Hidden Variables and Bell's two Theorems

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With

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For the 65th birthday of
Prof. Henk W.J. Blöte

Basic postulates in orthodox quantum mechanics

- State space—Hilbert space
- Evolution—unitary operation (Schrödinger equation)
- Measurement:

  1, Any physical quantity is described by a Hermitian operator A;

  2, outcome of measurement for A can only be a eigenvalue a_i of A;

  3, $|\psi\rangle = \sum_i \alpha_i |a_i\rangle \xrightarrow{M} |a_i\rangle$ with $p_i = \alpha_i^2$.

Puzzles about the Measurement postulate:

- nonrealistic (indeterministic)—physical quantity does not possess a definite value before **M**
- how does the wave collapse occur? instantaneous?
- nonlocal—wave is global
- subjective—what qualifies for “observer”?

Some quotations:

"We often discussed his notions on objective reality. I recalled that during one walk Einstein suddenly stopped, turned to me and asked whether I really believed that the moon exists only when I look at it" *---A. Pais*

"God does not play dice" *--- A. Einstein*

"*I cannot seriously believe in [the quantum theory] because it cannot be reconciled with the idea that physics should represent a reality in time and space, free from spooky actions at distance."* *--- A. Einstein*

"Was the wave function of the world waiting to jump for thousands of millions of years until a single-celled living creature appeared? Or did it have to wait a little longer, for some better qualified system ... with a Ph.D.?" *---J. Bell*

Reactions to these puzzles:

A1: “shut-up and calculate”

A2: construct collapse theories

B : Propose various interpretations

- B1: Many worlds interpretation (linked to consciousness)
- B2: Ithaca interpretation (linked to consciousness)
- B3: Hidden-variables theories--Bohmian mechanics
- B4: …

Ithaca interpretation: *The subject of physics is about correlation and only correlation. Thus, correlations have physical reality; that which they correlate does not. The statement might bear some similarity as “Fields in empty space have physical reality; the medium that supports them does not.”*

Many worlds interpretation: *In addition to the world we are aware of directly, there are many other similar worlds which exist parallel at the same space and time.*

Bohmian mechanics: *A complete description of an isolated system includes the spatial positions of each particle (hidden variables) and the wave function of the system (guiding waves). The wave function is governed by Schrödinger equation, and it guides the motion of particles. Each particle obeys a 1st order equation of motion specifying that its velocity is proportional to the spatial gradient of the wave function of the system, evaluated at the instantaneous positions of all the other particles (nonlocal and contextual).*

Dream of hidden variables (HV):

1, State vectors describe an ensemble of systems;
2, in each individual member, every observable does have a definite value.

Classical statistical mechanics $\leftrightarrow$ Newtonian mechanics

Quantum mechanics $\leftrightarrow$ Hidden-variables theories

--if two observables do not commute, the uncertainty principle does not prohibit both from having definite values in an individual system. It merely insists that it is impossible to prepare an ensemble of systems in which the values of neither observables fluctuate.

Plausible constraints on HV theories:

1, quantized---outcomes of ***A*** are to be eigenvalues of ***A***.

2, if ***A***,***B***,***C***,… commute, the values possessed by an individual member are the simultaneous eigenvalues. In particular

$$f(A, B, C, \cdots) = 0 \qquad \Rightarrow \qquad f(a, b, c, \cdots) = 0$$

Task of no-HV theorems (Bell's theorems)

Refuse HV models of some particular features—
noncontextual and local

Von Neumann's assumption:

$C = A + B \quad \Rightarrow \quad c = a + b$ even if A and B do not commute

Spin-1/2: Let $A = \sigma_x, B = \sigma_y$, then $C = A + B = \sqrt{2}\hat{n}\bullet\vec{\sigma}$.
We have $a = \pm 1, b = \pm 1, \quad c = \pm\sqrt{2}.\ \therefore c \neq a + b$.

Is it silly? $\because$ *QM only requires* $\langle C\rangle = \langle A\rangle + \langle B\rangle$ *(J. Bell)*

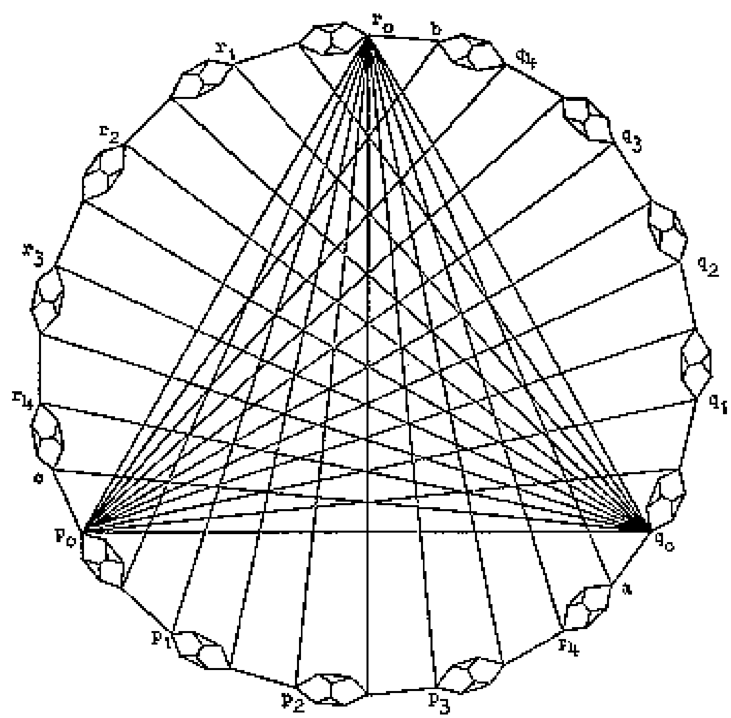
Bell-KS theorem--noncontextuality:

1, three dimensions—spin 1:

Let $A = S_x^{\ 2}, B = S_y^{\ 2}, C = S_z^{\ 2}$, $\therefore a = 0,1 \ \ (b,c)$. Further, $A + B + C = 2$,
$\therefore a + b + c = 2$ -- geometrically, in any orthogonal triad, two of the axis are 1, the third is 0.

Bell: *In 3D, a set of directions exists such that there is no way to consistently assign 1's and 0's to each orthogonal triad.*

Kochen-Specker: use 117 directions (1967);
Conway-Kochen—31; Peres—33.



M. C. E S C H E R.

2, four dimensions—2 spin-1/2:

σ_x^1	σ_x^2	$\sigma_x^1\sigma_x^2$
σ_y^2	σ_y^1	$\sigma_y^1\sigma_y^2$
$\sigma_x^1\sigma_y^2$	$\sigma_x^2\sigma_y^1$	$\sigma_z^1\sigma_z^2$

(a), Observables in each row (column) commute;
(b), Product for the 3rd column is -1, for all other columns and rows is +1.

3, eight dimensions—3 spin-1/2:

$$\sigma_y^1$$

$$\sigma_x^1\sigma_x^2\sigma_x^3 \quad \sigma_y^1\sigma_y^2\sigma_x^3 \quad \sigma_y^1\sigma_x^2\sigma_y^3 \quad \sigma_x^1\sigma_y^2\sigma_y^3$$

$$\sigma_x^3 \qquad \sigma_y^3$$

$$\sigma_x^1$$

$$\sigma_y^2 \qquad \sigma_x^2$$

(a), Observables in each of 5 lines commute;

(b), Product for the horizontal line is -1, for all other lines are +1.

-----the theorems are *state-independent*

Involoved assumption—noncontextuality:

$A, B, C, \cdots$ commute, $A, L, M, \cdots$ commute, the same value a is assigned to A regardless of which of two sets is chosen.

Bell's theorem--locality:

1, GHZ state: $|\psi\rangle = \frac{1}{\sqrt{2}}\left(|000\rangle - |111\rangle\right)$.

(a), Simultaneous eigenstate of $A = \sigma_x^1\sigma_x^2\sigma_x^3, B = \sigma_x^1\sigma_y^2\sigma_y^3, C = \sigma_y^1\sigma_x^2\sigma_y^3$, $D = \sigma_y^1\sigma_y^2\sigma_x^3$, with eigenvalues $a = -1, b = c = d = 1$.

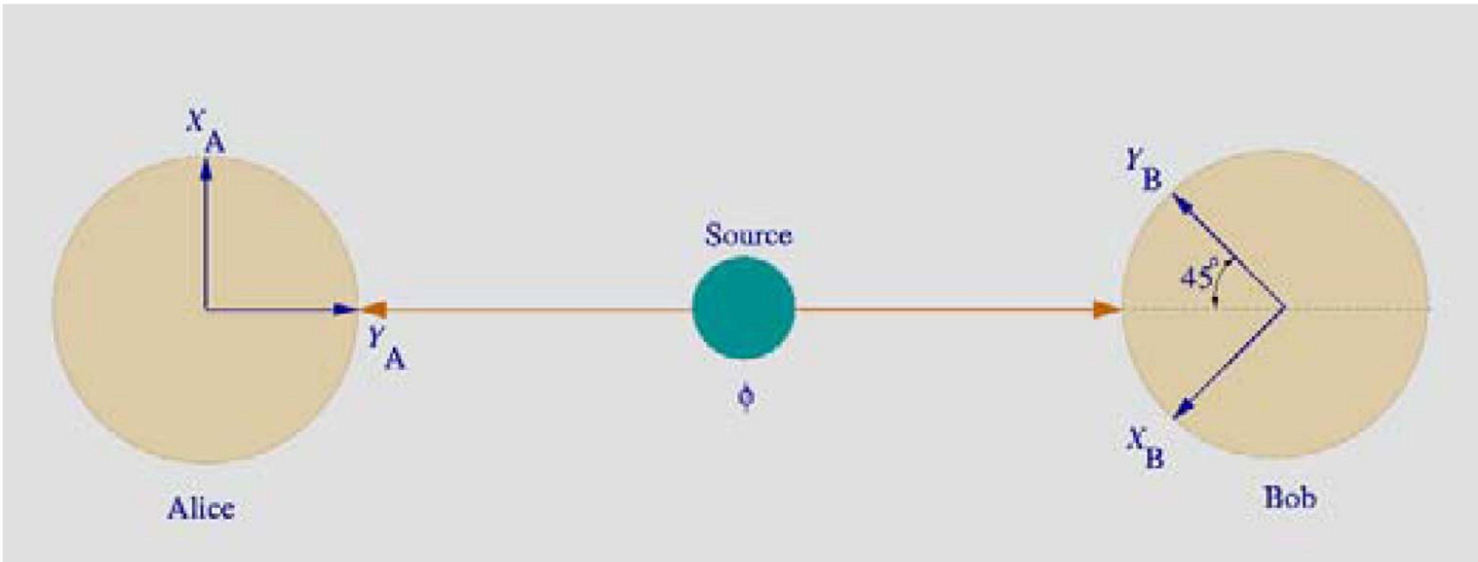
2, Hardy state:

$$
\begin{aligned}
|\psi\rangle &= \frac{1}{2\sqrt{3}}\left(3|00\rangle - |01\rangle - |10\rangle - |11\rangle\right) \\
&= \frac{1}{\sqrt{6}}\left(2|+0\rangle \qquad + |-0\rangle - |-1\rangle\right) \\
&= \frac{1}{\sqrt{6}}\left(2|0+\rangle + |0-\rangle \qquad - |1-\rangle\right) \\
&= \frac{1}{\sqrt{3}}\left(\quad |++\rangle + |+-\rangle - |-+\rangle \qquad \right)
\end{aligned}
$$

One has

$$
\begin{array}{lllll}
p(+_A, 1_B) = 0 \ \Rightarrow \ p(-_A | 1_B) = 1 & & p(1_A, +_B) = 0 \ \Rightarrow \ p(-_B | 1_A) = 1 \\
p(-_A, -_B) = 0 \ \Rightarrow \ p(+_B | -_A) = 1 & & p(-_A, -_B) = 0 \ \Rightarrow \ p(+_A | -_B) = 1 \\
p(1_A, +_B) = 0 \ \Rightarrow \ p(0_A | +_B) = 1 & \text{or} & p(+_A, 1_B) = 0 \ \Rightarrow \ p(0_B | +_A) = 1 \\
\Downarrow & & \Downarrow \\
p(0_A | 1_B) = 1 \ \text{Conflict with QM} & & p(0_B | 1_A) = 1 \ \text{Conflict with QM}
\end{array}
$$

3, Bell-inequality:



(a) $X_A = \pm 1, Y_A = \pm 1, X_B = \pm 1, Y_B = \pm 1$.

(b) $V_{Bell} = |\langle X_A X_B \rangle + \langle X_A Y_B \rangle + \langle Y_A Y_B \rangle - \langle Y_A X_B \rangle| \leq 2$.

Proof:

Assumption: $\langle X_A X_B \rangle = \int \rho(\lambda) x_A(\lambda) x_B(\lambda)\, d\lambda$ with

$$x_A(\lambda) = \pm 1, x_B(\lambda) = \pm 1,\; 1 = \int \rho(\lambda)\, d\lambda .$$

For each individual λ,

$$| x_A(\lambda)[x_B(\lambda) + y_B(\lambda)] + y_A(\lambda)[x_B(\lambda) - y_B(\lambda)] | = 2 .$$

Thus,

$$V_{Bell} = | \int \rho(\lambda) x_A(\lambda)[x_B(\lambda) + y_B(\lambda)] + y_A(\lambda)[x_B(\lambda) - y_B(\lambda)] d\lambda |$$

$$\leq \int \rho(\lambda) | x_A(\lambda)[x_B(\lambda) + y_B(\lambda)] + y_A(\lambda)[x_B(\lambda) - y_B(\lambda)] | d\lambda = 2$$

For a singlet state $|\psi^-\rangle = \left(|01\rangle - |10\rangle\right)/\sqrt{2}$, $V_{Bell} = 2\sqrt{2}$ **!**

Conclusion:

Local Hidden-variables (LHV) theories cannot reproduce all the predictions of QM!

Nonseparability and quantum nonlocality in a two spin-1/2 system

Defintions:

Given *a density operator* ρ *:*

Quantum nonlocality: *if there exist some predictions by QM that cannot be reproduced by LHV theories.*

Separable (classical correlated): *if* $\rho = \sum_i p_i \rho_i^1 \otimes \rho_i^1$ *; in particular,*

uncorrelated if $\rho = \rho^1 \otimes \rho^2$,

Nonseparable (EPR correlated, entangled): *if* $\rho \neq \sum_i p_i \rho_i^1 \otimes \rho_i^1$.

Pure state:

Quantum nonlocal $\Leftrightarrow$ Nonseparable

1, two spin-1/2 particles: $|\psi\rangle = c_1 |01\rangle + c_2 |10\rangle$.

Let $X_A :\equiv (0,0,1), Y_A :\equiv (1,0,0), X_B :\equiv (\sin\beta, 0, \cos\beta)$,

with $\sin\beta = \left(\frac{4|c_1 c_2|}{1+4|c_1 c_2|} \right)^{1/2}, \cos\beta = \left(\frac{1}{1+4|c_1 c_2|} \right)^{1/2}$,

then $V_{Bell} = 2/\cos\beta > 2$!

2, two higher-dimensional particles:

$$|\psi\rangle = c_1\,|01\rangle + c_2\,|10\rangle + c_3\,|22\rangle + \cdots \text{ (Schmidt)}$$

Method: test Bell's inequality for *partial* correlation.

If $V_{Bell} > 2$, no LHV theory is available.

X_A	X_B		X_A	Y_B		Y_A	X_B		Y_A	Y_B
-1	-1		-1	-1		o	o		-1	-1
o	o		-1	1		1	-1		1	-1
-1	-1		-1	1		-1	-1		1	1
1	-1		1	1		-1	1		o	o
1	1		o	o		o	o		o	o
o	o		o	o		1	-1		o	o
1	-1		-1	-1		o	o		-1	1
$\cdots$	$\cdots$		$\cdots$	$\cdots$		$\cdots$	$\cdots$		$\cdots$	$\cdots$

$\Downarrow$ Discard information by $o \to 1$

X_A	X_B		X_A	Y_B		Y_A	X_B		Y_A	Y_B
-1	-1		-1	-1		1	1		-1	-1
1	1		-1	1		1	-1		1	-1
-1	-1		-1	1		-1	-1		1	1
1	-1		1	1		-1	1		1	1
1	1		1	1		1	1		1	1
1	1		1	1		1	-1		1	1
1	-1		-1	-1		1	1		-1	1
…	…		…	…		…	…		…	…

Then, $V_{Bell} = (|c_1|^2 + |c_1|^2)V_{Bell}^{(01)} + (1 - |c_1|^2 - |c_1|^2)V_{Bell}^{(o)}$

$= (|c_1|^2 + |c_1|^2)(V_{Bell}^{(01)} - 2) + 2$. Thus,

$$V_{Bell}^{(01)} > 2 \quad \Rightarrow \quad V_{Bell} > 2\,!$$

Mixed state:

Quantum nonlocal $\Leftrightarrow$ Nonseparable?

1, two spin-1/2 particles:

$$\rho_W = x\,|\psi^-\rangle\langle\psi^-| + \frac{1-x}{4}1 \text{ (Werner state)}$$

Separability criterion (Peres's criterion):

$$\rho = \sum_i p_i \rho_i^1 \otimes \rho_i^1 \quad \Rightarrow \quad \rho^{\text{PT}} = \sum_i p_i \left(\rho_i^1\right)^T \otimes \rho_i^1$$

is also a density operator!

$\Rightarrow \rho_W$ is separable for $x < 1/3$.

Violation of Bell's inequality: if $x > 1/\sqrt{2}$.

$\exists$ **LHV model for** ***Bell-type*** **measurement**: if $x < 1/2$.

Teleportation: ρ_W with $x > 1/3$ can be used for quantum teleportation with nonperfect fidelity.

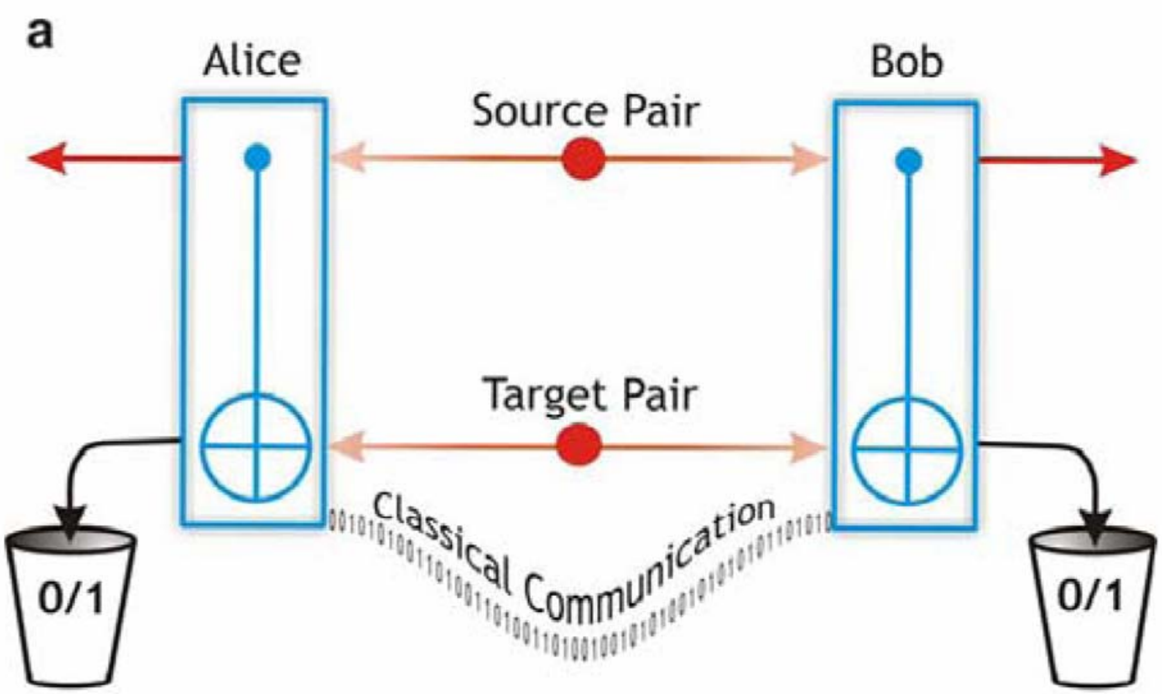
Purification: a large number of ρ_W with $x > 1/3$ can be purified to a small number of ρ_W' with $x' \to 1$.

Conclusion:

Results of the "purification + Bell-type measurement"
Cannot be simulated by LHV theories.

Proof:

1, Experimental setup



2, Involved operations:

Local Operation and Classical Communication (LOCC)

LO $\rightarrow$ redistribute EPR correlations

CC $\rightarrow$ subensemble (sub-Hilbert-space) selection

$\therefore$ The totality of EPR correlations is unchanged.

3, Physical implication of LOCC in LHV theories:

$$\text{LHV Model} \xrightarrow{\text{Conditional Probability}} \text{LHV Model}$$

Proof:

◆ Assumption: $P(x_{A1}, x_{B1}) = \int \rho(\lambda_1) p_{A1}(x_{A1}, \lambda_1)\, p_{B1}(x_{B1}, \lambda_1) d\lambda_1$,

$P(x_{A2}, x_{B2}) = \cdots$

- After Local Operator:

$$P_{LO}(x_{A1},x_{A2};x_{B1},x_{B2}) = \int \rho(\lambda)\, p_A(x_{A1},x_{A2},\lambda)\, p_B(x_{B1},x_{B2},\lambda)\, d\lambda$$

- After Classical Communication:

Let the marginal probabilities be

$$P^M(x_{A1},x_{B1}) = \sum_{x_{B2}=\pm1}\sum_{x_{A2}=\pm1} P_{LO}(x_{A1},x_{A2};x_{B1},x_{B2}),$$

$$p_A^M(x_{A1},\lambda) = \sum_{x_{A2}=\pm1} p_A(x_{A1},x_{A2},\lambda),$$

$$p_B^M(x_{B1},\lambda) = \sum_{x_{B2}=\pm1} p_B(x_{B1},x_{B2},\lambda),$$

Then, the conditional probabilities are

$$p_A(x_{A1},|_{x_{A2}=1},\lambda)=\frac{p_A(x_{A1},1,\lambda)}{p_A^M(x_{A1},\lambda)},$$

$$p_B(x_{B1},|_{x_{B2}=1},\lambda)=\frac{p_B(x_{B1},1,\lambda)}{p_B^M(x_{B1},\lambda)}$$

And

$$P^X(x_{A1},x_{B1})=P_{LO}(x_{A1},x_{A2};x_{B1},x_{B2})\Big|_{x_{A2}=1,x_{B2}=1}$$

$$=\frac{\int\rho(\lambda)p_A(x_{A1},1,\lambda)\,p_B(x_{B1},1,\lambda)d\lambda}{P^M(x_{A1},x_{B1})}$$

$$=\int p_A(x_{A1}\,|_{x_{A2}=1},\lambda)p_B(x_{B1}\,|_{x_{B2}=1},\lambda)\frac{p_A^M(x_{A1},\lambda)p_B^M(x_{B1},\lambda)}{P^M(x_{A1},x_{B1})}\rho(\lambda)d\lambda$$

$$=\int p_A^X(x_{A1},\lambda)p_B^X(x_{B1},\lambda)\rho^X(\lambda)d\lambda$$

Done!