Cluster simulations of loop models on planar lattices.

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OUTLINE

- Swendsen-Wang simulation of the Potts model
- Cluster simulation of the $O(N)$ loop model on Honeycomb lattice
- Cluster simulation of the N-component corner-cubic model
- Cluster simulation of the N-component face-cubic model

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Cluster method for the Potts model

- *Potts model*

$$\mathcal{H}_{\text{Potts}}(q,K) = -K \sum_{\langle ij \rangle} \delta_{\sigma_i,\sigma_j} \quad (\sigma = 1, 2, \cdots, q)$$

- *Metropolis Method*

  ⋆ Integrated autocorrelation time $\tau \propto L^z$.

  $z \sim 2.2$: severe critical slowing down

- *Swendsen-Wang (SW) Method*

  ⋆ Map onto random-cluster model

$$\mathcal{Z}_{\text{Potts}}(q,v) = \sum_{\{\sigma\}} \prod_{\langle ij \rangle} (1 + v\delta_{\sigma_i,\sigma_j})$$

$$= \sum_{\{b_{ij}\}} v^{\#\text{bonds}} q^{\#\text{clusters}}$$

  $(v = e^K - 1)$; bond variable: $b_{ij} = 0, 1$

  ⋆ A Swendsen-Wang step:

  `Start from bond configuration` $\{b_{ij}\}$:

(1): `Identify Fortuin-Kasteleyn clusters for bond configuration` $\{b_{ij}\}$.

(2): `Randomly and independently assign for each cluster all spin values to be` $\sigma \in (1, 2, \cdots, q)$

(3) For each edge $\langle ij \rangle$, set $b_{ij} = 1$ with probability

$$p = \begin{cases} 1 - e^{-K} & \text{if } \sigma_i = \sigma_j \\ 0 & \text{if } \sigma_i \neq \sigma_j \end{cases}$$

- *Chayes-Machta (CM) Method*
  (for any real value $q \geq 1$)

Start from bond configuration $\{b_{ij}\}$:

(1): Identify Fortuin-Kasteleyn clusters for bond configuration $\{b_{ij}\}$.

(2): Randomly and independently set for each cluster all spin values to be ``active'' with probability $p = 1/q$, ``inactive'' with $p = 1 - 1/q$

(3) For each edge $\langle ij \rangle$, update b_{ij} only if σ_i and σ_j are both ``active''.

⋆ Critical slowing down is strongly suppressed in SW and CM cluster methods. Exponent z satisfies Li-Sokal bound: $z \geq \alpha/\nu$

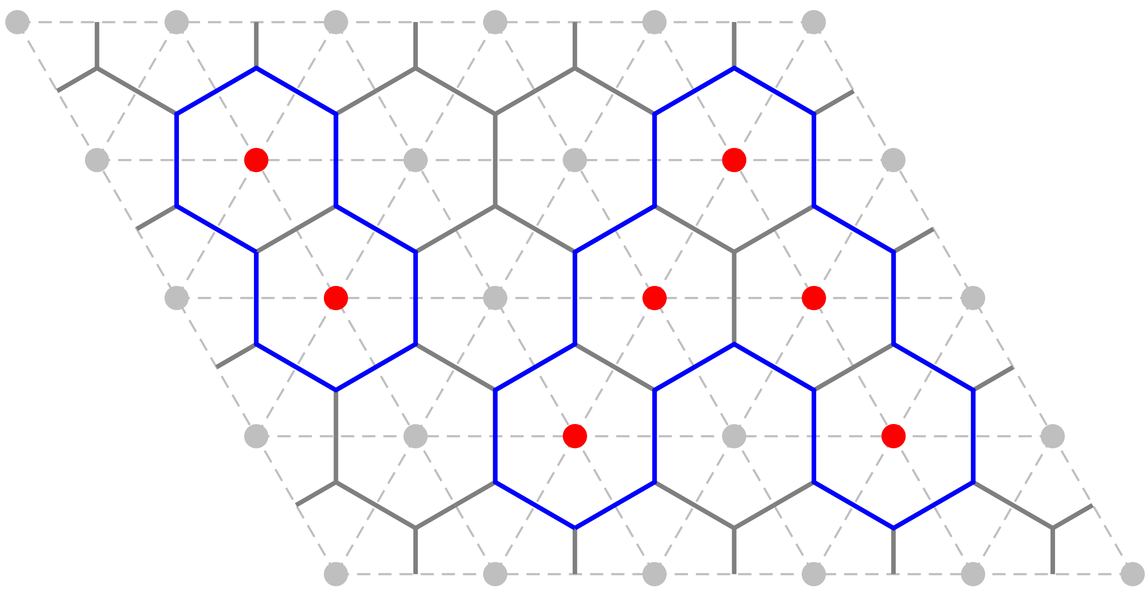
Honeycomb-lattice $O(N)$ loop model

- *$O(N)$ loop model*

$$\mathcal{Z}_{\text{loop}}(\lambda, N) = \int \prod_k d\,\vec{S}_k \prod_{\langle ij \rangle} (1 + \lambda N \vec{S}_i \cdot \vec{S}_j) \ ,$$

On Honeycomb lattice:

$$\mathcal{Z}_{\text{loop}}(\lambda, N) = \sum_{\{b_{ij}\}:\text{Eulerian}} \lambda^{\#\text{bonds}} N^{\#\text{loops}}$$



$\star$: critical line

$$\lambda_c(N) = \left(2 + \sqrt{2 - N}\right)^{-1/2}$$

$\star$: Universality: a *critical* $O(N)$ loop model corresponds with a *tricritical* N^2-state Potts model

- *Cluster Simulation*

Start from spin configuration $\{s\}$ on $\mathcal{T}$:

(1): Construct loops on $\mathcal{H}$ based on $\{s\}$: low-temperature graph of spin config.$\{s\}$.

(2): For each loop, set it to be ''active'' with probability $p = 1/N$, and to be ''inactive'' with $p = 1 - 1/N$.

(3): For each edge $\langle ij \rangle$, place bond with probability

$$p = \begin{cases} 1-\lambda & \text{if } s_i = s_j \\ 1 & \text{if } s_i \neq s_j \text{ and } \langle ij \rangle \text{ is inactive} \\ 0 & \text{if } s_i \neq s_j \text{ and } \langle ij \rangle \text{ is active} \end{cases}$$

(4) Identify clusters for spins s; randomly flip for each cluster all spins s with probability $1/2$.

$\star$ *Why does it work?*

- *N-color Ashkin-Teller (AT) model*

On triangular lattice $\mathcal{T}$:

$$\mathcal{H}_{\text{AT}}(J_2, J_4, N) = -J_2 \sum_{m=1}^{N} \sum_{\langle ij \rangle} \sigma_i^{(m)} \sigma_j^{(m)}$$

$$-J_4 \sum_{m>n} \sum_{\langle ij \rangle} \sigma_i^{(m)} \sigma_j^{(m)} \sigma_i^{(n)} \sigma_j^{(n)}$$

Let $J_2 \to \infty$, $J_4 \to -\infty$, but $J_2 + (N-1)J_4 = J$ held finite.

For edge $\langle ij \rangle$, let k denote the number of pairs of unequal Ising spins $\sigma_i^{(m)} \neq \sigma_j^{(m)}$, the statistical weight of $\langle ij \rangle$ is

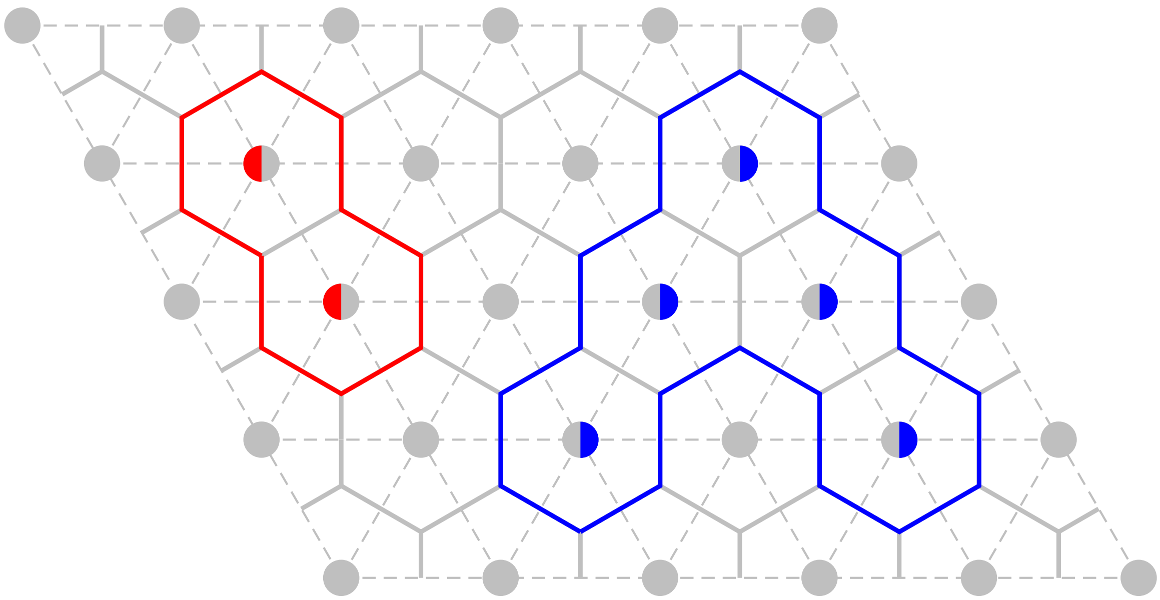
Configuration	Normalized weight
$k = 0$	1
$k = 1$	$e^{-2[J_2+(N-1)J_4]} \equiv e^{-2J}$
$k \geq 2$	$e^{-2kJ} \cdot e^{k(k-1)J_4} \to 0$

In words, an edge can be occupied by at most one pair of unequal Ising variable $\sigma_i^{(m)} \neq \sigma_j^{(m)}$.

Low-temperature expansion of $\mathcal{Z}_{\mathsf{AT}}$ leads to:

$$\mathcal{Z}_{\mathsf{AT}}(\lambda, N) = N^2 \sum_{\{b_{ij}\}:\text{Eulerian}} \lambda^{\#\text{bonds}} N^{\#\text{loops}}$$

$\star$: the $N = 2$ case at criticality is equivalent with the *critical* Baxter-Wu model with 3-spin interactions.



- *Cluster Simulation of the N-color AT model*

Effective Hamiltonian for variable $\sigma^{(m)}$:

$$\mathcal{H}_{\text{eff}}(J_2, J_4, N; \sigma^{(m)}) \; = \; -J_{\text{eff}} \sum_{\langle ij \rangle} \sigma_i^{(m)} \sigma_j^{(m)} \; ,$$

with

$$J_{\text{eff}} = J_2 + J_4 \sum_{n \neq m} \sigma_i^{(n)} \sigma_j^{(n)} \ .$$

$\sigma^{(m)}$ can be updated by an embedding SW method:

Start from spin config. $\sigma^{(m)}$ and $\sigma^{(n)}$ for $n \neq m$.

(1) Place bonds with probability

$$p = \begin{cases} 1 - e^{-2J} & \text{if } \sigma_i^{(m)} = \sigma_j^{(m)} \text{ and } \sigma_i^{(n)} = \sigma_j^{(n)} \\ 1 & \text{if } \sigma_i^{(m)} = \sigma_j^{(m)} \text{ and } \sigma_i^{(n)} \neq \sigma_j^{(n)} \\ 0 & \text{if } \sigma_i^{(m)} \neq \sigma_j^{(m)} \ . \end{cases}$$

(2) Identify clusters for spins $\sigma^{(m)}$.

(3) For each cluster, independently flip for each cluster all spins $\sigma^{(m)}$ with $1/2$.

Reformulation

Define Ising variable $s = \prod_m \sigma^{(\mathrm{m})}$. Keep colors of loops in computer memory $\{\text{loops}\}$.

Start from spin config. $\{s\}$ and loop config. $\{\texttt{loops}\}$

(1) Place bonds with probability

$$p = \begin{cases} 1 - e^{-2J} & \text{if } s_i = s_j \\ 1 & \text{if } s_i \neq s_j \text{ and } ij \text{ is of color } n \neq m \\ 0 & \text{if } s_i \neq s_j \text{ and } ij \text{ is of color } m \end{cases}$$

(2) Identify clusters for $\{s\}$. Randomly flip for each cluster all spins with probability $1/2$.

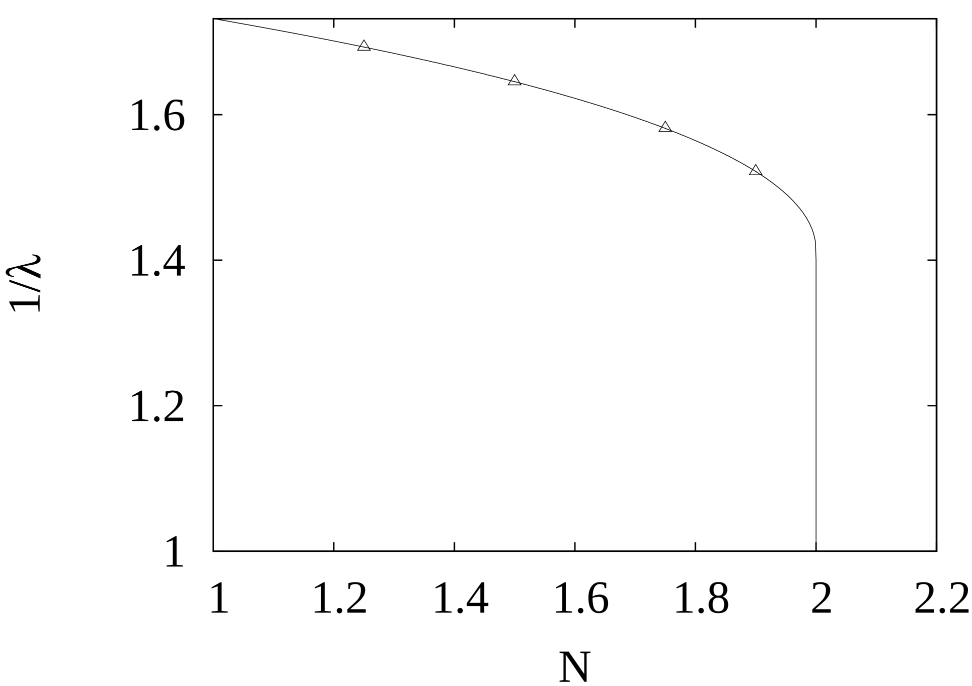
(3) Update loop information for color m (loops of color $n \neq m$ are frozen).

Static and Dynamic Critical Behavior

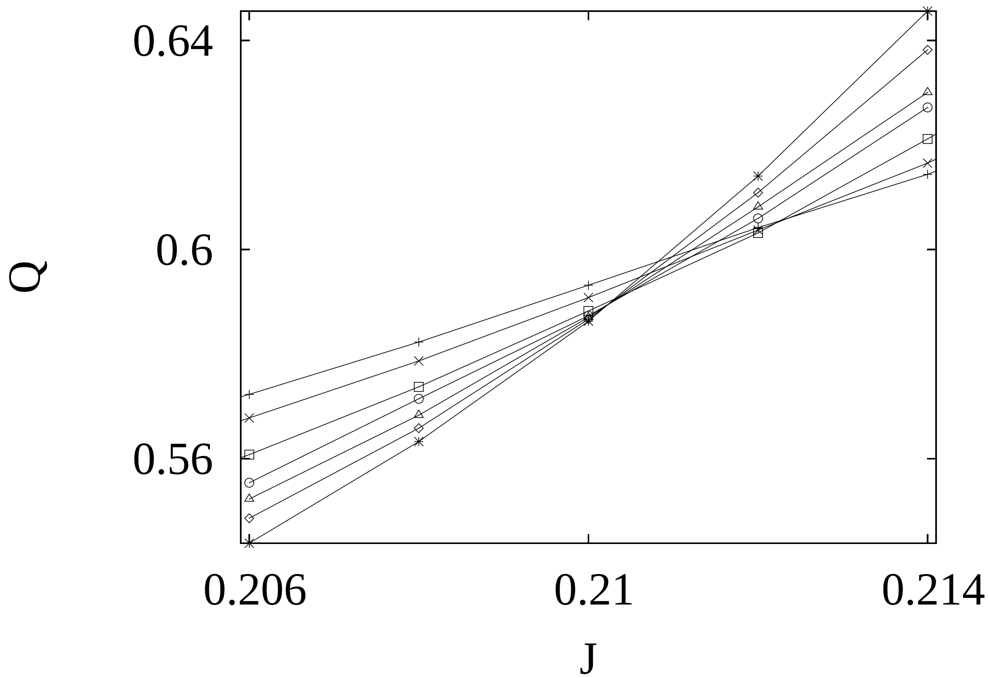
- *Sampled Quantities*

Specific heat C, susceptibility χ, and Binder ratio Q etc.

- *Critical Line*



$$\lambda_c(N) = \left(2 + \sqrt{2 - N}\right)^{-1/2}$$



$$\lambda_c(N = 1.9) = 1.521899 \Rightarrow J_c = 0.20998.$$

- *Critical Exponent*

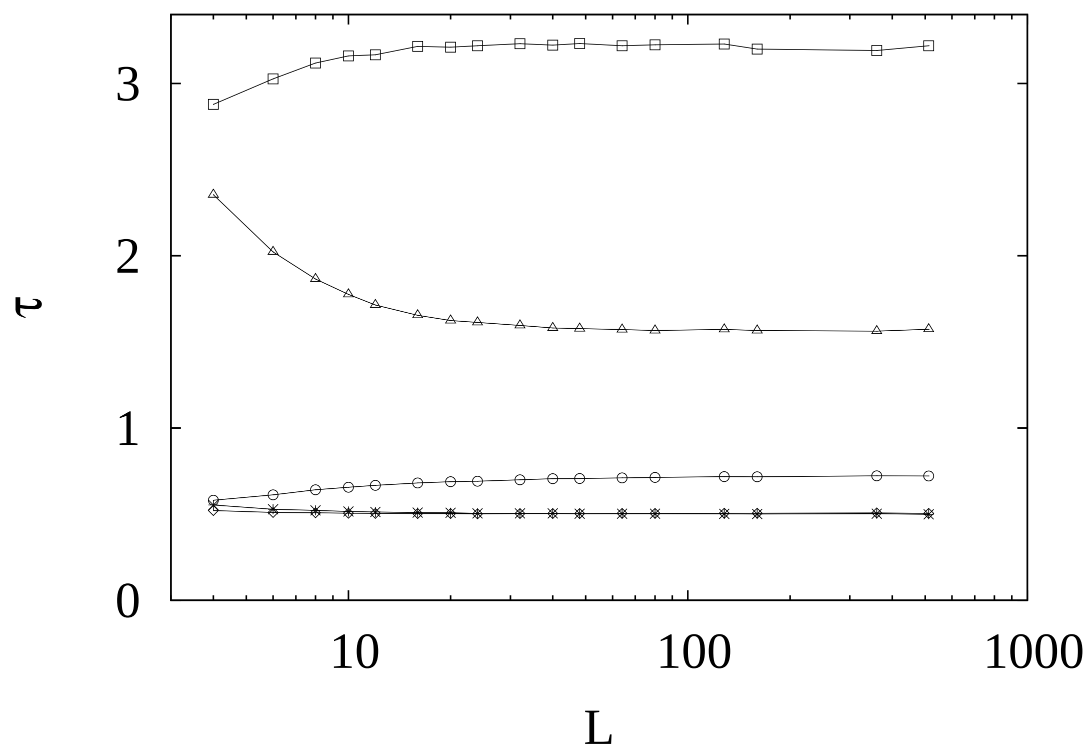
N		y_{t1}	y_{t2}	y_h
1.25	Exact	1.8327	0.8873	1.9343
	Num.	1.8327(2)	0.887(2)	1.9342(2)
1.50	Exact	1.7805	0.7481	1.9198
	Num.	1.7801(4)	0.745(4)	1.9199(2)
1.75	Exact	1.7078	0.5542	1.9034
	Num.	1.7079(4)	0.54(1)	1.9032(2)
2.00	Exact	1.5	0	1.875
	Num.	1.5002(4)	0.1(2)	1.8751(1)

- *Dynamic Critical Behavior*

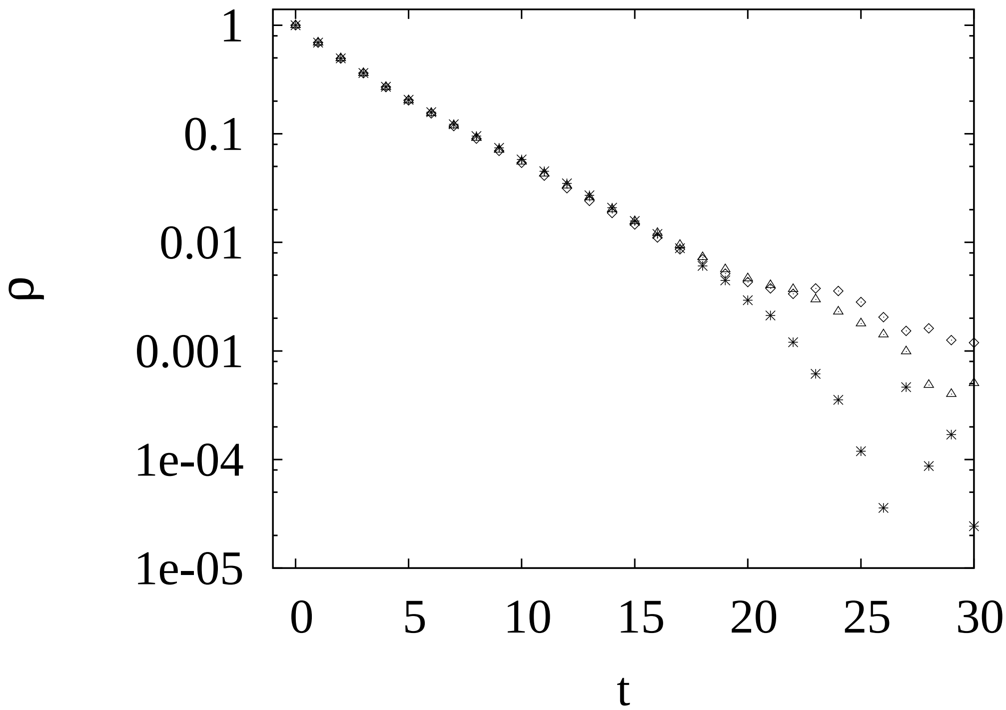
For quantity f, integrated autocorrelation time τ:

$$\tau_{int,f} = \frac{1}{2} \sum_{t=-\infty}^{\infty} \rho_f(t) = \frac{1}{2} + \sum_{t=1}^{\infty} \rho_f(t)$$

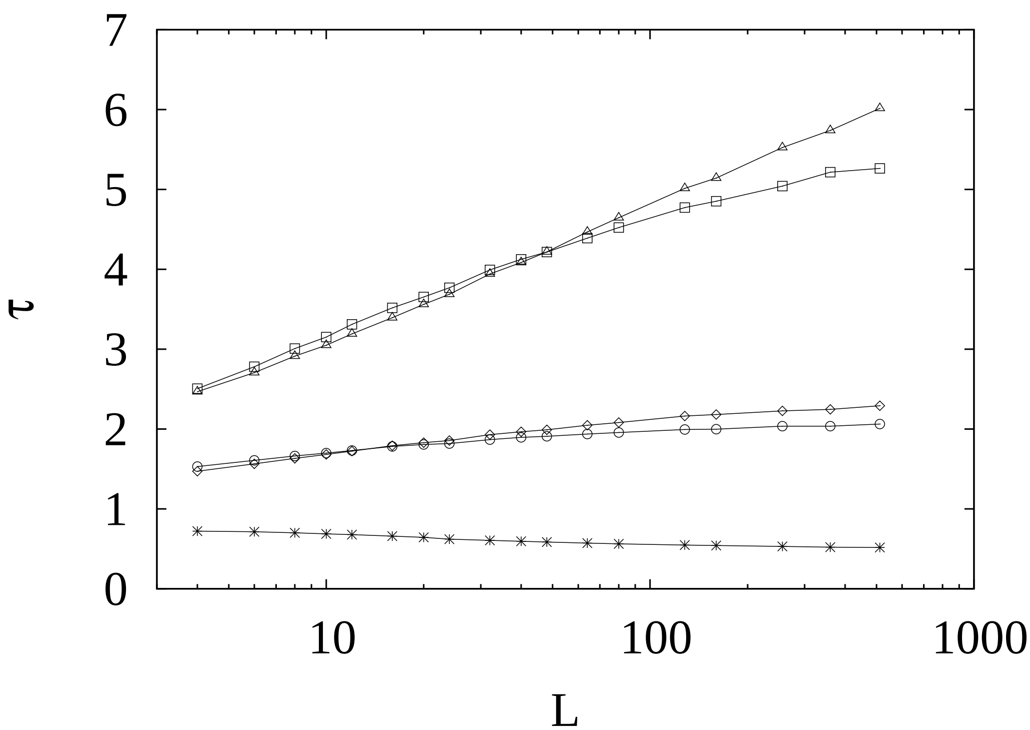
ρ_f: normalized autocorrelation function



$N = 2$



$N = 2$



$N = 1.25$

Critical slowing down hardly exits !

N-component Corner-Cubic Model

- *N-component corner-cubic model*

  $O(N)$ model:

$$\mathcal{Z}_{\text{loop}}(\lambda, N) = \int \prod_k d\,\vec{S}_k \prod_{\langle ij \rangle} (1 + \lambda N \vec{S}_i \cdot \vec{S}_j) \ ,$$

  Approximate $\vec{S}$ by corner-cubic spin:

$$\mathcal{Z}_{\text{CC}} = \sum_{\{\sigma^{(m)}\}} \prod_{\langle ij \rangle} [1 + \lambda(\sigma_i^{(1)} \sigma_j^{(1)} + \cdots \sigma_i^{(N)} \sigma_j^{(N)})]$$

$$= \underbrace{\sum \sum \cdots \sum}_{\text{Loop colors}} \lambda^{\#\text{bonds}}$$

- *N-color Ashkin-Teller (AT) model*

  On each site, put N independent copies of Ising spins $\sigma^{(m)}$. Let f be the number of pairs of unequal spins, the normalized weight is

Configuration	Normalized weight
$k = 0$	1
$k = 1$	λ
$k \geq 2$	0

In words, an edge can be occupied at most one pair of unequal Ising variable $\sigma_i^{(m)} \neq \sigma_j^{(m)}$.

Low-temperature expansion of $\mathcal{Z}_{\mathsf{AT}}$ leads to:

$$\mathcal{Z}_{\mathsf{AT}}(\lambda, N) \propto \underbrace{\sum\sum\cdots\sum}_{\text{Loop colors}} \lambda^{\#\text{bonds}}$$

- *Embedding SW method*

Start from spin config. $\sigma^{(m)}$ and $\sigma^{(n)}$ for $n \neq m$.

(1) Place bonds with probability

$$p = \begin{cases} 1-\lambda & \text{if } \sigma_i^{(m)} = \sigma_j^{(m)} \text{ and } \sigma_i^{(n)} = \sigma_j^{(n)} \\ 1 & \text{if } \sigma_i^{(m)} = \sigma_j^{(m)} \text{ and } \sigma_i^{(n)} \neq \sigma_j^{(n)} \\ 0 & \text{if } \sigma_i^{(m)} \neq \sigma_j^{(m)} . \end{cases}$$

(2) Construct clusters for $\{\sigma^{(m)}\}$; randomly flip for each cluster all spins with $1/2$.

- *Special Symmetries for $N = 2$*

  Ising domains: Areas enclosed by loops

  Given spin config. $\sigma^{(1)}$ and $\sigma^{(2)}$

  (1) Calculate $s = \sigma^{(1)} \cdot \sigma^{(2)}$, and Construct Ising domains for s.

  (2) For each domain, randomly set $\sigma^{(1)} = \pm 1$, let $\sigma^{(2)} = s \cdot \sigma^{(1)}$.

  ⋆ Random swapping between $\sigma^{(1)}$ and $\sigma^{(2)}$

- *Embedding SW method with Random Swapping* (integer $N \geq 2$)

  (1) Randomly choose a pair of spins $\sigma^{(m)}$ and $\sigma^{(n)}$, and carry out random swapping.

  (2) Update spins $\sigma^{(m)}$.

• *Embedding SW method with Random Swapping* (noninteger $N = k/2^l$)

Example for $N = 3/2$. On each site, put 3 independent spins $\sigma^{(1)}$, $\sigma^{(2)}$, and $\sigma^{(3)}$.

(1) `Randomly choose a pair of spins` $\sigma^{(i)}$ `and` $\sigma^{(j)}$`; the third copy of spin is` $\sigma^{(k)}$`. Calculate` $s = \sigma^{(i)} \cdot \sigma^{(j)}$.

(2) `Update spins` s `by the embedding SW method. Bond probability is:`

$$p = \begin{cases} 1-\lambda & \text{if } s_i = s_j \text{ and } \sigma_i^{(k)} = \sigma_j^{(k)} \\ 1 & \text{if } s = s_j \text{ and } \sigma_i^{(k)} \neq \sigma_j^{(k)} \\ 0 & \text{if } s_i \neq s_j \; . \end{cases}$$

(3) `For each Ising domain of` s, `randomly set` $\sigma^{(i)} = \pm 1$, `and set` $\sigma^{(j)} = s \cdot \sigma^{(j)}$.

⋆ For noninteger N, the corner-cubic model has not yet been well defined.

⋆ The embedding SW method with random swapping suffers little from critical slowing down on the square lattice for $N \leq 2$.

N-component Face-Cubic Model

- *Embedding SW method*

$O(N)$ model:

$$\mathcal{Z}_{\text{loop}}(\lambda, N) = \int \prod_k d\,\vec{S}_k \prod_{\langle ij \rangle} (1 + \lambda N \vec{S}_i \cdot \vec{S}_j) \; ,$$

Approximate $\vec{S}$ by face-cubic spin:

$$\begin{aligned} \mathcal{Z}_{\text{fc}} &= \sum_{\{\sigma^{(m)}\}} \prod_{\langle ij \rangle} [1 + \lambda(\sigma_i^{(1)} \sigma_j^{(1)} + \cdots \sigma_i^{(N)} \sigma_j^{(N)})] \\ &= \sum_{\{s,\tau\}} \prod_{\langle ij \rangle} (1 + \lambda s_i s_j \delta_{\tau_i, \tau_j}) \qquad (1) \end{aligned}$$

$\sigma^{(m)} = 0, \pm 1$, $s = \pm 1$, and $\tau = 1, 2, \cdots N$.

Expansion of Eq. (1) leads to

$$\mathcal{Z}_{\text{fc}} = \sum_{\text{Eulerian}} \lambda^{\#\text{bonds}} N^{\#\text{clusters}}$$

$$\propto \sum_{\text{Eulerian}} \lambda^{\#\text{bonds}} N^{\#\text{domains}}$$

- *SW-type simulation*

  Let loop config. on lattice $\mathcal{L}$ be the low-temperature graph of Ising spins s on the dual lattice $\mathcal{L}^*$.

  (1) Construct Ising domains for s. For each domain, let it be ‘‘active" with probability $p = 1/N$ and be ‘‘inactive" with $p = 1-1/N$.

  (2) Update spins s by SW method. Bond probability is

$$p = \begin{cases} 1-\lambda & s_i = s_j;\ \text{both } s_i, s_j \text{ are active} \\ 1 & s_i \text{ or } s_j \text{ is inactive} \\ 0 & s_i \neq s_j;\ \text{both } s_i, s_j \text{ are active}\,. \end{cases}$$

  ⋆ Simulation results for $N \leq 2$ agree with those by transfer-matrix techniques.

  ⋆ Suffers little from critical slowing down

Conclusions

- A set of cluster methods is developed for some loop models

- Little critical slowing down is observed for $1 < N \leq 2$.

Work to be done

- Directly prove the validity of the cluster methods

- Some versions of cluster methods may be improved

- Develop cluster method for $O(N)$ loop model on square lattice

- Explore new physics.