{s-1} (d - 1 - 2i) = (d - s)s \leq \frac{d^2}{4}. \tag{3.5}$$

This finishes the proof. □

We are now ready to prove our main result in the case $\chi = \Psi$:

Proposition 3.4. *If $p > N(d)$, then the q-adic Newton polygon of $L^*(f, t)$ has slopes $\{w_0, w_1, \cdots, w_{d-1}\}$.*

Proof. Write

$$\det(I - tA_1) = \sum_{i \geq 0} c_i t^i, \quad \det(I - tA_h) = \sum_{i \geq 0} C_i t^i.$$

If $p > \frac{d^2(d-1)}{4} + 1$, then (3.4) implies that

$$\operatorname{ord} \det A_1[s] < \frac{s^2 - s}{2d} + \frac{1}{d}$$

holds for $0 \leq s < d$. By (2.8), $\operatorname{ord} a_{s,r}(f) \geq \frac{s}{d}$. Then for $\{i_1, \cdots, i_s\} \neq \{0, 1, \cdots, s-1\}$, one has

$$\det A_1(i_1, \cdots, i_s) \equiv 0 \mod p^{\frac{s^2 - s + 2}{2d}}.$$

Therefore for $0 \leq s < d$,

$$\operatorname{ord} c_s = \operatorname{ord}(\det A_1[s]) = \sum_{i=0}^{s-1} w_i.$$

Then $w_0, w_1, \cdots, w_{d-1}$ are d slopes of $\mathrm{NP}_p(\det(I - tA_1))$, all of which are less than 1.

Moreover, if $p > \frac{d^2(d-1)}{2} + 1$, then (3.4) implies that

$$\operatorname{ord} \det A_1[s] < \frac{s^2 - s}{2d} + \frac{1}{2d}$$

holds for $0 \le s < d$. Let $\beta_s = \frac{s}{d}$. Then the assumptions of Theorem 2.2 are satisfied, $\operatorname{ord} C_s = \operatorname{ord} c_s$ for $0 \le s < d$ and $\mathrm{NP}_q(\det(I - tA_h[s])) = \mathrm{NP}_p(\det(I - tA_1[s]))$. Hence $w_0, w_1, \cdots, w_{d-1}$ are d slopes of $\mathrm{NP}_q(\det^{\varphi^{-1}}(I - tA_h))$, all of which are less than 1.

By Theorem 2.1,

$$\det{}^{\varphi^{-1}}(I - tA_h) = L^*(f, t) \det{}^{\varphi^{-1}}(I - tqA_h).$$

As the valuation of any item in A_h is ≥ 0, the q-adic slopes of the Newton polygon of $\det(I - tA_h)$ are all ≥ 0. Hence the q-adic slopes of $\det^{\varphi^{-1}}(I - tA_h)$ are also ≥ 0 and those of $\det^{\varphi^{-1}}(I - tqA_h)$ are all ≥ 1. Consequently, the q-adic slopes of the Newton polygon of $\det^{\varphi^{-1}}(I - tA_h)$ less than 1 must be the q-adic slopes of the Newton polygon of its factor $L^*(f, t)$. However the degree of $L^*(f, t)$ is d, $\{w_i\}$ must be all the q-adic slopes of $L^*(f, t)$. □

4. Slopes of Newton polygons of $L^*(f, \chi, t)$

In this section, we fix a monic polynomial $f(x) = x^d + \bar{b}_{d-1}x^{d-1} + \cdots + \bar{b}_0 \in \mathbb{F}_q[x]$ whose degree d is not divisible by p. We will use Davis–Wan–Xiao's result [2] to study Newton polygons of the L-functions $L^*(f, \chi, t)$ for general finite characters χ. For such a χ, we set $\pi_\chi = \chi(1) - 1$ and recall $m_\chi = \log_p(\#\chi(\mathbb{Z}_p))$.

4.1. T-adic L-function

For a positive integer k, the T-adic exponential sum of f over $\mathbb{F}_{q^k}^\times$ is the sum:

$$S_k^*(f, T) := \sum_{x \in \mathbb{F}_{q^k}^\times} (1 + T)^{\mathrm{Tr}_{\mathbb{Q}_{q^k}/\mathbb{Q}_p} \widehat{f}(\widehat{x})}. \tag{4.1}$$

The associated T-adic L-function of f over $\mathbb{G}_{m,\mathbb{F}_q}$ is the generating function

$$L^*(f, T, t) = \exp\left(\sum_{k=1}^{\infty} S_k^*(f, T) \frac{t^k}{k}\right) \in 1 + t\mathbb{Z}_p[[T]][[t]]. \tag{4.2}$$

Note that $L^*(f,T,t)$ is the L-function associated to the character $\mathbb{Z}_p \to \mathbb{Z}_p[[T]]^\times$ sending 1 to $1+T$. It is clear that for a finite character χ, we have

$$L^*(f,T,t)|_{T=\pi_\chi} = L^*(f,\chi,t). \tag{4.3}$$

The T-adic characteristic function of f over $\mathbb{G}_{m,\mathbb{F}_q}$ is the generating function

$$C^*(f,T,t) = \exp\left(\sum_{k=1}^{\infty} \frac{1}{1-q^k} S_k^*(f,T)\frac{t^k}{k}\right). \tag{4.4}$$

Clearly, we have

$$C^*(f,T,t) = L^*(f,T,t)L^*(f,T,qt)L^*(f,T,q^2t)\cdots, \tag{4.5}$$

and

$$L^*(f,T,t) = \frac{C^*(f,T,t)}{C^*(f,T,qt)}. \tag{4.6}$$

In particular, $C^*(f,T,t) \in 1+t\mathbb{Z}_p[[T]][[t]]$. Evaluating $C^*(f,T,t)$ at $T=\pi_\chi$, we have

$$C^*(f,\chi,t) = C^*(f,T,t)\ |_{T=\pi_\chi}\ .$$

It follows that

$$C^*(f,\chi,t) = L^*(f,\chi,t)L^*(f,\chi,qt)L^*(f,\chi,q^2t)\cdots, \tag{4.7}$$

and

$$L^*(f,\chi,t) = \frac{C^*(f,\chi,t)}{C^*(f,\chi,qt)}. \tag{4.8}$$

Liu–Wan [4] showed that the T-adic characteristic function $C^*(f,T,t)$ is T-adically entire in t. Thus one can write it in the form

$$C^*(f,T,t) = 1 + a_1(T)t + a_2(T)t^2 + \cdots \in 1+t\mathbb{Z}_p[[T]][[t]]. \tag{4.9}$$

Liu–Wan [4] also proved

$$v_{T^{h(p-1)}}(a_k(T)) \geq \frac{k(k-1)}{2d}, \tag{4.10}$$

where v_{T^m} is the normalized valuation on $\mathbb{Q}[[T]]$ such that $v_{T^m}(T^m)=1$. In other words, each $a_k(T)$ can be written as a power series in T:

$$a_k(T) = a_{k,\lambda_k}T^{\lambda_k} + a_{k,\lambda_k+1}T^{\lambda_k+1} + a_{k,\lambda_k+2}T^{\lambda_k+2} + \cdots,$$

with $a_{k,i} \in \mathbb{Z}_p$, $a_{k,\lambda_k} \neq 0$ and

$$\lambda_k \geq \frac{k(k-1)h(p-1)}{2d}.$$

Now we let $\mathrm{NP}(f, \chi, x)$ be the piecewise linear function whose graph is the $\pi_\chi^{h(p-1)}$-adic Newton polygon of $C^*(f, \chi, t)$, and let $\mathrm{HP}(f, x)$ be the piecewise linear function whose graph is the polygon with vertices

$$\left(k, \frac{k(k-1)}{2d}\right), \quad k = 0, 1, 2, \cdots.$$

Then we have $\mathrm{NP}(f, \chi, x) \geq \mathrm{HP}(f, x)$. Set

$$\mathrm{gap}(f, \chi) = \max_{x \geq 0}\{\mathrm{NP}(f, \chi, x) - \mathrm{HP}(f, x)\}, \tag{4.11}$$

which is the maximum gap between $\mathrm{NP}(f, \chi, x)$ and $\mathrm{HP}(f, x)$. Proposition 3.2(1) and Lemma 3.7 in [2] imply that for any finite character χ,

$$0 \leq \mathrm{gap}(f, \chi) \leq \frac{h(d-1)^2}{8d}. \tag{4.12}$$

Theorem 3.8 in [2] implies that $\mathrm{NP}(f, \chi, x)$ is independent of the choice of χ if $m_\chi \geq 1 + \log_p \frac{h(d-1)^2}{8d}$. We denote this function by $\mathrm{NP}(f, \chi_\infty, x)$. We make an improvement of this result in the following

Theorem 4.1. *If for some non-trivial finite character χ_0, $m_{\chi_0} > 1 + \log_p(h \cdot \mathrm{gap}(f, \chi_0))$, then for any finite character χ such that $m_\chi \geq m_{\chi_0}$,*

$$\mathrm{NP}(f, \chi, x) = \mathrm{NP}(f, \chi_\infty, x).$$

In particular, we have

$$\mathrm{NP}(f, \chi_0, x) = \mathrm{NP}(f, \chi_\infty, x).$$

Proof. We only need to show that $\mathrm{NP}(f, \chi, x) = \mathrm{NP}(f, \chi_0, x)$. Recall that

$$a_k(\pi_{\chi_0}) = a_{k,\lambda_k}{\pi_{\chi_0}}^{\lambda_k} + a_{k,\lambda_k+1}{\pi_{\chi_0}}^{\lambda_k+1} + a_{k,\lambda_k+2}{\pi_{\chi_0}}^{\lambda_k+2} + \cdots.$$

Firstly suppose $p \mid a_{k,\lambda}$ for all $\lambda \geq \lambda_k$. By definition of m_{χ_0}, $\chi_0(1)$ is a primitive root of unity of order $p^{m_{\chi_0}}$ and hence the π_{χ_0}-adic order of p is $(p-1)p^{m_{\chi_0}-1}$. As $m_{\chi_0} > 1 + \log_p(h \cdot \mathrm{gap}(f, \chi_0))$, we have $\mathrm{ord}_{\pi_{\chi_0}^{h(p-1)}}(p) > \mathrm{gap}(f, \chi_0)$. Thus

$$\mathrm{ord}_{\pi_{\chi_0}^{h(p-1)}}(a_k(\pi_{\chi_0})) > \mathrm{gap}(f, \chi_0) + \frac{k(k-1)}{2d} \geq \mathrm{NP}(f, \chi_0, k).$$

Similarly, as $m_\chi \geq m_{\chi_0}$, we have

$$\mathrm{ord}_{\pi_\chi^{h(p-1)}}(a_k(\pi_\chi)) > \mathrm{NP}(f, \chi_0, k).$$

Secondly suppose that there is some $\lambda \geq \lambda_k$ such that $a_{k,\lambda}$ is a p-adic unit. Denote by $\lambda_k' \geq \lambda_k$ the smallest integer such that $a_{k,\lambda_k'}$ is a p-adic unit. It is clear that

$$a_k(\pi_{\chi_0}) \equiv a_{k,\lambda_k'} {\pi_{\chi_0}}^{\lambda_k'} \quad \mathrm{mod}\ (p{\pi_{\chi_0}}^{\lambda_k}, {\pi_{\chi_0}}^{\lambda_k'+1}),$$

and

$$a_k(\pi_\chi) \equiv a_{k,\lambda_k'} {\pi_\chi}^{\lambda_k'} \quad \mathrm{mod}\ (p{\pi_\chi}^{\lambda_k}, {\pi_\chi}^{\lambda_k'+1}).$$

As $\mathrm{ord}_{\pi_{\chi_0}^{h(p-1)}}(p\pi_{\chi_0}^{\lambda_k}) > \mathrm{NP}(f, \chi_0, x)$ and $\mathrm{ord}_{\pi_{\chi_0}^{h(p-1)}}(a_k(\pi_{\chi_0})) \geq \mathrm{NP}(f, \chi_0, x)$, we have

$$\lambda_k' \geq h(p-1)\mathrm{NP}(f, \chi_0, x).$$

If $\lambda_k' = h(p-1)\mathrm{NP}(f, \chi_0, x)$, then

$$\mathrm{ord}_{\pi_{\chi_0}^{h(p-1)}}(a_k(\pi_{\chi_0})) = \frac{\lambda_k'}{h(p-1)} = \mathrm{NP}(f, \chi_0, x),$$

and

$$\mathrm{ord}_{\pi_\chi^{h(p-1)}}(a_k(\pi_\chi)) = \frac{\lambda_k'}{h(p-1)} = \mathrm{NP}(f, \chi_0, x).$$

On the other hand, if $\lambda_k' > h(p-1)\mathrm{NP}(f, \chi_0, x)$, then

$$\mathrm{ord}_{\pi_{\chi_0}^{h(p-1)}}(a_k(\pi_{\chi_0})) \geq \min\left\{\frac{\lambda_k'}{h(p-1)}, \mathrm{ord}_{\pi_{\chi_0}^{h(p-1)}}(p\pi_{\chi_0}^{\lambda_k})\right\} > \mathrm{NP}(f, \chi_0, x),$$

and, as $m_\chi \geq m_{\chi_0}$,

$$\mathrm{ord}_{\pi_\chi^{h(p-1)}}(a_k(\pi_\chi)) \geq \min\left\{\frac{\lambda_k'}{h(p-1)}, \mathrm{ord}_{\pi_\chi^{h(p-1)}}(p\pi_\chi^{\lambda_k})\right\} > \mathrm{NP}(f, \chi_0, x).$$

Thus the $\pi_\chi^{h(p-1)}$-adic Newton polygon of $C^*(f, \chi, t)$ is the same as that of $C^*(f, \chi_0, t)$, which means that $\mathrm{NP}(f, \chi, x) = \mathrm{NP}(f, \chi_0, x)$. □

If χ_0 is a finite character such that the assumption $m_{\chi_0} > 1 + \log_p(h \cdot \mathrm{gap}(f, \chi_0))$ holds, by Theorem 4.1, then the slopes of $L^*(f, \chi, t)$ for $m_\chi \geq m_{\chi_0}$ are determined by the slopes of $L^*(f, \chi_0, t)$ just as in [2, Theorem 1.2].

Moreover, if $\mathrm{gap}(f, \chi_0) < \frac{1}{h}$, then $m_{\chi_0} \geq 1 > 1 + \log_p(h \cdot \mathrm{gap}(f, \chi_0))$. The assumption in Theorem 4.1 trivially holds. In particular, if $\mathrm{gap}(f, \Psi) < \frac{1}{h}$, we apply Theorem 4.1 to get a variation of [2, Theorem 1.2]:

Theorem 4.2. *Let $f(x) \in \mathbb{F}_q[x]$ be a monic polynomial of degree d. Let $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_{d-1} < 1$ be the slopes of the q-adic Newton polygon of $L^*(f,t)$. If $\mathrm{gap}(f,\Psi) < \frac{1}{h}$, then the q-adic Newton polygon of $L^*(f,\chi,t)$ has slopes*

$$\bigcup_{i=0}^{p^{m_\chi-1}-1} \left\{ \frac{i+\alpha_0}{p^{m_\chi-1}}, \frac{i+\alpha_1}{p^{m_\chi-1}}, \ldots, \frac{i+\alpha_{d-1}}{p^{m_\chi-1}} \right\},$$

for any non-trivial finite character χ.

Proof. As $C^*(f,\Psi,t) = L^*(f,\Psi,t)L^*(f,\Psi,qt)L^*(f,\Psi,q^2t)\cdots$,

$$\bigcup_{i\geq 0} \{i+\alpha_0, i+\alpha_1, \cdots, i+\alpha_{d-1}\} \tag{4.13}$$

are the slopes of the q-adic Newton polygon of $C^*(f,\Psi,t)$. As $\mathrm{gap}(f,\Psi) < \frac{1}{h}$, the assumption $1 = m_\Psi > 1 + \log_p(h \cdot \mathrm{gap}(f,\Psi))$ in Theorem 4.1 holds. For any finite character χ, we have $m_\chi \geq 1 = m_\Psi$. Theorem 4.1 implies that the slopes of the $\pi_\chi^{h(p-1)}$-adic Newton polygon of $C^*(f,\chi,t)$ are also given by (4.13) and hence the slopes of the q-adic Newton polygon of $C^*(f,\chi,t)$ are

$$\bigcup_{i\geq 0} \left\{ \frac{i+\alpha_0}{p^{m_\chi-1}}, \frac{i+\alpha_1}{p^{m_\chi-1}}, \ldots, \frac{i+\alpha_{d-1}}{p^{m_\chi-1}} \right\}.$$

Then the theorem follows from the relation

$$L^*(f,\chi,t) = \frac{C^*(f,\chi,t)}{C^*(f,\chi,qt)}. \qquad \square$$

Remark. Suppose that Wan's Conjecture (see [8]) holds for $f(x) \in \mathbb{Z}[x]$, which means that $\lim_{p\to\infty} \mathrm{gap}(f(x) \bmod p, \Psi) = 0$. Then there is a positive integer N_h such that $\mathrm{gap}(f(x) \bmod p, \Psi) < \frac{1}{h}$ for all $p > N_h$.

Proof of Theorem 1.1. In our situation $f(x) = x^d + ax$, the case $\chi = \Psi$ is just Proposition 3.4. For χ general, by Theorem 4.2, it suffices to show $\mathrm{gap}(f,\Psi) < \frac{1}{h}$ for $p > \max\{N(d), \frac{hd(d-1)}{4} + 1\}$. For $p > N(d)$, the slopes of the q-adic Newton polygon of $C^*(f,\Psi,t)$ are

$$\bigcup_{i\geq 0} \{i+w_0, i+w_1, \cdots, i+w_{d-1}\}.$$

Denote $w_{kd+s} = k + w_s$ for all $k \in \mathbb{N}$ and $0 \leq s < d$. It is easy to see that

$$\mathrm{NP}(f,\Psi,kd+s) = w_0 + w_1 + \cdots + w_{kd+s-1},$$

and

$$\mathrm{HP}(f, kd+s) = \frac{0}{d} + \frac{1}{d} + \cdots + \frac{kd+s-1}{d}.$$

As $w_0 + w_1 + \cdots + w_{d-1} = \frac{0}{d} + \frac{1}{d} + \cdots + \frac{d-1}{d}$, $\mathrm{NP}(f, \Psi, x) - \mathrm{HP}(f, x)$ is a periodic function of period d. For all $0 \le k < d$,

$$\begin{aligned}\mathrm{NP}(P, \Psi, k) - \mathrm{HP}(P, k) &= (w_0 + w_1 + \cdots + w_{k-1}) - (\frac{0}{d} + \frac{1}{d} + \cdots \frac{k-1}{d}) \\ &= \frac{d-1}{d(p-1)} \sum_{i=0}^{k-1} (r_i - i) \le \frac{d(d-1)}{4(p-1)} < \frac{1}{h}\end{aligned}$$

by (3.5) if $p > \frac{hd(d-1)}{4} + 1$. This finishes the proof. □

5. Note added in proof

After the paper was accepted, we were informed by the authors of [3] that Theorem 1.1 was also proved in [3, Theorem 1.6].

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