Conformal invariance and a quantum Monte Carlo method

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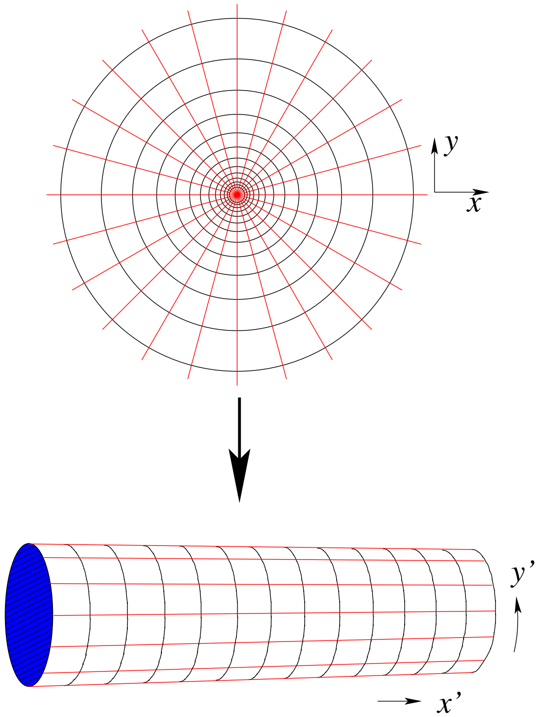
- Introduction
  - Conformal mappings
  - Critical phenomena
  - Conformal invariance
  - Monte Carlo simulations
- A quantum Monte Carlo method
- Numerical results
- Conclusions

Conformal mappings

- *Definition:* A conformal mapping $\vec{r} \to \vec{r}\,' = \vec{r}\,'(\vec{r})$

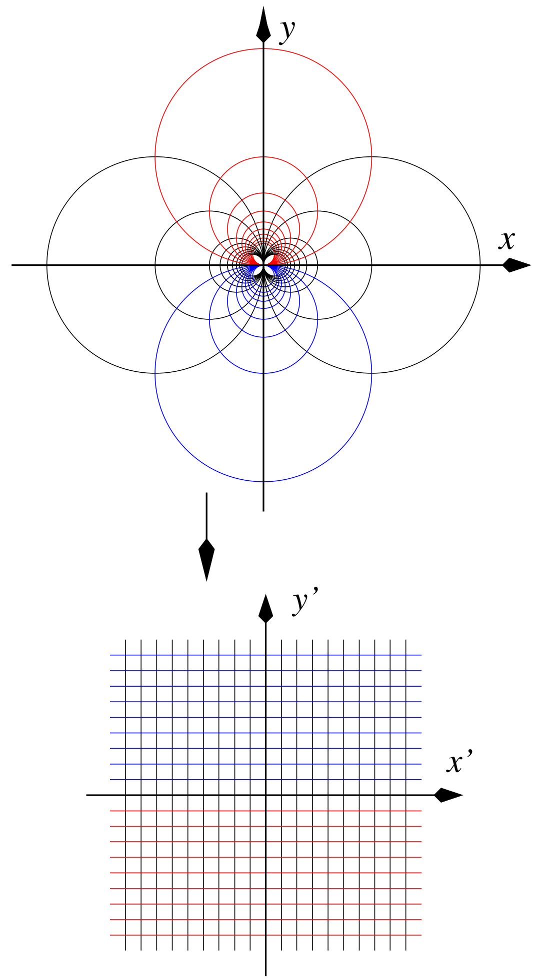
  $\Longleftrightarrow$ infinitesimal objects keep their shape.

- *Examples:*



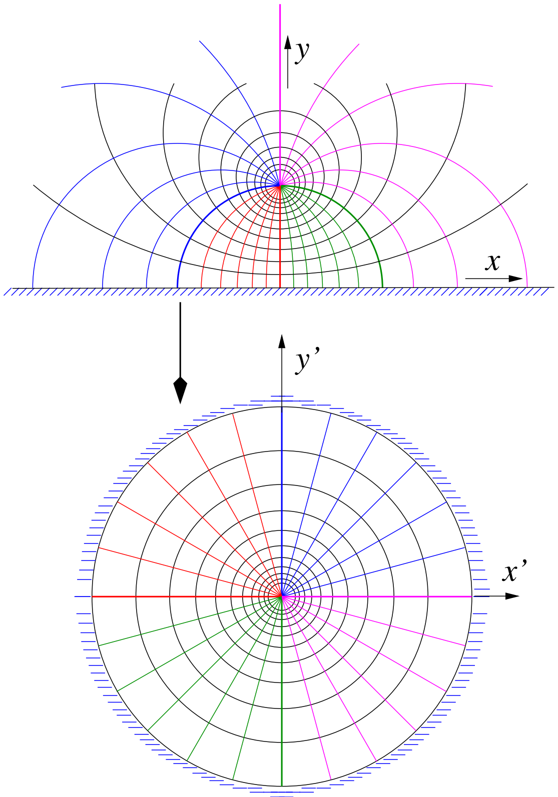
(1) Cardy's mapping

$z' = \ln z = \ln(x + iy)$



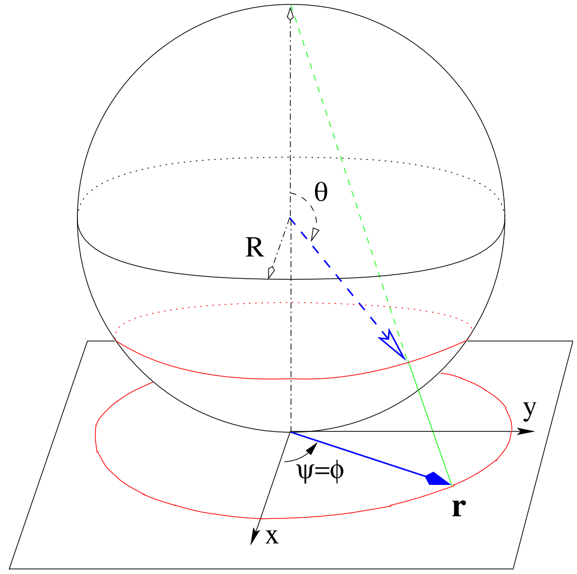
(2) Inversion transformation

$z' = 1/z$



(3) Interior of a circle

$z' = (z-1)/(z+1)$



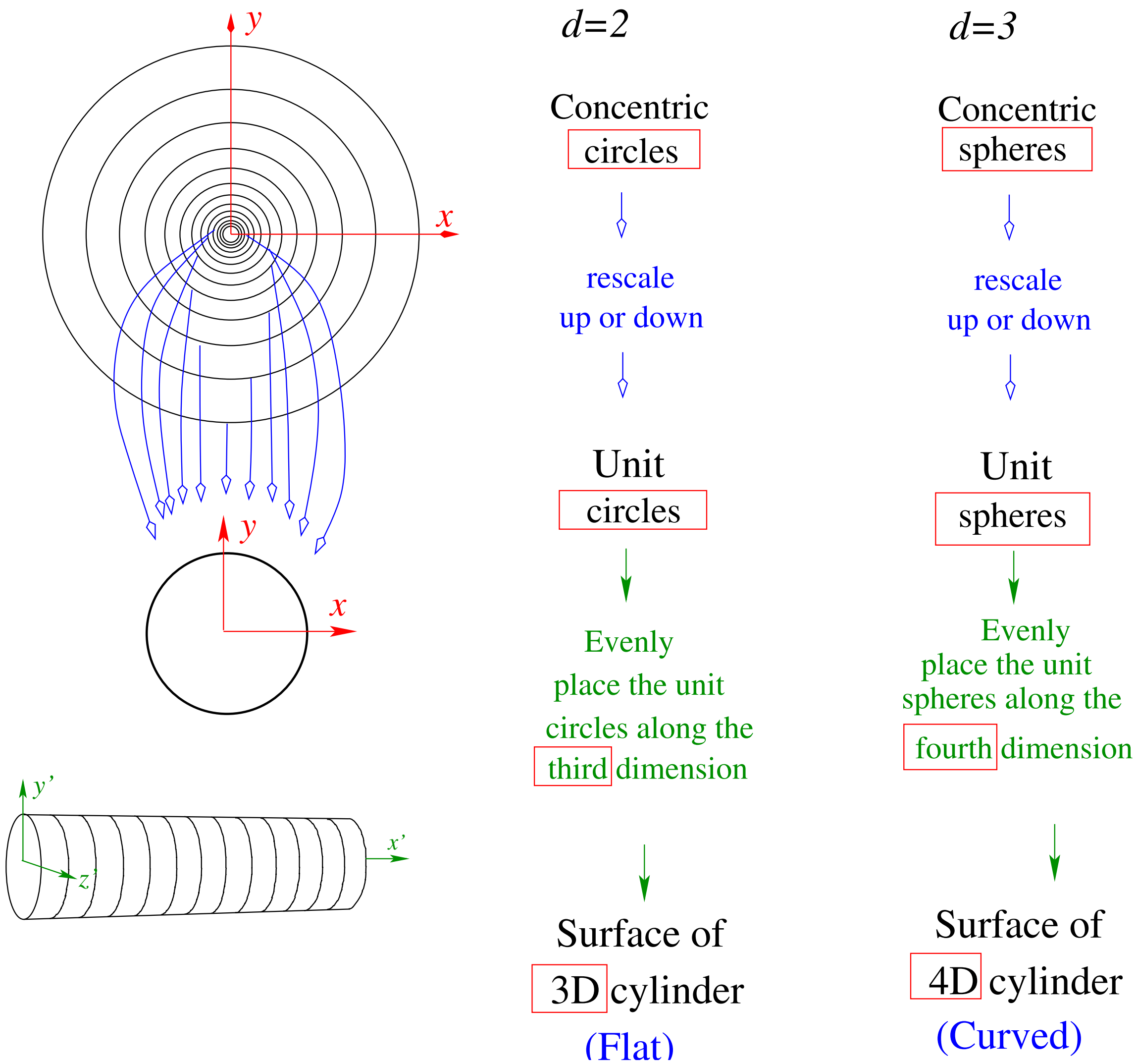
(4) Surface of a sphere

$(r, \psi) \to (2R\cot\theta, \psi)$

‡ For $d = 2$, any analytical complex function is conformal

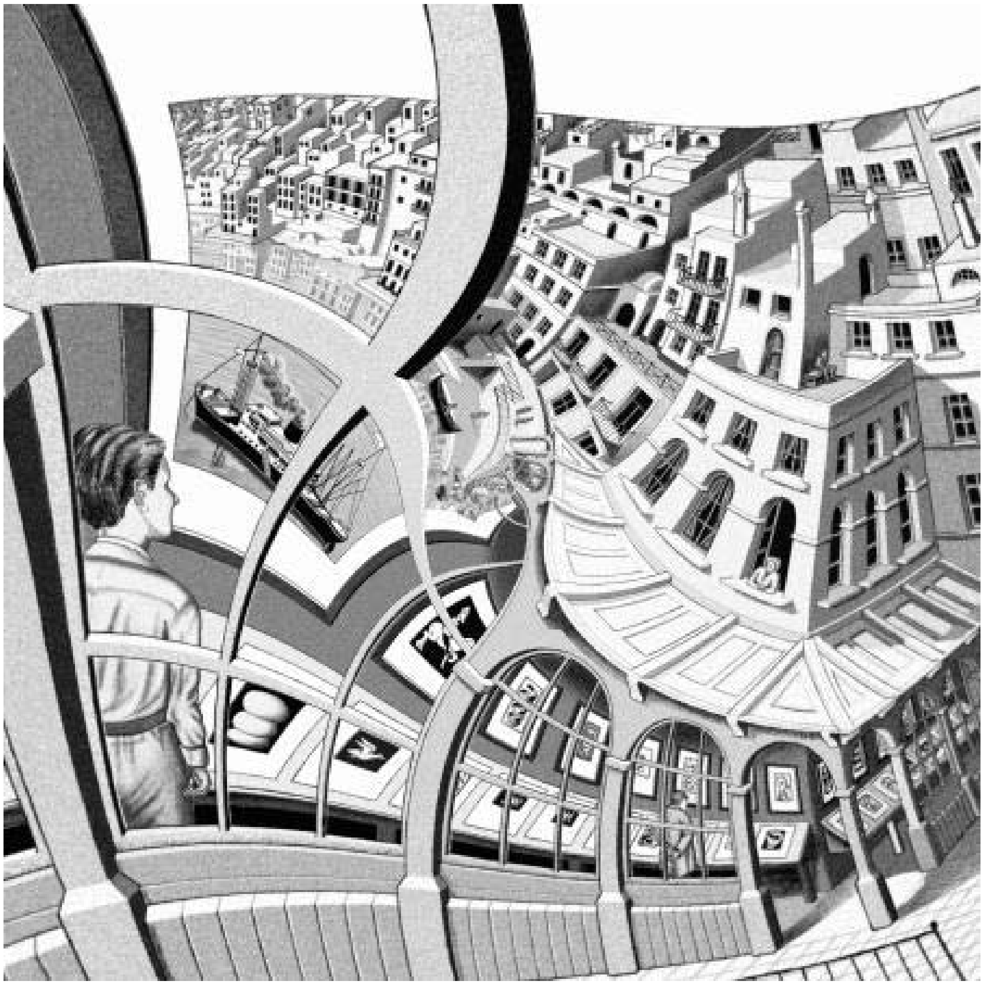
‡ Mapping (4) can be generalized to the surface of a spheroid (Y.J. Deng and H.W.J. Böte, Phys. Rev. E 036107, 2003).

‡ Mappings (1) and (2) ⇒ *flat* geometries; mappings (4) and (3) ⇒ *curved* geometries or curved boundaries

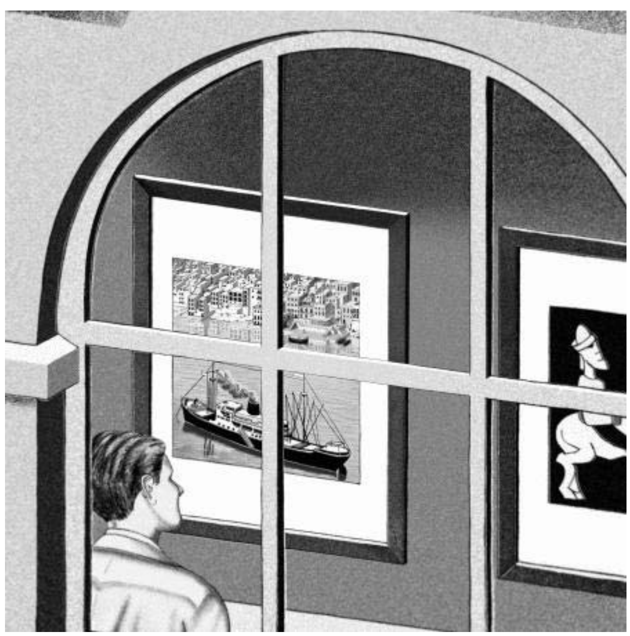


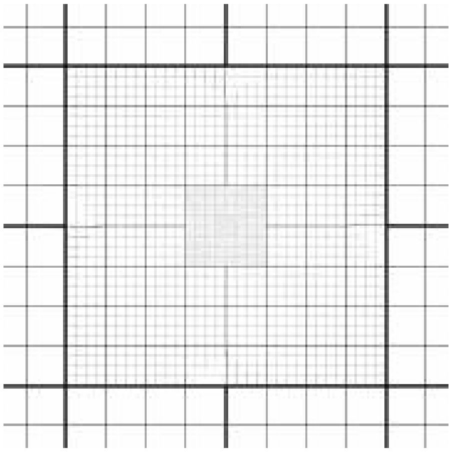
Cardy's mapping for $d = 3$

‡ Surface of the 4D cylinder is named a *spherocylinder* (Y.J. Deng and H.W.J. Böte, Phys. Rev. Lett. 190602, 2002).

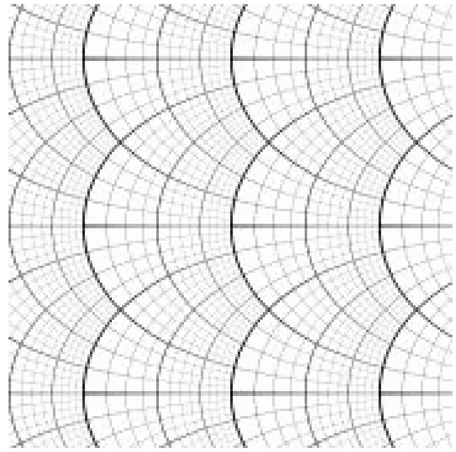


By M.E. Escher (1898-1972, Holland)

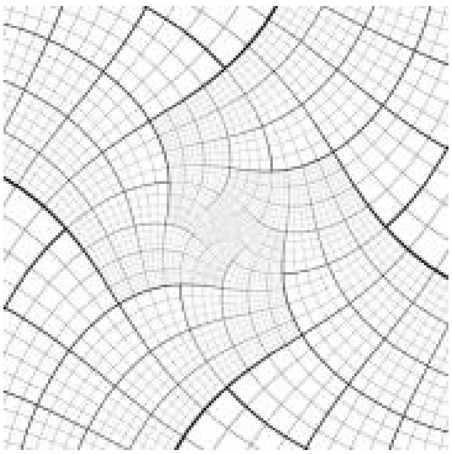




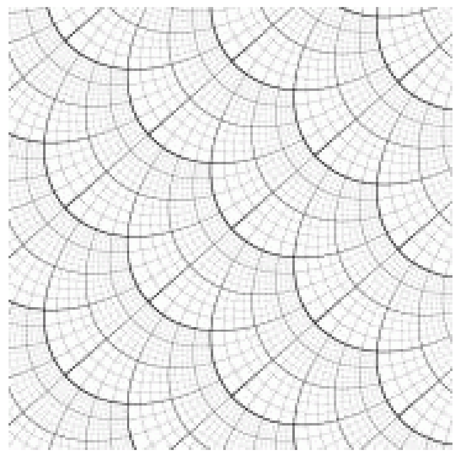
(1)



(2)



(4)



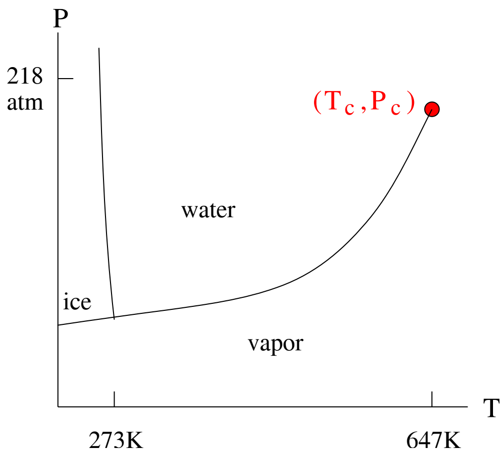
(3)

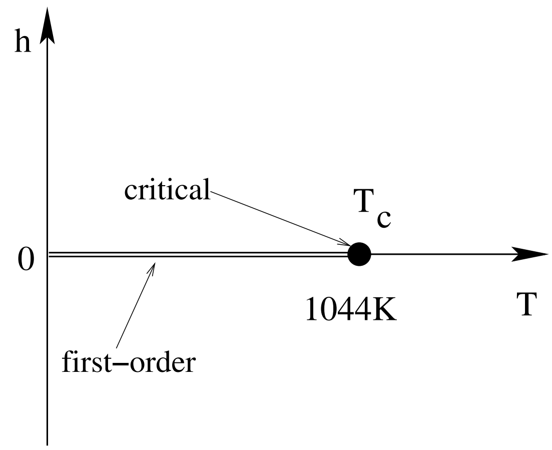
By H. Lenstra, B. de Smit *et. al* (Leiden University)

(See. *e.g.*, *New York Times* July 30, 2002)

Critical phenomena

- *Examples:*





Liquid-gas transition   Ferro-paramagnet transition

Near T_c: Specific-heat $C \propto |T - T_c|^{-\alpha}$

Susceptibility $\chi \propto |T - T_c|^{-\gamma}$

‡ α, γ take same values: same universality class

- *Theoretical descriptions*

Ising model

$$\mathcal{H}/k_{\mathrm{B}}T = -K \sum_{\langle i,j \rangle} s_i s_j \;, \qquad (s_i = \pm 1 \;, K : \text{interactions})$$

q-state Potts model

$$\mathcal{H}/k_{\mathrm{B}}T = -K \sum_{\langle i,j \rangle} \delta_{\sigma_i \sigma_j} \;, \qquad (\sigma_i = 1, 2, \cdots, q)$$

Conformal invariance

• For $d = 2$, conformal field theory has <u>*predicted*</u> a series of critical exponents for Potts models.

• At criticality, under a conformal mapping,

$$\langle \varphi_1(\vec{r}_1)\varphi_2(\vec{r}_2)\cdots\rangle_{\vec{r}} =$$

$$b(\vec{r}_1)^{-X_1} b(\vec{r}_2)^{-X_2}\cdots\langle \varphi_1(\vec{r}\,'_1)\varphi_2(\vec{r}\,'_2)\cdots\rangle_{\vec{r}\,'}$$

φ_i: scaling operators (m, e etc),

X_i: scaling dimensions (related to $\alpha, \gamma, \cdots$),

$b(\vec{r})$: scaling factor ($b(\vec{r})^d = \det(\partial\vec{r}/\partial\vec{r}\,')$)

‡ Examples:

(1) Along (sphero)cylinder ($|u| \gg 1$)

$\langle \varphi(0)\varphi(u)\rangle \propto \exp\left(-|u|/\xi_R\right)$, $\quad \xi_R = R/X$

(2) On surface of a sphere

$\langle \varphi(0,0)\varphi(\theta,\psi)\rangle \propto \sin^{-2X}(\theta/2)$

(3) On interior of a circle

$\langle \varphi(r)\rangle \propto [1-(r/R)^2]^{-X}$

‡ <u>Difficulties</u>:

(1) *Can one numerically verify these assumptions?*

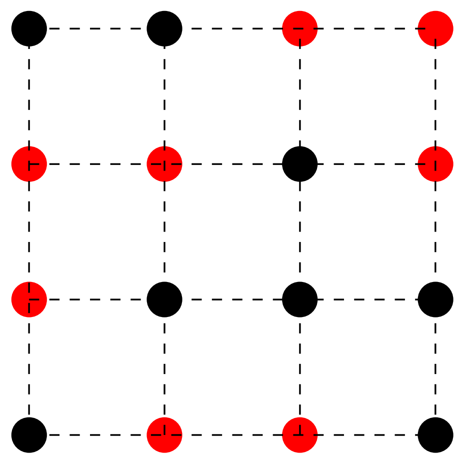
(2) *How can one simulate such curved geometries?*

△ <u>Solution</u>: Make use of a quantum Hamiltonian and develop a quantum Monte Carlo method.

Monte Carlo simulations

Boltzmann distribution: $w \propto \exp(-E/k_B T)$

- Metropolis algorithm



1, choose a spin s_i, and

2, calculate ΔE as $s_i \to -s_i$

3, draw random number $0 \leq r < 1$, if $r < e^{-\Delta E/k_b T}$, then $s_i \to -s_i$; else, do nothing

suffer from critical slowing down

- Swendsen-Wang and Wolff cluster algorithm

spin up-down (↑↓) symmetry

- Geometric cluster algorithm

spatial geometry (inversion invariance) (H.W.J. Böte and J.R. Heringa, Phys. Rev. E **57** 4976, 1997)

A quantum Monte Carlo method

- Classical Ising model on $N \times M$ square lattice:

$$\mathcal{H}/k_{\rm B}T = -\sum_{i}^{N}\sum_{j}^{M}[K_x s_{i,j}s_{i+1,j} + K_y s_{i,j}s_{i,j+1}] \; ,$$

Take *anisotropic* limit $\epsilon \to 0$:

$$K_y = \epsilon/t, \;\; \exp[-2K_x] = \epsilon \; , \;\; (t : \text{temperature-like})$$

- Quantum transverse Ising chain

$$\mathcal{H}_Q = -\sum_{i}^{N}(s_i^z\sigma_{i+1}^z + t\sigma_i^x) \; , \;\; (\sigma^z, \, \sigma^x : \text{Pauli matrices}).$$

t: transverse field, criticality at: $t_c = 1$.

- A continuous Wolff algorithm (H.W.J. Böte and Y.J. Deng, Phys. Rev. E 066110, 2002)

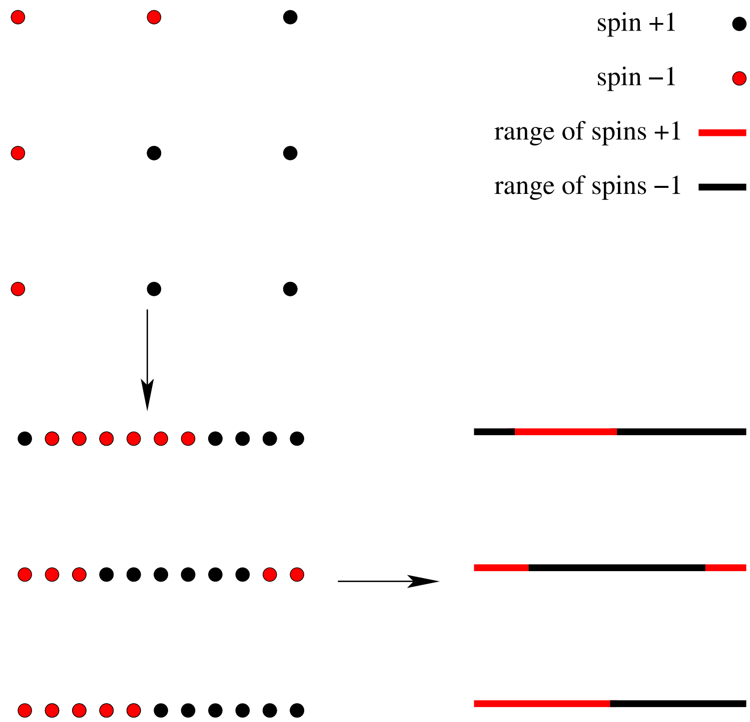
[Difficulty] For $\epsilon \to 0$, $M \sim 1/\epsilon$, since $\xi_x \sim 1/\epsilon$

[Solution] Rescale x-coordinate

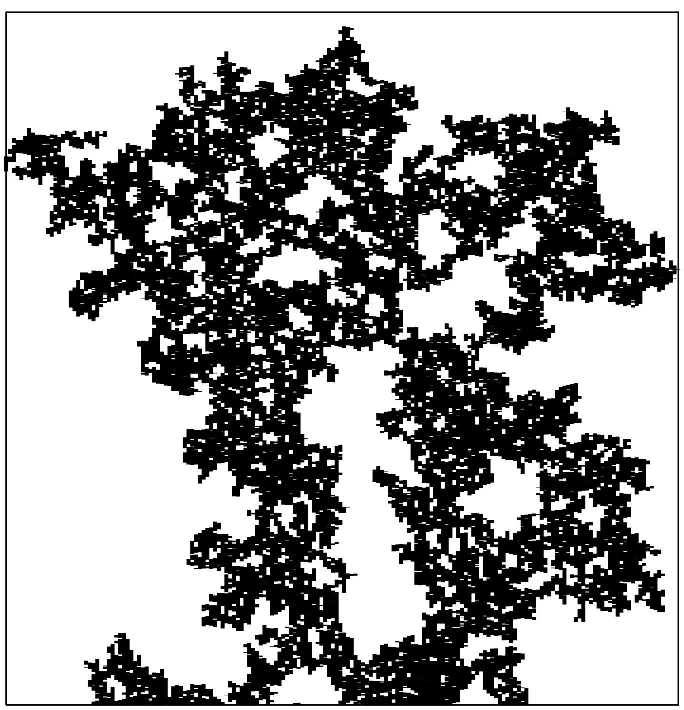
$\downarrow \quad x' = x/\epsilon$

x-dimension continuous

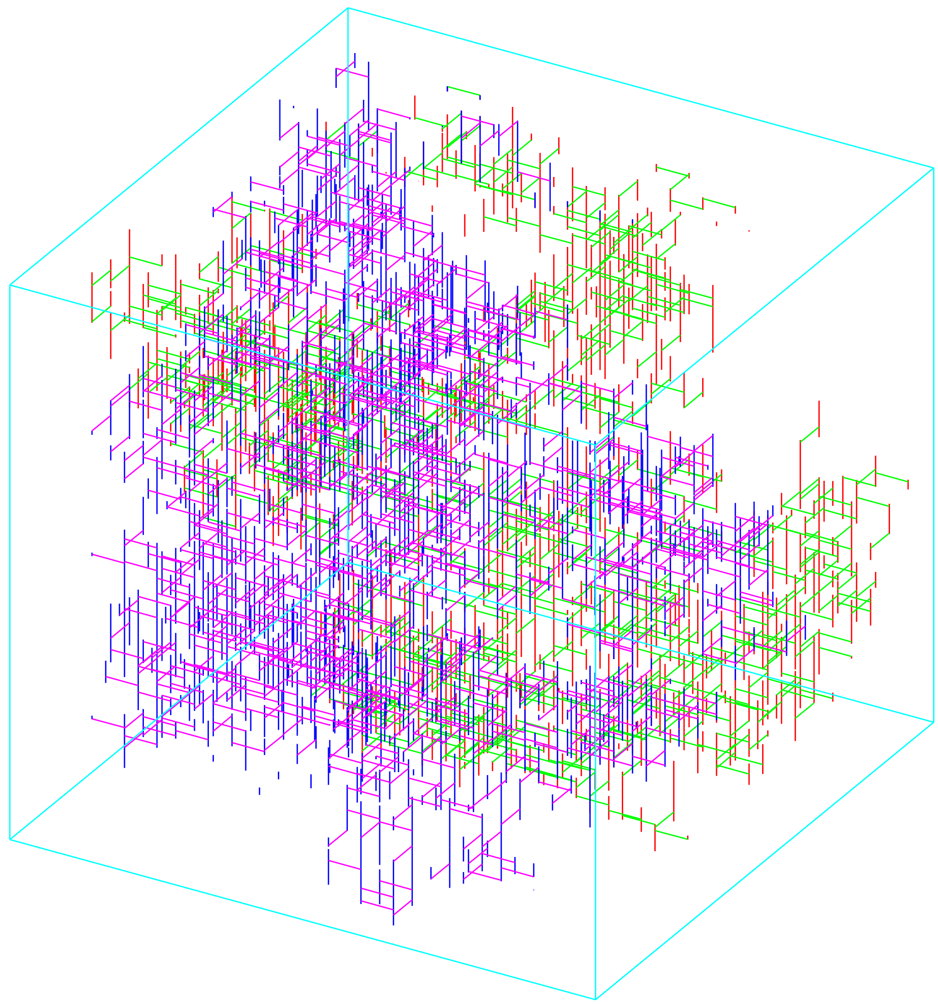
Dynamical variables: $\pm$ interfaces

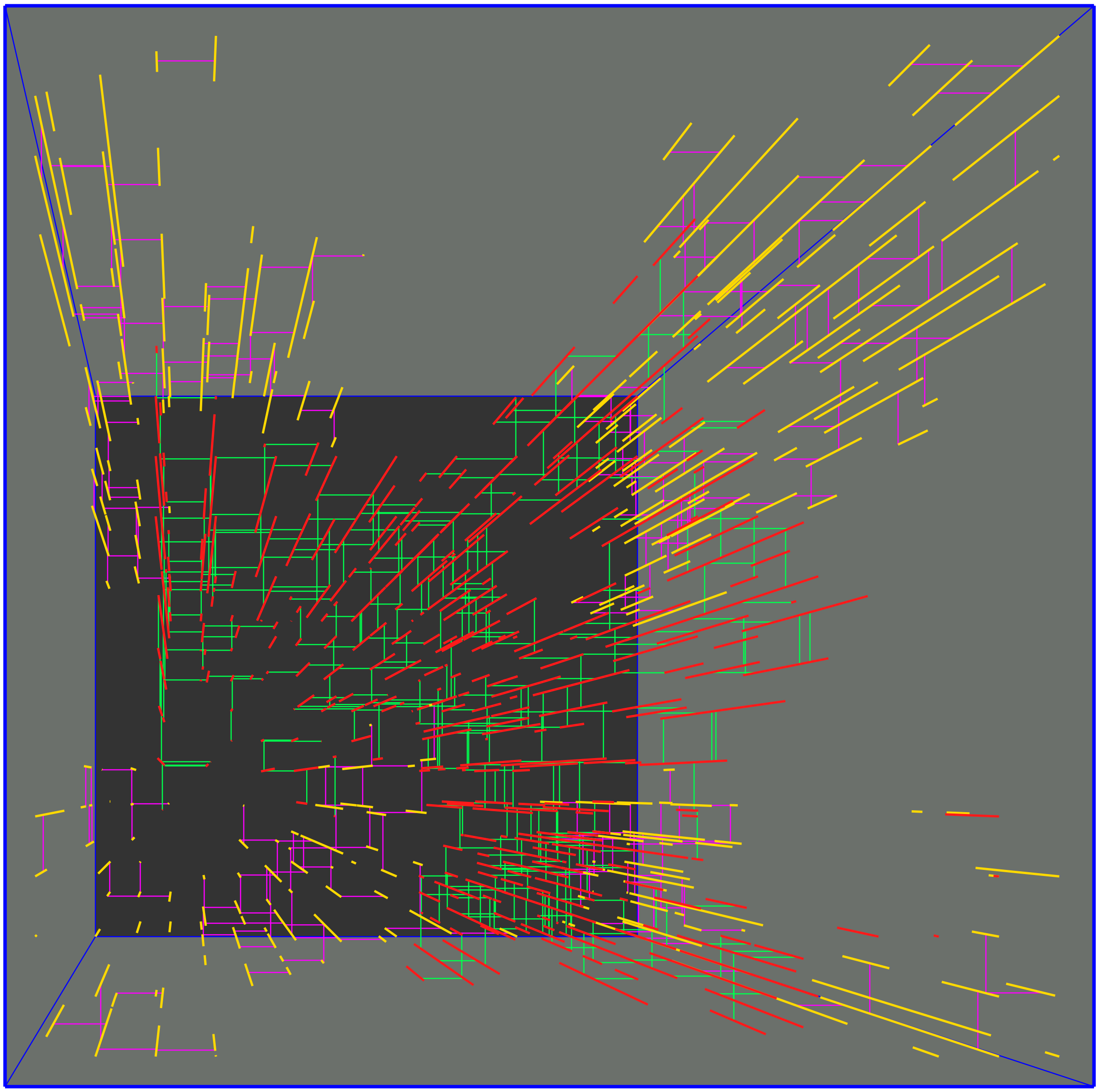


Procedures of rescaling



A Wolff cluster





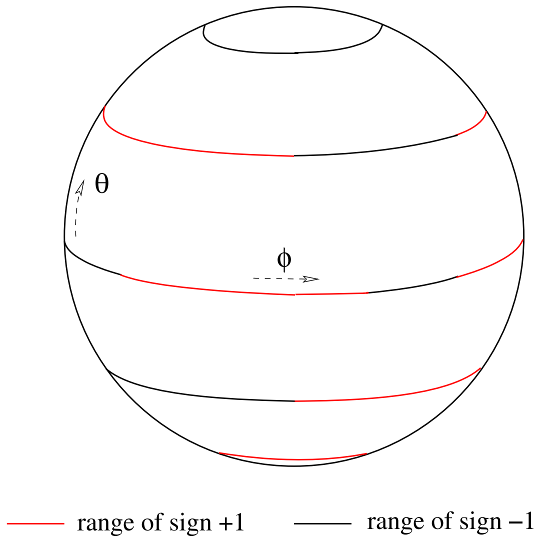
Quantum criticality

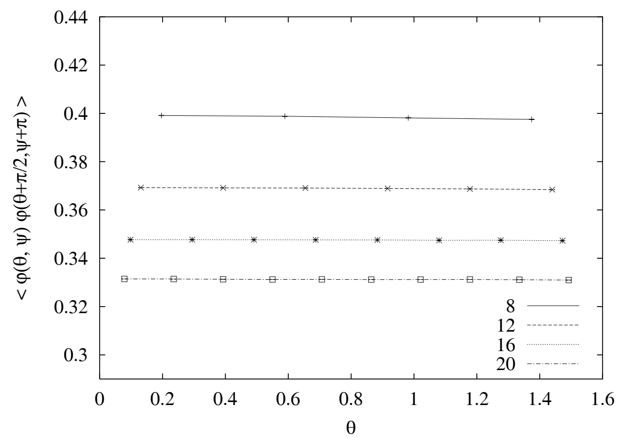
Simulations of a $d = 2$ quantum spin model as viewed from the so-called imaginary time direction. The critical clusters are essentially fractal in nature.

By Y.J. Deng, J.R. Heringa, and H.W.J. Böte

(see *e.g.* http://www.fom.nl)

- simulations on a sphere





Numerical results

- Critical points ($d = 2$):

  $t_c = 3.04438(2)$: quantum transverse Ising model

  $t_c = 4.3233(1)$: quantum transverse percolation model

- Simulations on sphere and disc geometries ($d = 1$):

  Ising: $X_h = 0.12496(9)$ (conformal invariance)

  $X_t = 0.996(6)$

  $X_h = 1/8$, $X_t = 1$ (exact solutions)

  Percolation: $X_h = 0.10420(5)$ (conformal invariance)

  $X_h = 5/48 \approx 0.10416\cdots$ (exact solutions)

- Simulations on spherocylinder ($d = 2$)

  Ising: $X_h = 0.5178(6)$ (conformal invariance)

  $X_t = 1.419(7)$

  $X_h = 0.5184(1)$, $X_t = 1.4132(3)$ (other sources)

  Percolation: $X_h = 0.479(1)$ (conformal invariance)

  $X_h = 0.476(4)$ (other sources)

Conclusions

- We have verified assumptions of conformal invariance in curved geometries.

- We show that conformal invariance is also a powerful tool to study critical phenomena for $d = 3$.

- We have developed an efficient Monte Carlo algorithm for descrete quantum spin models.