Further details of "Space of Ricci flows(II)"

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In this note, we provide further details for some properties claimed in Chen-Wang [16] without detailed proofs, following the sketch given by Chen-Wang [16]. No new idea is needed. We set $m = 2n$. Namely, m is the real dimension and n is the complex dimension.

1 Proof of Excess estimate

Theorem 1.1 (**Abresch-Gromoll type estimate**, Lemma 2.37 of [16])**.** *Suppose $x_0 \in X$, γ is a line segment centered at x_0 with length 2, end points p_+ and p_-. Let $e(x)$ be the excess function $d(x, p_+) + d(x, p_-) - 2$. Then we have*

$$\sup_{x \in B(x_0,\epsilon)} e(x) \leq C\epsilon^{\frac{2n}{2n-1}} \tag{1.1}$$

for each $\epsilon \in (0, 1)$ and some universal constant $C = C(n)$.

Theorem 1.1 originates from the excess estimate of Abresch-Gromoll(c.f. Proposition 2.3 of [1]). The main ingredient of the proof is an application of maximum principle for subharmonic functions, and the existence of a poled function $\underline{L}$ such that $\Delta \underline{L} \geq 1$. In our case, both maximum principle (Proposition 2.28 of [16]) and the existence of $\underline{L}$(Lemma 2.36 of [16]) are known. For the convenience of readers, we provide the full details in the following proof.

Proof. In order to prove (1.1), it suffices to prove the following inequality

$$\sup_{x \in B(x_0,\epsilon)\backslash B(x_0,0.5\epsilon)} e(x) \leq C\epsilon^{\frac{2n}{2n-1}}. \tag{1.2}$$

We shall focus on the proof of (1.2) in the following argument. Fix $y_0 \in B(x_0, \epsilon)\backslash B(x_0, 0.5\epsilon)$ and consider the function $f = 4(m-1)\underline{L}_\epsilon \circ d(\cdot, y_0) - e$.

Recall that the excess function $e(x)$ satisfies the following inequality

$$\Delta e(x) \leq (m-1)\left(\frac{1}{d(x, p_+)} + \frac{1}{d(x, p_-)}\right) \leq 4(m-1). \tag{1.3}$$

The function $\underline{L}_\epsilon$ is defined as(c.f. (2.67) of [16])

$$\underline{L}_\epsilon(\rho) = \frac{\rho^{2-m}\epsilon^m - \epsilon^2}{m(m-2)} + \frac{\rho^2 - \epsilon^2}{2m} \tag{1.4}$$

and satisfies

$$\Delta \underline{L}_\epsilon \circ d(\cdot, y_0) \geq 1. \tag{1.5}$$

By (1.3) and (1.5), it is clear that $\Delta f \geq 0$ on $B(y_0, \epsilon)\backslash\{y_0\}$. Let c be arbitrary number in $(0, d(y_0, x_0))$. Then $x_0 \in B(y_0, \epsilon)\backslash B(y_0, c)$. Note that $\underline{L}_\epsilon = 0$ on $\partial B(y_0, \epsilon)$. Therefore, $f < 0$ on $\partial B(y_0, \epsilon)$. However, $e(x_0) = 0$, which implies that $f(x_0) = 4(m-1)\underline{L}_\epsilon \circ d(x_0, y_0) > 0$. Consequently, the maximum principle for subharmonic function forces that $\sup_{z \in \partial B(y_0, c)} f(z) > 0$. In particular, there is a point $z_0 \in \partial B(y_0, c)$ such that $f(z_0) > 0$, which means that

$$e(z_0) < 4(m-1)\underline{L}_\epsilon(c).$$

Note that the excess function e is a Lipschitz function with Lipschitz constant 2. Then it follows that

$$e(y_0) < 4(m-1)\underline{L}_\epsilon(c) + 2d(z_0, y_0) = 4(m-1)\underline{L}_\epsilon(c) + 2c.$$

Since c can be arbitrary number in $(0, d(y_0, x_0))$ and $0.5\epsilon < d(y_0, x_0) < \epsilon$, we obtain

$$e(y_0) < \inf_{c \in (0, d(y_0, x_0))} \left\{4(m-1)\underline{L}_\epsilon(c) + 2c\right\} \leq C\epsilon^{\frac{m}{m-1}},$$

where we used the definition of $\underline{L}_\epsilon$ in (1.4). Clearly, (1.2) follows from the above inequality. The proof of the theorem is complete. □

2 Proof of parabolic approximation

Theorem 2.1 (**Parabolic approximation of local Buseman function**, Lemma 2.40 of [16])**.** *There exist two constants $c = c(n)$, $\bar{\epsilon} = \bar{\epsilon}(n)$ with the following properties.*

Suppose $x_0 \in X$, γ is a line segment whose center point locates in $B(x_0, 0.2\epsilon)$, with end points p_+ and p_-, with length 2. Let h_t be the heat approximation of b which is one of $b_\pm$. Suppose the excess value $d(x_0, p_+) + d(x_0, p_-) - 2 < \epsilon^2$ for some $\epsilon \in (0, \bar{\epsilon})$. Then there exists $\lambda \in [0.5, 2]$ such that

- $|h_{\lambda\epsilon^2} - b|(x_0) \leq c\epsilon^2$.
- $⨍_{B(x_0,\epsilon)} ||\nabla h_{\lambda\epsilon^2}|^2 - 1| \leq c\epsilon$.
- $\int_{0.1}^{1.9} ⨍_{B(\gamma(s),\epsilon)} ||\nabla h_{\lambda\epsilon^2}|^2 - 1| \leq c\epsilon^2$.

Most importantly, we have

$$\int_{0.1}^{1.9} ⨍_{B(\gamma(s),\epsilon)} |Hess_{h_{\lambda\epsilon^2}}|^2 \leq c.$$

In the smooth Riemannian manifold case, Theorem 2.1 is a simplified version of Theorem 2.19 of Colding-Naber [14]. The key ingredients are applications of heat kernel estimates and integration by parts, which both hold in our setting(c.f. Proposition 2.20, Proposition 2.17 and Lemma A.7. of [16]). We shall follow the proof of Colding-Naber verbatim in the following argument.

We first recall some estimates related to heat kernel. The following Lemma is implied by the proof of Proposition 2.20 of [16].

Lemma 2.2. *The heat kernel satisfies the following Gaussian estimate*

$$\frac{1}{C}\frac{1}{|B(x,\sqrt{t})|}e^{-\frac{d^2(x,y)}{\alpha t}} \leq p(t,x,y) \leq \frac{C}{|B(x,\sqrt{t})|}e^{-\frac{d^2(x,y)}{5t}} \tag{2.1}$$

for some positive constants $C = C(m)$, $\alpha = \alpha(m)$.

Proof. We basically apply the results of Sturm. Recall that Corollary 4.2 and Corollary 4.10 of Sturm's paper imply that

$$\frac{1}{C}\frac{1}{|B(x,\sqrt{t})|}e^{-\frac{Cd^2(x,y)}{t}} \leq \rho(x,y,t) \leq \frac{C}{|B(x,\sqrt{t})|^{\frac{1}{2}}}\cdot\frac{1}{|B(y,\sqrt{t})|^{\frac{1}{2}}}\cdot e^{-\frac{d^2(x,y)}{4t}}\cdot\left(1+\frac{d^2(x,y)}{t}\right)^n. \tag{2.2}$$

The number $n = \frac{m}{2}$ and m is the constant of volume doubling(c.f. the number N in Property (Ib) on page 276 of Sturm [28]), which coincides with the real dimension of X, due to the Bishop-Gromov volume comparison

$$\frac{|B(x,2r)|}{|B(x,r)|} \leq 2^m.$$

The coefficient C in (2.2) depends on m and C_P, the Poincaré constant. However, the Poincaré constant(c.f. Proposition 2.7 of [16] and Property (Ic) on page 278 of Sturm [28]) depends only on the dimension m. Therefore, the C appeared in (2.2) depends only on the dimension $m = 2n$.

We first use (2.2) to prove the upper bound in (2.1). The Bishop Gromov volume comparison implies that

$$t^{-n}|B(y,\sqrt{t})| \geq (\sqrt{t}+d(x,y))^{-2n}|B(y,d(x,y)+\sqrt{t})| \geq (\sqrt{t}+d(x,y))^{-2n}|B(x,\sqrt{t})|,$$

which means that

$$|B(y,\sqrt{t})| \geq \left(1+\frac{d(x,y)}{\sqrt{t}}\right)^{-2n}|B(x,\sqrt{t})|.$$

Then we have

$$\rho(x,y,t) \leq \frac{C}{|B(x,\sqrt{t})|}\cdot e^{-\frac{d^2(x,y)}{4t}}\cdot\left(1+\frac{d^2(x,y)}{t}\right)^n\cdot\left(1+\frac{d(x,y)}{\sqrt{t}}\right)^{2n} \leq \frac{C}{|B(x,\sqrt{t})|}\cdot e^{-\frac{d^2(x,y)}{5t}},$$

which is the upper bound in (2.1).

The Gaussian lower bound of (2.1) is a direct application of the lower bound in (2.2). □

Lemma 2.3. *Suppose u is a smooth nonnegative function supported on $\Omega \times [0, r^2]$ for some bounded set Ω. Suppose*

$$(\partial_t - \Delta)u \geq -c_0$$

in the distribution sense. Then

$$\fint_{B(x_0,r)} u_0 \leq c(m)\left\{u_{r^2}(x_0) + c_0 r^2\right\}. \tag{2.3}$$

Proof. It follows from integration by parts and the δ-function property of the heat kernel(c.f. Proposition 2.3 of [16]) that

$$u(r^2, x_0) - \int_X u(0, x)p(r^2, x, x_0)d\mu_x = \int_0^{r^2} \int_X \left\{\left(\frac{\partial}{\partial t} - \Delta\right)u(t, x)\right\} p(s, x, x_0)d\mu_x ds \geq -c_0 r^2,$$

which is equivalent to

$$\int_X u_0(x)p(r^2, x, x_0)d\mu_x \leq u_{r^2}(x_0) + c_0 r^2.$$

Since $u \geq 0$, we can apply the Gaussian lower bound in Lemma 2.2 to obtain that

$$\begin{aligned}\int_X u_0(x)p(r^2, x, x_0)d\mu_x &\geq \int_{B(x_0,r)} u_0(x)p(r^2, x, y_0)dv_x \geq \frac{1}{C}\cdot\frac{1}{|B(x_0, r)|}\cdot\int_{B(x_0,r)} u_0(x)dv_x \\ &\geq \frac{1}{C}\fint u_0(x)dv_x\end{aligned}$$

for some $C = C(m)$. Then (2.3) follows from the combination of the previous two steps. □

Remark 2.4. *Lemma 2.2 and Lemma 2.3 are only needed when we need to show the constants $\bar{\epsilon}, c$ in Theorem 2.1 depends only on n. If we allow $\bar{\epsilon}, c$ to depend on n and κ(which is enough for our remaining argument, as we are dealing with κ-non-collapsing spaces), then these two lemmas are not necessary. By further work(c.f. e.g. Theorem 5.6.3 of [20]), the α in (2.1) can be improved to be very close to 4. However, such precise improvement is not needed in our application.*

Proof of Theorem 2.1. Let us recall the construction of h. For each $r_1 << 1$ and $r_2 > 4$, we define

$$X_{r_1,r_2} = B(x_0, r_2)\backslash\left\{B(p_+, r_1) \cup B(p_-, r_1)\right\}. \tag{2.4}$$

Let ψ be a cutoff function supported on $X_{\frac{\delta}{4},16}$, and constantly equal to 1 on $X_{\frac{\delta}{2},8}$ with

$$\delta|\nabla\psi| + \delta^2|\Delta\psi| \leq C. \tag{2.5}$$

Here δ is a small but fixed positive number, say $\delta = 0.01$. We say a few words about the existence of ψ, which is guaranteed by Lemma 2.38 of [16]. Actually, we can find a cutoff function ϕ_+ which is equivalent to 1 on $B(p_+, 0.01)$ and vanishes outside $B(p_-, 0.02)$ with $|\nabla\phi_+| + |\Delta\phi_+|$ uniformly bounded. Similar, we can define ϕ_-. Also, we set ϕ_0 be a cutoff function supported on $B(x_0, 8)$

and vanishes outside $B(x_0, 16)$ with bounded value of $|\nabla\phi_0| + |\Delta\phi_0|$. Let ψ be $(1-\phi_+)\cdot(1-\phi_-)\cdot\phi_0$. It is easy to check that ψ satisfies all the requirements.

Recall that $h_{t,\pm}$ solve the heat equation with initial data $h_{0,\pm} = \psi b_\pm$, where $b_\pm$ are local Buseman function defined as(c.f. Lemma 2.39 of [16])

$$b_+(x) = d(x, p_+) - d(\gamma(0), p_+), \quad b_-(x) = d(x, p_-) - d(\gamma(0), p_-).$$

The function $e_t(x)$ is the heat solution starting from $e_0 = h_{0,+} + h_{0,-} = (b_{0,+} + b_{0,-})\psi$. Let b be one of $b_\pm$. Then it follows from (2.5) and the Laplacian comparison that

$$\Delta h_0 = b\Delta\psi + 2\langle\nabla\psi, \nabla b\rangle + \psi\Delta b \le C, \tag{2.6}$$

$$\Delta e_0 = \Delta h_{0,+} + \Delta h_{0,-} \le C, \tag{2.7}$$

on $X_{\frac{\delta}{4},16}$. Let h or h_t denote one of $h_{t,\pm}$ when convenient.

In order to finish the proof of this theorem, it suffices to prove the following inequalities.

$$|h_{\epsilon^2} - b|(x_0) \le c\epsilon^2; \tag{2.8}$$

$$\fint_{B(x_0,\epsilon)} ||\nabla h_{\epsilon^2}|^2 - 1| \le c\epsilon; \tag{2.9}$$

$$\int_{0.1}^{1.9} \fint_{B(\gamma(s),\epsilon)} ||\nabla h_{\epsilon^2}|^2 - 1| \le c\epsilon^2; \tag{2.10}$$

$$\int_{0.1}^{1.9} \fint_{B(\gamma(s),\epsilon)} |Hess_{h_{\lambda\epsilon^2}}|^2 \le c, \quad \text{for some } \lambda \in [0.5, 2]. \tag{2.11}$$

We shall prove (2.8)-(2.11) step by step. In each step, we shall focus on the proof of one inequality. Since each singular point can be approximated by a sequence of smooth points, we can assume x_0 be a regular point without loss of generality.

Step 1. Proof of (2.8).

For each $z \in X_{\frac{\delta}{4},16} \cap \mathcal{R}$, we have

$$\begin{aligned}\Delta_z h(t,z) &= \int_{\mathcal{R}} h(0,y)\Delta_z p(t,z,y)dv_y = \int_{\mathcal{R}} h(0,y)\dot{p}(t,z,y)dv_y = \int_{\mathcal{R}} h(0,y)\Delta_y p(t,z,y)dv_y \\ &= \int_{\mathcal{R}} p(t,z,y)\Delta_y h(0,y)dv_y \le C\end{aligned}$$

where we used (2.6) in the last step. Consequently, on $X_{\frac{\delta}{2},8}$, we have

$$\frac{\partial}{\partial t}(h_{t,-}(x) - b_-(x)) = \frac{\partial}{\partial t}h_{t,-}(x) = \Delta h_{t,-}(x) \le C, \quad \frac{\partial}{\partial t}(h_{t,+}(x) - b_+(x)) = \Delta h_{t,+}(x) \le C.$$

It follows from the previous two steps that

$$h_{t,-} - b_- \le Ct, \quad h_{t,+} - b_+ \le Ct \tag{2.12}$$

on $X_{\frac{\delta}{2},8} \cap \mathcal{R}$ for each $t > 0$. Recall that $e_t = h_{t,+} + h_{t,-} \ge 0$ for each t. Therefore, for each $t \in [0, 4\epsilon^2]$, we have

$$-C\epsilon^2 \le -e_0(x_0) \le (h_{t,+} - h_{0,+})(x_0) + (h_{t,-} - h_{0,-})(x_0) = e_t(x_0) - e_0(x_0) \le Ct \le C\epsilon^2. \tag{2.13}$$

Then (2.8) follows from the combination of (2.12) and (2.13).

Step 2. Proof of (2.9).

Note that on $[0,\infty)\times\mathcal{R}$, we have

$$(\partial_t-\Delta)|\nabla h|=\frac{-|\nabla\nabla h|^2+|\nabla|\nabla h||^2}{|\nabla h|}\leq 0.$$

Using the maximum principle for heat sub-equation(c.f. Lemma A.4 of [16], the high codimension of $\mathcal{S}$ enables that we only need to study the regular part) to obtain for all $x\in X_{\frac{\delta}{2},4}\cap\mathcal{R}$ and $t\leq 4\epsilon^2$, that

$$|\nabla h_t|(x)\leq\int_{X_{\frac{\delta}{16},16}}\rho(x,y,t)|\nabla h_0|(y)\leq\int_{X_{\frac{\delta}{4},8}}\rho(x,y,t)+c(m)\int_{X_{\frac{\delta}{16},16}\backslash X_{\frac{\delta}{4},8}}\rho(x,y,t).$$

All the integral actually happens on the corresponding regular part, we omitted "$\cap\mathcal{R}$" for simplicity of notations. Note that we used the fact that $|\nabla h_0|$ vanishes outside $X_{\frac{\delta}{16},16}$, is constantly 1 on $X_{\frac{\delta}{4},8}$, and is bounded above by $c(m)$ in between. Using the Stochastically completeness(c.f. Proposition 2.20 of [16]), we then obtain

$$\begin{aligned}|\nabla h_t|(x)&\leq 1-\int_{\left(X_{\frac{\delta}{4},8}\right)^c}\rho(x,y,t)+c(m)\int_{X_{\frac{\delta}{16},16}\backslash X_{\frac{\delta}{4},8}}\rho(x,y,t)\\&\leq 1+c(m)\int_{X\backslash B(x,0.1\delta)}\rho(x,y,t)dv_y.\end{aligned}$$

Suppose $t<\frac{\delta^2}{10000}$, then we can apply the heat kernel estimate to obtain

$$\begin{aligned}|\nabla h_t|(x)&\leq 1+\frac{C}{|B(x_0,\sqrt{t})|}\int_{X\backslash B(x,0.1\delta)}e^{-\frac{d^2(x,y)}{5t}}dv_y\\&\leq 1+\frac{C}{|B(x_0,\sqrt{t})|}\sum_{k=1}^{\infty}\int_{B(x,\frac{(k+1)\delta}{10})\backslash B(x,\frac{k\delta}{10})}e^{-\frac{d^2(x,y)}{5t}}dv_y\\&\leq 1+Ct^{-n}\sum_{k=1}^{\infty}(k\delta)^{2n}\cdot e^{-\frac{k^2\delta^2}{100t}}\leq 1+Ct,\end{aligned}$$

where we have used the Bishop-Gromov volume comparison in the above deduction. Recall δ is chosen as 0.01. Therefore, for each $x\in X_{\frac{\delta}{2},4}\cap\mathcal{R}$ and $t\leq 4\epsilon^2\leq 4\bar{\epsilon}^2(m)<<\delta^2$, we have

$$|\nabla h_t|(x)\leq 1+ct, \tag{2.14}$$

for some $c=c(m)$.

The inequality (2.14) provides the point-wise upper bound of $|\nabla h_t|$. In order to obtain the integral lower bounds, we consider the quantity $w_t=1+ct-|\nabla h_t|^2$, with $c=c(m,\delta)>0$ so small that $w_t\geq 0$ on $X_{\frac{\delta}{2},4}\cap\mathcal{R}$ for any $t\in[0,4\epsilon^2]$. Note that $|w_t|$ is bounded and

$$(\partial_t-\Delta)w_t=c+2|Hess_{h_t}|^2\geq 0.$$

Let φ be a cutoff function supported on $X_{\frac{\delta}{2},4}$ and equals 1 on $X_{\delta,2}$. Then we have

$$
\begin{aligned}
(\partial_t - \Delta)\left\{\varphi^2 w_t\right\} &= \varphi^2(\partial_t - \Delta)w_t + w_t(\partial_t - \Delta)\varphi^2 - 2\langle\nabla\varphi^2, \nabla w_t\rangle \\
&= \varphi^2(c + 2|\nabla\nabla h_t|^2) - 2w_t(\varphi\Delta\varphi + |\nabla\varphi|^2) + 4\varphi\left\langle\nabla\varphi, \nabla|\nabla h_t|^2\right\rangle \\
&= \left\{c\varphi^2 - 2w_t(\varphi\Delta\varphi + |\nabla\varphi|^2)\right\} + 2\varphi^2|Hess_{h_t}|^2 + 8\varphi Hess_{h_t}(\nabla\varphi, \nabla h_t) \\
&\geq \left\{c\varphi^2 - 2w_t(\varphi\Delta\varphi + |\nabla\varphi|^2)\right\} - 8|\nabla\varphi|^2|\nabla h_t|^2.
\end{aligned}
$$

In particular, this means that

$$
(\partial_t - \Delta)\left\{\varphi^2 w_t\right\} \geq -C, \quad \text{on } X_{\frac{\delta}{2},4}.
$$

Then we can apply Lemma 2.3 to obtain that

$$
⨍_{B(y,\sqrt{t})} w_t = ⨍_{B(y,\sqrt{t})} \varphi^2 w_t \leq C\left\{\inf_{B(y,\sqrt{t})} w_{2t} + t\right\} \leq C\left\{w_{2t}(y) + t\right\} \tag{2.15}
$$

for each $t \in [0, 4\epsilon^2]$ and every $y \in X_{\delta,2}$.

The purpose of the following paragraph is to derive an average estimate of $w_{2\epsilon^2}$ along geodesic segment. Fix $0.1 < s_1 < s_2 < 1.9$ and integrate $w_{2\epsilon^2}$ along the geodesic γ, we have

$$
\begin{aligned}
\int_{s_1}^{s_2} w_{2\epsilon^2} &= \int_{s_1}^{s_2}(1 + cs - |\nabla h_{2\epsilon^2}|^2)(\gamma(s))\mathrm{d}s \leq (1 + c\epsilon^2)|s_2 - s_1| - \frac{1}{|s_2 - s_1|}\left(\int_{s_1}^{s_2} \nabla_{\dot\gamma} h_{2\epsilon^2}\mathrm{d}s\right)^2 \\
&= (1 + c\epsilon^2)|s_2 - s_1| - \frac{1}{|s_2 - s_1|}\left|h_{2\epsilon^2}(\gamma(s_2)) - h_{2\epsilon^2}(\gamma(s_1))\right|^2.
\end{aligned} \tag{2.16}
$$

Note that $e(\gamma(s)) \equiv 0$ whenever $s \in [0.1, 1.9]$. Therefore, we can apply (2.8) on each $\gamma(s)$ and obtain that

$$
\left||h_{2\epsilon^2}(\gamma(s_2)) - h_{2\epsilon^2}(\gamma(s_1))| - |s_2 - s_1|\right| < c\epsilon^2, \tag{2.17}
$$

which implies that

$$
\left|h_{2\epsilon^2}(\gamma(s_2)) - h_{2\epsilon^2}(\gamma(s_1))\right|^2 \geq |s_2 - s_1|^2 - c|s_2 - s_1|\epsilon^2.
$$

Plugging the above inequality into (2.16) implies that

$$
⨍_{s_1}^{s_2} w_{2\epsilon^2} ds \leq \frac{c\epsilon^2}{|s_2 - s_1|} \tag{2.18}
$$

We then apply (2.18) on a particular short geodesic segment around x_0. Recall that $B(x_0, 0.2\epsilon) \cap \gamma \neq \emptyset$. Without loss of generality, let $\gamma(\theta) \in B(x_0, 0.2\epsilon)$. Let $s_1 = \theta - 0.1\epsilon$ and $s_2 = \theta + 0.1\epsilon$. Then (2.18) and the mean value theorem imply that there must exist some $s_0 \in [\theta - 0.1\epsilon, \theta + 0.1\epsilon]$ such that

$$
w_{2\epsilon^2}(\gamma(s_0)) < c\epsilon.
$$

Furthermore, the triangle inequality implies that $\gamma(s_0) \in B(x_0, \epsilon)$, which yields that

$$\fint_{B(x_0,\epsilon)} w_{\epsilon^2} \le c\left\{\inf_{B(x_0,\epsilon)} w_{2\epsilon^2} + \epsilon^2\right\} < c\left\{w_{2\epsilon^2}(s_0) + \epsilon^2\right\} < c\epsilon.$$

Recall that $w_t = 1 + ct - |\nabla h_t|^2$ and w_t is point-wisely bounded from below by $-c\epsilon^2$, due to (2.14). Therefore, (2.9) follows from the above inequality.

Step 3. Proof of (2.10).

In (2.18), denoting $s_1 = 0.1$ and $s_2 = 1.9$, we obtain

$$\int_{0.1}^{1.9} w_{2\epsilon^2}(\gamma(s))ds \le c\epsilon^2.$$

Combining the above inequality with the second inequality of (2.15), we arrive at

$$\int_{0.1}^{1.9} \fint_{B(\gamma(s),\epsilon)} 1 - |\nabla h_{\epsilon^2}|^2 \le \int_{0.1}^{1.9} \fint_{B(\gamma(s),\epsilon)} w_{\epsilon^2} \le c\int_{0.1}^{1.9} (w_{2\epsilon^2}(\gamma(s)) + \epsilon^2) \le c\epsilon^2,$$

which implies (2.10) by the point-wise upper bound of $|\nabla h_{\epsilon^2}|^2$ in (2.14).

Step 4. Proof of (2.11).

The proof of Hessian bound is an application of Bochner formula, together with the integral gradient bounds and the use of a good cutoff function on space-time, in a similar way as was done for harmonic approximation. Now for any fixed s define a cutoff function ϕ_s supported on $B(\gamma(s), 2\epsilon)$, such that $\phi_s \equiv 1$ on $B(\gamma(s), \epsilon)$ and have the bounds $\epsilon|\nabla\phi_s| < c$, $\epsilon^2|\Delta\phi_s| < c$. Moreover, let $a(t)$ be a smooth cutoff function defined on $[\frac{1}{4}\epsilon^2, 4\epsilon^2]$, $a \equiv 1$ on $[\frac{1}{2}\epsilon^2, 2\epsilon^2]$ and $\epsilon^2|a'| < c$. Then

$$\begin{aligned} 2\int_{B(\gamma(s),\epsilon)} a(t)|Hess_{h_t}|^2 &\le 2\int_{B(\gamma(s),2\epsilon)} a(t)\phi_s|Hess_{h_t}|^2 = \int_{B(\gamma(s),2\epsilon)} a(t)\phi_s(\Delta - \partial_t)\left(|\nabla h_t|^2 - 1|\right) \\ &= \int_{B(\gamma(s),2\epsilon)} a(t)\left\{|\nabla h_t|^2 - 1\right\}\Delta\phi_s - \int_{B(\gamma(s),2\epsilon)} a(t)\phi_s\partial_t\left(|\nabla h_t|^2 - 1\right) \\ &\le \int_{B(\gamma(s),2\epsilon)} c\epsilon^{-2}\left||\nabla h_t|^2 - 1\right| - \int_{B(\gamma(s),2\epsilon)} a(t)\phi_s\partial_t\left(|\nabla h_t|^2 - 1\right). \end{aligned}$$

Integration by parts in time gives

$$\left|\int_{\frac{1}{4}\epsilon^2}^{4\epsilon^2}\left(\int_{B(\gamma(s),2\epsilon)} a(t)\phi_s\partial_t\left(|\nabla h_t|^2 - 1\right)\right)\mathrm{d}t\right| \le c\int_{B(\gamma(s),2\epsilon)}\left||\nabla h_t|^2 - 1\right|.$$

Thus for any $s \in [0.1, 1.9]$ we have

$$\int_{\frac{1}{2}\epsilon^2}^{2\epsilon^2}\left(\fint_{B(\gamma(s),\epsilon)} |Hess_{h_t}|^2\right)\mathrm{d}t \le c\epsilon^{-2}\int_{\frac{1}{4}\epsilon^2}^{4\epsilon^2}\left(\fint_{B(\gamma(s),2\epsilon)}\left||\nabla h_t|^2 - 1\right|\right)\mathrm{d}t.$$

Integrating s over $[0.1, 1.9]$ gives

$$\int_{\frac{1}{2}\epsilon^2}^{2\epsilon^2}\left(\int_{0.1}^{1.9}\fint_{B(\gamma(s),\epsilon)} |Hess_{h_t}|^2\right)\mathrm{d}t \le c\epsilon^{-2}\int_{\frac{1}{4}\epsilon^2}^{4\epsilon^2}\left(\int_{0.1}^{1.9}\fint_{B(\gamma(s),2\epsilon)}\left||\nabla h_t|^2 - 1\right|\right)\mathrm{d}t \le c\epsilon^2,$$

where we used (2.10) in the last inequality. Clearly, (2.11) follows from the above inequality and the mean value theorem of Calculus. □

3 Slicing property

Theorem 3.1 (**Approximation slices**, Lemma 2.41 of [16]). *Suppose $x_0 \in X$, the pointed-Gromov-Hausdorff distance between (X, x_0) and $(Y \times \mathbb{R}^k, (\hat{y}, 0))$ is less than $\psi(L^{-1})$ for some metric space Y. Suppose $\gamma_1, \gamma_2, \cdots, \gamma_k$ are k line segments with length $2L >> 2$ such that the center point of γ_i locates in $B(x_0, 1)$ for each i. Furthermore, the Gromov-Hausdorff distance between $\gamma_1 \cup \gamma_2 \cdots \cup \gamma_k$ and $\tilde{\gamma}_1 \cup \tilde{\gamma}_2 \cup \cdots \tilde{\gamma}_k$ is bounded by $\psi(L^{-1})$, where $\tilde{\gamma}_i$ is the line segment on the i-th coordinate axis of $\mathbb{R}^k$, centered at the origin and with length $2L$, ψ is a nonnegative monotonically increasing function satisfying $\psi(0) = 0$. Suppose the end points of γ_i are $p_{i,+}$ and $p_{i,-}$. Let $b_{i,\pm}$ be the corresponding local Buseman functions with respect to γ_i. Let u_i be the harmonic function on $B(x_i, 4)$ with the same value as $b_{i,\pm}$ on $\partial B(x_0, 4)$. Then we have*

$$\int_{B(x_0,1)} \left\{ \sum_{1 \le i \le k} |\nabla u_i - 1|^2 + \sum_{1 \le i < j \le k} |\langle \nabla u_i, \nabla u_j \rangle| + \sum_{1 \le i \le k} |Hess_{u_i}|^2 \right\} \le \bar{\psi}(L^{-1}),$$

where $\bar{\psi}$ is also a nonnegative monotonically increasing function satisfying $\bar{\psi}(0) = 0$, depending on ψ.

In the smooth setting, Theorem 3.1 first appeared as inequality (2.6) on page 884 of Cheeger-Colding-Tian [10], whose proof is referred to Theorem 6.62 of Cheeger-Colding [8] and Section 9 of Cheeger [6]. However, Theorem 6.62 of Cheeger-Colding explicitly only deals with the case $k = 1$. Section 9 of Cheeger [6] only states(c.f. (9.30) on page 44 of Cheeger [6]) the inequality for the case $k = n$ without a proof. Therefore, for general $1 < k < n$, the proof of the smooth version of Theorem 3.1 seems not available in literature, although the basic idea is well-known to experts. We shall provide the full details to treat the general case of k in our singular setting, following the basic idea of Colding [13]. There are some extra difficulty caused by the existence of singularity. However, due to the high codimension of $\mathcal{S}$, such difficulty can be overcome by a standard method, which is used repeatedly in our paper(c.f. the proof of Lemma 2.31 of [16]).

Proof. In order to prove Theorem 3.1, it suffices to prove the following three inequalities for each i and $\{i, j\}$ with $i < j$.

$$\int_{B(x_0,1)} ||\nabla u_i| - 1|^2 \le \bar{\psi}(L^{-1}), \tag{3.1}$$

$$\int_{B(x_0,1)} |Hess_{u_i}|^2 \le \bar{\psi}(L^{-1}), \tag{3.2}$$

$$\int_{B(x_0,1)} |\langle \nabla u_i, \nabla u_j \rangle| \le \bar{\psi}(L^{-1}). \tag{3.3}$$

We shall prove them term by term. Since $|\nabla b_i| = 1$, we have $||\nabla u_i| - 1| \le |\nabla u_i - \nabla b_i|$. It follows from the second inequality of Lemma 2.39 in [16] that

$$\int_{B(x_0,1)} ||\nabla u_i| - 1|^2 \le \int_{B(x_0,1)} |\nabla u_i - \nabla b_i|^2 \le cL^{-\alpha} \le \bar{\psi}(L^{-1}), \tag{3.4}$$

whence we obtain (3.1). Similarly, because of the existence of line segment γ_i, up to rescaling, it follows from the third inequality of Lemma 2.39 in [16] that

$$\int_{B(x_0,1)} |Hess_{u_i}|^2 \leq cL^{-\alpha} \leq \bar{\psi}(L^{-1}),$$

which is exactly (3.2).

In the following discussion, we focus on the proof of (3.3). We shall follow the argument of Colding [13] and Cheeger [6], with slight modification to deal with the extra trouble caused by the existence of the singularities. For the convenience of readers, we provide compete details, even more than the original papers of Cheeger and Colding(For example the Step 2 in the following argument). Basically, the proof is divided into 5 steps. The extra effort for the singularities only appear in Step 1, as we shall see below. For simplicity of notation, we choose $i = 1, j = 2$.

Step 1. For each continuous function f which is smooth on $\mathcal{R} \cap B(x_0, 2)$ and has bounded $|\nabla f|$, we have

$$\frac{1}{|SB(x_0,1)|}\int_{SB(x_0,1)} |\langle \nabla f, v\rangle - (f(\gamma_v(1)) - f(\gamma_v(0)))| < 2^m ⨍_{B(x_0,2)} |Hess_f|. \tag{3.5}$$

In particular, let u be one of u_1 or u_2, the harmonic approximation functions of b_1 and b_2, then we have

$$\frac{1}{|SB(x_0,1)|}\int_{SB(x_0,1)} |\langle \nabla u, v\rangle - (u(\gamma_v(1)) - u(\gamma_v(0)))| < CL^{-\frac{\alpha}{2}} \tag{3.6}$$

for some $C = C(m)$.

Notice that we use $SB(x_0, 1)$ to denote the unit sphere bundle over $B(x_0, 1) \cap \mathcal{R}$, for simplicity of notations. We also need to make sense of the integral on both sides of (3.5). The right hand side integral actually happens on $B(x_0, 2) \cap \mathcal{R} = B(x_0, 2)\backslash\mathcal{S}$, which is a full measure subset of $B(x_0, 2)$. Since the subset $B(x_0, 2) \cap \mathcal{S}$, where $|Hess_f|$ is not defined, is only a measure-zero set, we abuse notation by using $\int_{B(x_0,2)} |Hess_f|$ to denote $\int_{B(x_0,2)\backslash\mathcal{S}} |Hess_f|$. The situation of the left hand side of (3.5) is similar. Note that $\gamma_v(1)$ may not be defined since it is possible that $\gamma_v(t) \in \mathcal{S}$ for some $t < 1$ even if $v \in T_x(X)$ for some regular point $x \in B(x_0, 1)\backslash\mathcal{S}$. Then the geodesic cannot proceed beyond t. We call v to be exceptional if $\gamma_v(t) \in \mathcal{S}$ for some $t \in [0, 1]$. We collect all such exceptional v's together and call the collection as the exceptional set, denoted by E. In general, $E \neq \emptyset$. However, it is not hard to see that E is a measure zero subset of $SB(x_0, 1)$.

Actually, due to the high codimension of $\mathcal{S}$, following similar argument of the proof of Lemma 2.31 of [16], for each small number ξ, we can find a smooth hyper surface Σ_ξ (c.f. Claim 2.32 of [16]) such that

$$|B(x_0, 10) \cap \Sigma_\xi| \leq C\xi; \qquad \frac{1}{C}\xi < d(x, \mathcal{S}) < C\xi, \quad \forall\, x \in \Sigma_\xi \cap B(x_0, 10).$$

Here C may depends on x_0. Let E_ξ be the subset of $S(B(x_0, 1) \cap \mathcal{R})$ such that $\gamma_v(t) \in \Sigma_\xi$ for some $t \in [0, 2]$. Then E_ξ can be regarded as a bundle over the $S(\Sigma_\xi \cap B(x_0, 2))$, the collection of $v \in T_yX$

such that $y \in \Sigma_\xi \cap B(x_0, 2)$ and $v \in T_y\Sigma_\xi$. We equip $\left\{\Sigma_\xi \cap B(x_0, 2)\right\} \times [0, 2]$ with the obvious product measure and define a map φ from E_ξ to $\left\{\Sigma_\xi \cap B(x_0, 2)\right\} \times [0, 2]$ as follows:

$$\begin{aligned} \varphi : E_\xi &\mapsto S\left\{\Sigma_\xi \cap B(x_0, 10)\right\} \times [0, 2] \\ v &\mapsto (\gamma_v'(t_v), t_v), \end{aligned}$$

where t_v is the first time t such that $\gamma_v(t) \in \Sigma_\xi$. Clearly, $d(\gamma_v(t_v), \pi(v)) < |t_v| \leq 2$, it follows from triangle inequality that $\gamma_v(t_v) \in B(x_0, 3)$. Therefore, the above map is well defined. In light of Liouville's theorem(c.f. Excercise 14 on page 86 of [21]), the geodesic flow on sphere bundle preserves the volume form, as $|\gamma_v'(t)| \equiv 1$, it is clear that φ is volume expanding. It follows that

$$|E_\xi|_{\mathcal{H}^{2m-1}} \leq 2\left|S\left\{\Sigma_\xi \cap B(x_0, 10)\right\}\right|_{\mathcal{H}^{2m-2}} \leq C|\Sigma_\xi|_{\mathcal{H}^{m-1}} \leq C\xi. \tag{3.7}$$

Suppose $v \in E$, then $v \in T_xX$ for some $x \in B(x_0, 1) \cap \mathcal{R}$ and $\gamma_v(t_0) \in \mathcal{S}$ for some $t_0 \in [0, 1]$. Recall that Σ_{x1} can be regarded as the boundary of ξ-neighborhood of $\mathcal{S}$, then it follows from connectedness of γ_v that $\gamma_v(s) \in \Sigma_\xi$ for some $s \in [0, t_0] \subset [0, 2]$. Consequently, $v \in E_\xi$. This means that $E \subset E_\xi$ for every small positive ξ. It follows from (3.7) that E is a measure zero subset, of the sphere bundle over $B(x_0, 1) \cap \mathcal{R}$, which is denoted by $SB(x_0, 1)$ for simplicity.

We proceed to prove (3.5). For each $v \in SB(x_0, 1)\backslash E$, intermediate value theorem implies

$$f(\gamma_v(1)) - f(\gamma_v(0)) = (f \circ \gamma_v)'(t_0)$$

for some $t_0 \in [0, 1]$. Consequently, we have

$$\begin{aligned} \langle \nabla f, v\rangle - (f(\gamma_v(1)) - f(\gamma_v(0))) &= -(f \circ \gamma_v)'(t_0) + (f \circ \gamma_v)'(0) = -\int_0^{t_0} (f \circ \gamma)' dt \\ &= -\int_0^{t_0} \int_0^t \frac{\partial^2}{\partial \tau^2}(f \circ \gamma) d\tau dt. \end{aligned}$$

Taking absolute value on both sides yields that

$$|\langle \nabla f, v\rangle - (f(\gamma_v(1)) - f(\gamma_v(0)))| \leq t_0 \int_0^{t_0} |Hess_f| dt \leq \int_0^1 |Hess_f| dt.$$

Integrating both sides of the above inequality on $SB(x_0, 1)\backslash E$, we obtain

$$\begin{aligned} &\int_{SB(x_0,1)\backslash E} |\langle \nabla f, v\rangle - (f(\gamma_v(1)) - f(\gamma_v(0)))| \\ \leq &\int_{SB(x_0,1)\backslash E} \left(\int_0^1 |Hess_f| dt\right) \leq \int_0^1 \left\{\int_{SB(x_0,2)\backslash E} |Hess_f|\right\} dt = \int_{SB(x_0,2)\backslash E} |Hess_f|. \end{aligned}$$

Note that we have used the fact that the geodesic flow is volume preserving in the above inequality. It is clear that

$$\int_{SB(x_0,2)\backslash E} |Hess_f| = |S_{m-1}| \int_{B(x_0,2)\backslash \mathcal{S}} |Hess_f| = m\omega_m \int_{B(x_0,2)\backslash \mathcal{S}} |Hess_f|.$$

By abusing of notation, the combination of the previous inequalities implies that

$$\begin{aligned}
&\frac{1}{|SB(x_0,1)|}\int_{SB(x_0,1)} |\langle\nabla f, v\rangle - (f(\gamma_v(1)) - f(\gamma_v(0)))| \\
&< \frac{m\omega_m}{|SB(x_0,1)|}\int_{B(x_0,2)} |Hess_f| = \frac{1}{|B(x_0,1)|}\int_{B(x_0,2)} |Hess_f| = \frac{|B(x_0,2)|}{|B(x_0,1)|}\fint_{B(x_0,2)} |Hess_f| \\
&\le 2^m \fint_{B(x_0,2)} |Hess_f|,
\end{aligned}$$

where the last step follows from the Bishop-Gromov volume comparison. The above inequality is nothing but (3.5).

We continue to prove (3.6). Recall that u_1 and u_2 are harmonic functions satisfying(c.f. Lemma 2.39 of [16] and the discussion after it, or (3.2)):

$$\fint_{B(x_0,2)} |Hess_{u_1}|^2 + |Hess_{u_2}|^2 < CL^{-\alpha}.$$

It follows from Hölder inequality that

$$\fint_{B(x_0,2)} |Hess_{u_j}| < CL^{-\frac{\alpha}{2}}$$

for each $j \in \{1, 2\}$. The Cheng-Yau estimate guarantees that the $|\nabla u_j|$ is uniformly bounded on $B(x_0, 2)$. Since $u = u_j$, plugging the above inequality into (3.5) implies (3.6). Therefore, we finish the proof of Step 1. We remind the reader that the proof of this step is almost the same as Proposition 1.32 of [13].

Step 2. Let $\mathbf{U} = (u_1, u_2)$*. For every pair of points* $x, y \in B(x_0, 3)$*, we have*

$$|\mathbf{U}(x) - \mathbf{U}(y)| < d(x, y) + \xi \tag{3.8}$$

for some $\xi = \xi(L^{-1}|m)$*.*

We first explain the rough motivation behind (3.8). Actually, it follows from the Gromov-Hausdorff closeness between (X, x_0) and $(Y \times \mathbb{R}^k, (\hat{y}, 0))$ that (b_1, b_2) is an almost submersion from $B(x_0, 10)$ to its image in $\mathbb{R}^2$. Then we apply the C^0-closeness between b_i and u_i, we see that the map (u_1, u_2) is also an almost submersion. An almost submersion almost decreases distance, which is the meaning of (3.8).

Now we prove (3.8) in details. According to the given condition, it is clear to see that $B(x_0, 2L)$ is ψ-Gromov-Hausdorff close to a ball $B((\hat{y}, 0), 2L)$ in $Y \times \mathbb{R}^k$ for some metric space Y. Therefore there exists a map(not necessarily continuous) $F : B(x_0, 2L) \to Y$ such that

$$|d(x, y) - d(F(x), F(y))| < \psi, \forall\ x, y \in B(x_0, 2L). \tag{3.9}$$

Recall the construction of each b_i as

$$b_i(x) = d(x, \gamma_i(-L)) - d(x_0, \gamma_i(-L)).$$

Correspondingly, we define

$$\hat{b}_i(x) \triangleq d(F(x), F(\gamma_i(-L))) - d(F(x_0), F(\gamma_i(-L))). \tag{3.10}$$

On one hand side, by the property of F in (3.9), it is clear that

$$|b_i(x) - \hat{b}_i(x)| < 4\psi, \quad \forall\ x \in B(x_0, L).$$

On the other hand side, using the Gromov-Hausdorff closeness between X and $Y \times \mathbb{R}^k$ and the fact that γ_i is close to the line segment $[-L, L]$ on the i-th coordinate line, it follows from (3.10) that

$$|\hat{b}_i(x) - z_i \circ F(x)| < \psi'$$

for some $\psi' = \psi'(L^{-1})$. Consequently, we have

$$|b_i(x) - z_i \circ F(x)| < \psi'', \quad \forall\ x \in B(x_0, L). \tag{3.11}$$

Now we fix $x, y \in B(x_0, 3)$. It is clear from the splitting structure of $Y \times \mathbb{R}^k$ that

$$d^2(F(x), F(y)) \geq \{z_1 \circ F(x) - z_1 \circ F(y)\}^2 + \{z_2 \circ F(x) - z_2 \circ F(y)\}^2 .$$

Plugging (3.9) and (3.11) into the above inequality, noting that $|b_i(x) - b_i(y)| < 10$ in $B(x_0, 3)$, we arrive at

$$\{d(x, y) + \psi\}^2 \geq \{b_1(x) - b_1(y)\}^2 + \{b_2(x) - b_2(y)\}^2 - 200\psi''.$$

It follows that

$$\{b_1(x) - b_1(y)\}^2 + \{b_2(x) - b_2(y)\}^2 \leq d^2(x, y) + 200\psi'' + 100\psi. \tag{3.12}$$

Recall from Lemma 2.39 of [16] that $|u_i - b_i| < cL^{-\alpha}$ on $B(x_0, 3)$. Combing the previous inequalities implies that

$$\{u_1(x) - u_1(y)\}^2 + \{u_2(x) - u_2(y)\}^2 \leq d^2(x, y) + 200\psi'' + 100\psi + CL^{-\alpha}$$

whence we derive (3.8).

Step 3. Fix $\theta > 0$ small and set

$$C_\theta \triangleq \{v \in SB(x_0, 1) | \angle(v, \nabla u_1) < \theta\} . \tag{3.13}$$

Then we have

$$\int_{C_\theta} |u_2(\gamma_v(1)) - u_2(\gamma_v(0))| \leq 2\theta |C_\theta| \tag{3.14}$$

whenever $L > L_0(m, \theta)$.

Let Ω_θ be the set of points y in the unit sphere $S^m \subset \mathbb{R}^{m+1}$ such that $\angle(y, \vec{e}_1) < \theta$. Let $|\Omega_\theta|$ be the volume of Ω_θ. Then it is clear that

$$|C_\theta| = |B(x_0, 1)| \cdot |\Omega_\theta|. \tag{3.15}$$

On one hand, we have

$$\int_{C_\theta} ||u_1(\gamma_v(1)) - u_1(\gamma_v(0))| - 1| \le \int_{C_\theta} |u_1(\gamma_v(1)) - u_1(\gamma_v(0)) - 1|$$
$$\le \int_{C_\theta} |u_1(\gamma_v(1)) - u_1(\gamma_v(0)) - \langle \nabla u_1, v\rangle| + \int_{C_\theta} |\langle \nabla u_1, v\rangle - 1|.$$

Using the equations (3.6) and (3.13), we obtain

$$\frac{1}{|SB(x_0,1)|}\int_{C_\theta} ||u_1(\gamma_v(1)) - u_1(\gamma_v(0))| - 1| \le CL^{-\frac{\alpha}{2}} + (1-\cos\theta)\frac{|C_\theta|}{|SB(x_0,1)|}.$$

By (3.15) and the fact that $|S_{m-1}| = m\omega_m$, the above inequality can be simplified as

$$\frac{1}{|SB(x_0,1)|}\int_{C_\theta} |u_1(\gamma_v(1)) - u_1(\gamma_v(0))| \ge -CL^{-\frac{\alpha}{2}} + \frac{|\Omega_\theta|}{m\omega_m}\cdot\cos\theta.$$

By choosing L large enough, we can assume

$$CL^{-\frac{\alpha}{2}} < (1-\cos\theta)\cos\theta\cdot\frac{|C_\theta|}{|SB(x_0,1)|} = (1-\cos\theta)\cos\theta\cdot\frac{|\Omega_\theta|}{m\omega_m}.$$

Then we have

$$\int_{C_\theta} |u_1(\gamma_v(1)) - u_1(\gamma_v(0))| \ge |C_\theta|\cos^2\theta.$$

Then Hölder inequality implies that

$$\int_{C_\theta} |u_1(\gamma_v(1)) - u_1(\gamma_v(0))|^2 \ge \frac{1}{|C_\theta|}\left\{\int_{C_\theta} |u_1(\gamma_v(1)) - u_1(\gamma_v(0))|\right\}^2 \ge |C_\theta|\cos^4\theta. \tag{3.16}$$

On the other hand, it follows from (3.8) that

$$\int_{C_\theta}\left\{|u_1(\gamma_v(1)) - u_1(\gamma_v(0))|^2 + |u_2(\gamma_v(1)) - u_2(\gamma_v(0))|^2\right\} = \int_{C_\theta} |\mathbf{U}(\gamma_v(1)) - \mathbf{U}(\gamma_v(0))|^2 \le (1+\xi)|C_\theta|,$$

which in turn implies that

$$\int_{C_\theta} |u_2(\gamma_v(1)) - u_2(\gamma_v(0))|^2 \le (1+\xi)|C_\theta| - \int_{C_\theta} |u_1(\gamma_v(1)) - u_1(\gamma_v(0))|^2.$$

Plugging (3.16) into the above inequality, we arrive at

$$\frac{1}{|C_\theta|}\int_{C_\theta} |u_2(\gamma_v(1)) - u_2(\gamma_v(0))|^2 \le 1 + \xi - \cos^4\theta,$$

whence the Hölder inequality implies that

$$\frac{1}{|C_\theta|}\int_{C_\theta} |u_2(\gamma_v(1)) - u_2(\gamma_v(0))| \le \left\{1 + \xi - \cos^4\theta\right\}^{\frac{1}{2}}.$$

Note that $1 - \cos^4\theta \le 2(1 - \cos^2\theta) \le 2\theta^2$. By choosing L large enough, we can assume $\xi \le 2\theta^2$. Then (3.14) follows from the above inequality.

Step 4. We have

$$\int_{C_\theta} |\langle \nabla u_2, v\rangle| \le 3\theta|C_\theta| \tag{3.17}$$

whenever $L > L_1(m,\theta)$.

It follows from (3.6) that

$$\int_{SB(x_0,1)} |\langle \nabla u_2, v\rangle - (u_2(\gamma_v(1)) - u_2(\gamma_v(0)))| < CL^{-\frac{\alpha}{2}}|SB(x_0,1)|.$$

Recall that $C_\theta \subset SB(x_0,1)$. Plugging the above inequality into (3.14) yields that

$$\begin{aligned}
\int_{C_\theta} |\langle \nabla u_2, v\rangle| &\le \int_{C_\theta} |u_2(\gamma_v(1)) - u_2(\gamma_v(0))| + \int_{C_\theta} |\langle \nabla u_2, v\rangle - (u_2(\gamma_v(1)) - u_2(\gamma_v(0)))| \\
&\le \int_{C_\theta} |u_2(\gamma_v(1)) - u_2(\gamma_v(0))| + \int_{SB(x_0,1)} |\langle \nabla u_2, v\rangle - (u_2(\gamma_v(1)) - u_2(\gamma_v(0)))| \\
&\le 2\theta|C_\theta| + CL^{-\frac{\alpha}{2}}|SB(x_0,1)| = 2\theta|C_\theta|\left\{1 + \frac{|SB(x_0,1)|}{2\theta|C_\theta|}\cdot CL^{-\frac{\alpha}{2}}\right\} \\
&= 2\theta|C_\theta|\left\{1 + \frac{m\omega_m}{2\theta|\Omega_\theta|}\cdot CL^{-\frac{\alpha}{2}}\right\}.
\end{aligned} \tag{3.18}$$

By choosing $L > L_1$ for some $L_1 = L_1(m,\theta)$ sufficiently large, it is clear that (3.18) yields (3.17) directly. We finish the proof of Step 4.

Step 5. We have

$$⨍_{B(x_0,1)} |\langle \nabla u_1, \nabla u_2\rangle| < 5\theta \tag{3.19}$$

whenever $L > L_1(m,\theta)$.

It follows from the definition of C_θ in (3.13) that $|\langle \nabla u_1, v\rangle| \le 2\sin\frac{\theta}{2} \le 2\theta$. Consequently, we have

$$\int_{C_\theta} |\langle \nabla u_1, \nabla u_2\rangle| \le \int_{C_\theta} |\langle v, \nabla u_2\rangle| + \int_{C_\theta} |\langle \nabla u_1 - v, \nabla u_2\rangle| \le \int_{C_\theta} |\langle v, \nabla u_2\rangle| + 2\theta|C_\theta|. \tag{3.20}$$

Plugging (3.17) into (3.20) implies that

$$|\Omega_\theta| \int_{B(x_0,1)} |\langle \nabla u_1, \nabla u_2\rangle| = \int_{C_\theta} |\langle \nabla u_1, \nabla u_2\rangle| \le 5\theta|C_\theta|.$$

Dividing both sides of the above inequalities by $|C_\theta|$, with (3.15) in mind, we obtain (3.19). Therefore, we finish the proof of Step 5.

The inequality (3.3) follows from (3.19) since θ can be arbitrarily small, whenever L is very large. The proof of the theorem is complete. □

4 Proof of volume continuity

Theorem 4.1 (**Volume continuity**, Proposition 2.42 of [16]). *For every* $(X, x_0, g) \in \widetilde{\mathscr{K}\mathscr{S}}(n, \kappa)$ *and* $\epsilon > 0$*, there is a constant* $\xi = \xi(X, \epsilon)$ *such that*

$$\left|\log \frac{|B(y_0, 1)|}{|B(x_0, 1)|}\right| < \epsilon$$

for any $(Y, y_0, h) \in \widetilde{\mathscr{K}\mathscr{S}}(n, \kappa)$ *satisfying* $d_{PGH}((X, x_0, g), (Y, y_0, h)) < \xi$.

The key of volume convergence is the following property.

Proposition 4.2. *Given* $\epsilon > 0$*, there exists* $L = L(\epsilon, n) > 1$ *such that*

$$||B(x_0, 1)| - \omega_{2n}| < \epsilon \tag{4.1}$$

whenever $d_{GH}(B(x_0, L), B(0, L)) < L^{-1}$*. Here* $B(0, L)$ *is the standard ball of radius* L *in* $\mathbb{R}^{2n}$*,* $B(x_0, L)$ *is a geodeisc ball of radius* L *in some* $Y \in \widetilde{\mathscr{K}\mathscr{S}}(n, \kappa)$*.*

The smooth version of Proposition 4.2 is Lemma 2.1 of Colding [13], whose proof used directly the multi-Buseman fuctions. The proof was refined by Cheeger(c.f. Theorem 9.31 on page 44 of [6]), replacing the Buseman fuctions by their harmonic approximations and using the slicing theorem for $k = n$. One key new ingredient of Cheeger's proof is to make use of the mod-2 degree theory. We shall follow the route of Cheeger to prove Proposition 4.2. The mod-2 degree theory still works here, due to the high codimension of $\mathcal{S}$ and the gradient estimate of harmonic functions.

Since $B(x_0, L)$ is Gromov-Hausdorff close to the corresponding sized ball in $\mathbb{R}^m$, we can find m-line segments with length $2L$, almost perpendicular to each other. Following its construction, we obtain a harmonic map(c.f. Lemma 2.41 of [16]):

$$\vec{u} = (u_1, \cdots, u_m) : B(x_0, 1) \to B(0, 1 + \psi) \subset \mathbb{R}^m.$$

Note that $\vec{u}$ is a ψ -Gromov-Hausdorff approximation from $B(x_0, 1)$ to $B(0, 1)$. Actually, by Lemma 2.39 of [16], we have $|\vec{u} - \vec{b}|$ point-wisely very small in $B(x_0, 1)$, where $\vec{b} = (b_{1,-}, b_{2,-}, \cdots, b_{m,-})$. Similar to the argument in Step 2 of the proof of Theorem 3.1, under the help of triangle inequality, one can easily show that $\vec{b}$ is a ψ' -Gromov-Hausdorff approximation from $B(x_0, 1)$ to $B(0, 1)$. Therefore, $\vec{u}$ is a ψ -Gromov-Hausdorff approximation from $B(x_0, 1)$ to $B(0, 1)$. This means that $B(0, 1)$ is contained in the ψ-neighborhood of $\vec{u}(B(x_0, 1))$ and $|d(\vec{u}(x), \vec{u}(y)) - d(x, y)| < \psi$ for every $x, y \in B(x_0, 1)$. Up to a slight shifting, we can assume that $\vec{u}(x_0) = \vec{0}$. Furthermore, $\vec{u}$ satisfies

$$\int_{B(x_0,1)} \left\{ \sum_{1 \le i \le m} |\nabla u_i - 1|^2 + \sum_{1 \le i < j \le m} |\langle \nabla u_i, \nabla u_j \rangle| + \sum_{1 \le i \le m} |Hess_{u_i}|^2 \right\} \le \psi. \tag{4.2}$$

Here $\psi = \psi(L^{-1}|n)$ is a nonnegative continuous function defined on $[0, \infty)$ with $\psi(0) = 0$. Together with the gradient estimate of each u_i, the above estimate implies that

$$\int_{B(x_0,1)} ||\nabla u_1 \wedge \nabla u_2 \wedge \cdots \nabla u_m| - 1| \le \psi.$$

Consequently, we have

$$|B(x_0,1)| \geq \int_{B(x_0,1)} |\nabla u_1 \wedge \nabla u_2 \cdots \wedge \nabla u_m| - \psi \geq |\Omega| - \psi, \tag{4.3}$$

where $\Omega = \vec{u}\left(\overline{B(x_0,1)}\right)$. We claim that

$$\Omega \supset B(0, 1-5\psi). \tag{4.4}$$

We focus our attention on the set $B(0,1-5\psi)$ and its preimage under $\vec{u}$. By the fact that $\vec{u}$ is a ψ-Gromov-Hausdorff approximation, the preimage set must locate in $B(x_0, 1-4\psi)$.

Definition 4.3. *A point $v \in B(0,1-5\psi)$ is called a critical value of $\vec{u}$ if $v = \vec{u}(x)$ for some x with one of the following properties.*

- *$x \in \mathcal{S}$;*
- *$x \in \mathcal{R}$ and $D\vec{u}(x)$ degenerate.*

A point $v \in B(0,1-5\psi)$ is called a regular value if it is not a critical value.

Lemma 4.4. *The critical values of $\vec{u}$ form a measure-zero set, which we denote by E.*

Proof. Note that the singular set $\mathcal{S}$ has Hausdorff dimension at most $m-3$, which clearly has measure zero. Therefore, the proof is reduced to the case of Sard's theorem concerning smooth maps between smooth manifolds. □

Lemma 4.5. $\sharp\left\{\vec{u}^{-1}(\cdot)\right\}$ *is a locally constant function defined on $B(0,1-5\psi)\backslash E$.*

Proof. Fix $v \in B(0,1-5\psi)\backslash E$. Following definitions, we know $\vec{u}^{-1}(v)$ consists of smooth points with nondegenerate $D\vec{u}$. Therefore, it is a union of discrete points. Since $\vec{u}$ is a ψ-Gromov-Hausdorff approximation and $v \in B(0,1-5\psi)$, we know that $\vec{u}^{-1}(v) \subset B(x_0, 1-\psi)$. Note that $\vec{u}^{-1}(v)$ is a finite set. For otherwise, we can find $x_i \in \vec{u}^{-1}(v)$ such that $x_i \to x_\infty \in B(x_0, 1-\psi)$. Consequently, $\vec{u}(x_\infty) = \lim_{i\to\infty} \vec{u}(x_i) = v$. Since $v \in B(0,1-5\psi)\backslash E$, it follows from definition that $x_\infty \in \mathcal{R}$ and $D\vec{u}$ is non-degenerate. Then we see that $\vec{u}$ is a diffeomorphism on a small neighborhood of x_∞, which contradicts the fact that $\vec{u}(x_i) = v$ for each large i.

Since $\vec{u}^{-1}(v)$ is a finite set, by inverse function theorem, we obtain that v has a small neighborhood V such that $\vec{u}$ is a diffeomorphism from each component of $\vec{u}^{-1}(V)$ to V. This implies that $\sharp\left\{\vec{u}^{-1}(\cdot)\right\}$ is a constant on V. □

The set E may have codimension 1 and $B(0,1-5\psi)\backslash E$ may have many path-connected components. Therefore, $\sharp\left\{\vec{u}^{-1}(\cdot)\right\}$ may not be a constant on $B(0,1-5\psi)$, but just a constant function on each path connected component of $B(0,1-5\psi)\backslash E$. However, if we consider $\sharp\left\{\vec{u}^{-1}(\cdot)\right\}$ *mod* 2, we can exclude the effect of E.

Lemma 4.6. $\sharp\left\{\vec{u}^{-1}(\cdot)\right\}$ *mod* 2 *is a constant function defined on $B(0,1-5\psi)\backslash E$.*

Proof. The proof is a slight modification of the standard proof in differential topology to show mod-2 degree is well-defined for smooth map between two smooth manifolds of the same dimension.

Fix $v_0 \in B(0, 1-5\psi)\backslash E$. Let $v_1 \in B(0, 1-5\psi)\backslash E$ be close enough to v_0 such that

$$|v_0 - v_1| < \frac{1}{100}\min\{\psi, d(v_0, \vec{u}(\mathcal{S}))\}. \tag{4.5}$$

We can easily obtain a smooth map $\mathbf{U} : B(x_0, 1) \times [0, 1] \to \mathbb{R}^m$ by

$$\mathbf{U}(x, t) = \vec{u}(x) + t(v_0 - v_1).$$

From the above construction, the ψ-Gromov-Hausdorff approximation of $\vec{u}$, and the smallness of $|v_0 - v_1|$, it is clear that $\mathbf{U}^{-1}(B(0, 1-5\psi))$ locates in $B(x_0, 1-3\psi) \times [0, 1]$. Note that $\mathcal{S} \times [0, 1]$ has dimension less than $m-2$. Similar to Lemma 4.4, we obtain that away from a mesaure zero set $\mathbf{E}$(this is not E), evey point in $B(0, 1-5\psi)$ is the collection of smooth points satisfying that $D\mathbf{U}$ has rank m.

Note that v_0 is a regular point of $\mathbf{U}(\cdot, 0)$ and $\mathbf{U}(\cdot, 1)$. There is a very small number ϵ such that v_0 is a regular point of $\mathbf{U}$ when restricted on $B(x_0, 1) \times \{[0, 2\epsilon] \cup [1-2\epsilon, 1]\}$. Note that $\vec{u}^{-1}(v_0)$ and $\vec{u}^{-1}(v_1)$ are compact subset of $\mathcal{R}$, by gradient estimate of $\vec{u}$ and the choice of $|v_0 - v_1|$. The existence of ϵ can be obtained by a contradiction and compactness argument. Similar but more complicated argument will appear in the following. So we omit the details in this step. From previous discussion, we have a kind of Sard theorem in our situation. For every δ, we can find a value $c \in \mathbb{R}^m$ such that $v_0 + c$ is a regular value of $\mathbf{U}$ and $|c| < \delta$. Now we define

$$\tilde{\mathbf{U}} \triangleq \mathbf{U}(x, t) - c\eta(t), \tag{4.6}$$

where η is a smooth cutoff function with value 1 on $[2\epsilon, 1-2\epsilon]$ and value 0 outside $[\epsilon, 1-\epsilon]$. Clearly, v_0 is a regular value of $\tilde{\mathbf{U}}$, when restricted on $B(x_0, 1) \times \{[0, \epsilon] \cup [2\epsilon, 1-2\epsilon] \cup [1-\epsilon, 1]\}$. We claim that for very small c chosen above, v_0 is a regular value of $\tilde{\mathbf{U}}$ on $B(x_0, 1) \times [0, 1]$. For otherwise, we can choose $c_i \to 0$ and $y_i \in B(x_0, 1)$, $t_i \in [\epsilon, 2\epsilon] \cup [1-2\epsilon, 1-\epsilon]$ such that $\tilde{\mathbf{U}}_i(y_i, t_i) = v_0$ and $D\tilde{\mathbf{U}}_i$ degenerates at (y_i, t_i). Let (y_∞, t_∞) be the limit of (y_i, t_i). Then

$$v_0 = \mathbf{U}(y_\infty, t_\infty) = \vec{u}(y_\infty) + t_\infty(v_0 - v_1).$$

This implies that $\vec{u}(y_\infty) = t_\infty v_1 + (1 - t_\infty)v_0$. Recall that the line segment connecting v_1 and v_0 is away from the image $\vec{u}(\mathcal{S})$. Therefore $y_\infty \notin \mathcal{S}$. So for large i, all y_i locates in a coordinate chart of a smooth point y_∞, we have higher order estimate of $D\vec{u}$ since $\vec{u}$ is harmonic. Consequently, we can take smooth limit and obtain that $D\mathbf{U}(y_\infty, t_\infty)$ degenerates. Note that $y_\infty \in B(x_0, 1-3\psi)$ by ψ-Gromov-Hausdorff approximation of $\vec{u}$, and $t_\infty \in [\epsilon, 2\epsilon] \cup [1-2\epsilon, 1-\epsilon]$. Therefore, the degeneration of $D\mathbf{U}(y_\infty, t_\infty)$ contradicts to the fact that v_0 is a regular value of $\mathbf{U}|_{B(x_0,1)\times\{[0,2\epsilon]\cup[1-2\epsilon,1]\}}$.

In conclusion, for every $\delta > 0$, there is a $c \in \mathbb{R}^m$ with $|c| < \delta$ such that the map $\tilde{\mathbf{U}}$ defined in (4.6) has v_0 as a regular value. Since $\tilde{\mathbf{U}} = \vec{u}(x) + t(v_0 - v_1) - c\eta(t)$, the ψ-Gromov-Hausdorff approximation property of $\vec{u}$ implies that $\tilde{\mathbf{U}}^{-1}(v_0)$ is uniformly inside $B(x_0, 1-2\psi) \times [0, 1]$. The gradient estimate of $\vec{u}$ yields that $\tilde{\mathbf{U}}^{-1}(v_0)$ is uniformly away from the singular set $\mathcal{S} \times [0, 1]$. Therefore, $\tilde{\mathbf{U}}^{-1}(v_0)$ is the union of finite number of connected, compact, smooth one-dimensional manifolds(or curves), possibly with boundary. Each curve either has no intersection with $B(x_0, 1) \times \{0, 1\}$, or intersect with $B(x_0, 1) \times \{0, 1\}$ transversely. This guarantees that

$$\sharp\left\{\tilde{\mathbf{U}}^{-1}(v_0) \cap \{B(x_0, 1) \times \{0\}\}\right\} mod\ 2 = \sharp\left\{\tilde{\mathbf{U}}^{-1}(v_0) \cap \{B(x_0, 1) \times \{1\}\}\right\} mod\ 2,$$

which is the same as

$$\sharp\left\{\vec{u}^{-1}(v_0)\right\}\ mod\ 2 = \sharp\left\{\vec{u}^{-1}(v_1)\right\}\ mod\ 2. \tag{4.7}$$

We have proved the above equality under the assumption (4.5).

Note that $B(0, 1 - 5\psi)\backslash\vec{u}(\mathcal{S})$ is connected, since $\dim_{\mathcal{H}}(\vec{u}(\mathcal{S})) \le \dim_{\mathcal{H}}(\mathcal{S}) \le m - 3$, due to the gradient estiamte of $\vec{u}$. For every pair of points $v_0, v_1 \in B(0, 1 - 5\psi)\backslash E$, we can find a curve $\gamma \subset B(0, 1 - 5\psi)\backslash\vec{u}(\mathcal{S})$ connecting v_0 and v_1. By the compactness of the curve γ, we can find an ϵ such that γ locates in $B(0, 1 - 5\psi)\backslash B(\vec{u}(\mathcal{S}), \epsilon)$, where $B(\vec{u}(\mathcal{S}), \epsilon)$ is the ϵ-neighborhood of $\vec{u}(\mathcal{S})$. Covering γ by finite number of balls with radii less than $\frac{\epsilon}{100}$, we then obtain that $\sharp\left\{\vec{u}^{-1}(\cdot)\right\}\ mod\ 2$ is a constant on each of these balls. Consequently, we obtain (4.7), under the assumption that v_0, v_1 are arbitrary points of $B(0, 1 - 5\psi)\backslash E$. □

Lemma 4.7. *Suppose $\vec{f} = (f_1, f_2, \cdots, f_m) : B(z, 10) \to \mathbb{R}^m$ is a smooth map such that*

$$\sup_{B(z,3)} |\nabla\vec{f}| \le 2\sqrt{m}. \tag{4.8}$$

Furthermore, we assume

$$⨍_{B(z,10)} \left\{ \sum_{1\le i\le m} |\nabla f_i - 1|^2 + \sum_{1\le i<j\le m} |\langle\nabla f_i, \nabla f_j\rangle| + \sum_{1\le i\le m} |Hess_{f_i}|^2 \right\} \le \xi \tag{4.9}$$

for some small ξ. Then we have

$$\sup_{y\in\overline{B(z,1)}} \left| \left|\vec{f}(y) - \vec{f}(z)\right|^2 - d^2(y, z) \right| \le \psi(\xi) \tag{4.10}$$

where $\psi(\xi)$ can be chosen as $C\xi^{\frac{1}{m+2}}$ for some $C = C(m)$.

Actually, up to a shifting of center point, it is not hard to see that (4.10) implies that $\vec{f}$ is a $\psi(\xi)$-Gromov-Hausdorff approximation map from $B(z, 1)$ to the standard unit ball in $\mathbb{R}^m$.

Proof. Without loss of generality, we may assume that $\vec{f}(z) = 0$. By the compactness of $\overline{B(z, 1)}$, we can assume that the supreme on the left hand side of (4.10) is achieved at $y_0 \in \overline{B(z, 1)}$.

Fix $\delta = \xi^{\frac{1}{m+2}}$. In other words, $\xi = \delta^{m+2}$. Let $A_1 = B(z, \delta)$ and $A_2 = B(y_0, \delta)$. Both A_1 and A_2 are subsets of $B(z, 2)$. Then we can apply segment inequality(c.f. Proposition 2.6 of [16]) for function

$$h \triangleq \sum_{1\le i\le m} |\nabla f_i - 1|^2 + \sum_{1\le i<j\le m} |\langle\nabla f_i, \nabla f_j\rangle| + \sum_{1\le i\le m} |Hess_{f_i}|^2 \tag{4.11}$$

to obtain that

$$\int_{A_2}\left\{\int_{A_1}\mathcal{F}_h(x,y)dv_x\right\}dv_y \leq 2^{m+1}(|A_1|+|A_2|)\int_{B(z,6)} h \leq 2^{m+1}(|A_1|+|A_2|)|B(z,6)|\xi. \tag{4.12}$$

By mean value theorem, there exists $y^* \in A_2 \cap \mathcal{R}$ such that

$$\begin{aligned}
⨍_{A_1}\mathcal{F}_h(x,y^*)dv_x &\leq 2^{m+2}\left\{\frac{1}{|A_2|}+\frac{1}{|A_1|}\right\}|B(z,6)|\xi = 2^{m+2}\left\{\frac{|B(z,6)|}{|B(y_0,\delta)|}+\frac{|B(z,6)|}{|B(z,\delta)|}\right\}\xi\\
&\leq 2^{m+2}\left\{\frac{|B(y_0,8)|}{|B(y_0,\delta)|}+\frac{|B(z,8)|}{|B(z,\delta)|}\right\}\xi \leq 2^{4m+2}\delta^{-m}\xi < 2^{5m}\xi^{\frac{2}{m+2}} = 2^{5m}\delta^2.
\end{aligned}$$

In particular, we can find some $z^* \in A_1 = B(z,\delta)$ such that

$$\int_\gamma h \leq 2^{5m}\delta^2, \tag{4.13}$$

where γ is a smooth shortest geodesic connecting y^* and w^*. Decompose γ into two parts γ_α and γ_β. Let γ_α be the part where $h > \delta$ and γ_β be the part where $h \leq \delta$. It follows from the above inequality that

$$|\gamma_\alpha| < 2^{5m}\delta. \tag{4.14}$$

Note that on γ_β, as $h \leq \delta$, some elementary properties of linear algebra implies that

$$1-C\delta \leq (1-C\delta)|\gamma'|^2 \leq \sum_{1\leq i\leq m}|\langle\nabla f_i,\gamma'\rangle|^2 \leq (1+C\delta)|\gamma'|^2 \leq 1+C\delta \tag{4.15}$$

for some $C = C(m)$. It is clear that

$$\begin{aligned}
\frac{d}{dt}\left|\vec{f}(\gamma(t))\right|^2 &= 2\sum_{1\leq i\leq m} f_i\langle\nabla f_i,\gamma'\rangle,\\
\frac{d^2}{dt^2}\left|\vec{f}(\gamma(t))\right|^2 &= 2\sum_{1\leq i\leq m}\left\{|\langle\nabla f_i,\gamma'\rangle|^2 + f_i Hess_{f_i}(\gamma',\gamma')\right\}.
\end{aligned}$$

For each $0 < t \leq |\gamma|$, we have

$$\frac{d}{dt}\left|\vec{f}(\gamma(t))\right|^2 - 2\sum_{1\leq i\leq m} f_i\langle\nabla f_i,\gamma'(0)\rangle - 2t = \int_0^t\left\{\frac{d^2}{ds^2}\left|\vec{f}(\gamma(s))\right|^2 - 2\right\}ds.$$

Decompose the last integral into two parts $\gamma([0,t]) \cap \gamma_\alpha$ and $\gamma([0,t]) \cap \gamma_\beta$. On the part γ_β, the absolute value of the integrand is bounded by $C\delta$. On the part γ_α, it is bounded by $10m + h$. It follows from (4.14) that

$$\left|\frac{d}{dt}\left|\vec{f}(\gamma(t))\right|^2 - 2\sum_{1\leq i\leq m} f_i\langle\nabla f_i,\gamma'(0)\rangle - 2t\right| < \int_\gamma h + C|\gamma_\alpha| + C\xi|\gamma_\beta| < C(\xi+\delta) < C\delta.$$

Recall that

$$\left|\sum_{1\le i\le m} f_i\langle\nabla f_i, \gamma'(0)\rangle\right| < 10\sqrt{m}|\vec{f}(w)| < 100m\delta.$$

by condition (4.8) and the fact $w = \gamma(0) \in B(z, \delta)$. Then it follows that

$$\left|\frac{d}{dt}\left|\vec{f}(\gamma(t))\right|^2 - 2t\right| < C\delta.$$

Integrating over $[0, |\gamma|]$, the above inequality implies that

$$\left||\vec{f}(y^*)|^2 - |\vec{f}(z^*)|^2 - |\gamma|^2\right| < C\delta.$$

Using gradient estimate of $\vec{f}$, we have

$$\begin{aligned}
&\left|\vec{f}(y^*) - \vec{f}(y_0)\right| \le C\delta, \quad \Rightarrow \quad \left||\vec{f}(y^*)|^2 - |\vec{f}(y_0)|^2\right| < C\delta.\\
&\left|\vec{f}(z^*) - \vec{f}(z)\right| \le C\delta, \quad \Rightarrow \quad \left||\vec{f}(z^*)|^2 - |\vec{f}(z)|^2\right| < C\delta.\\
&||\gamma| - d(y_0, z)| = |d(y^*, z^*) - d(y_0, z)| < 2\delta, \quad \Rightarrow \left||\gamma|^2 - d^2(y_0, z)\right| < C\delta.
\end{aligned}$$

Combining the previous two steps, we obtain

$$\left||\vec{f}(y_0)|^2 - d^2(y_0, z)\right| < C\delta,$$

which implies (4.10) by the choice of y_0 and $\delta = \xi^{\frac{1}{m+2}}$. □

Lemma 4.8. *There exists one point in $B(0, 1 - 5\psi)$ such that $\vec{u}^{-1}(\vec{v})$ contains exactly one point in $B(x_0, 1)\backslash\mathcal{S}$.*

Proof. According to (4.2) and a standard measure comparision argument(c.f. Theorem 9.31 of Cheeger [6]), we can find a point $z \in B(x_0, 0.5)\backslash\mathcal{S}$ such that for every $r \in (0, 0.5)$, we have

$$⨍_{B(z,r)}\left\{\sum_{1\le i\le m}|\nabla u_i - 1|^2 + \sum_{1\le i<j\le m}|\langle\nabla u_i, \nabla u_j\rangle| + \sum_{1\le i\le m}|Hess_{u_i}|^2\right\} \le \psi.$$

In particular, we see that $\{\nabla u_i\}$ are almost perpendicular to each other at z and $D\vec{u}$ non-degenerate. Then we can apply Lemma 4.7 to obtain

$$r^2 = \sum_i (u_i - u_i(z))^2 + \psi r^2$$

on $\partial B(z, r)$, for every $r \in (0, 0.5)$. Togeter with ψ-Gromov-Hausdorff approximation, the above equation implies that

$$\left\{\vec{u}^{-1}(\vec{u}(z))\right\} \cap B(x_0, 1) = \left\{\vec{u}^{-1}(\vec{u}(z))\right\} \cap B(z, 0.5) = \{z\}.$$

It is also clear that $\vec{u}^{-1}(z) \in B(0, 1)\backslash\mathcal{S}$ by its choice. □

Now we are able to finish the proof of Proposition 4.2.

Proof of Proposition 4.2. By Lemma 4.6, the function $\sharp\left\{\vec{u}^{-1}(\cdot)\right\} mod\ 2$ is a constant on $B(0, 1-5\psi)\backslash E$. Now Lemma 4.8 implies this constant function is 1. Therefore, away from a measure zero set E, every point in $B(0, 1-5\psi)$ is the image of some $x \in B(x_0, 1-4\psi)$. Now we claim that $E \cap B(0, 1-5\psi)$ is also contained in $\vec{u}(B(x_0, 1))$. Actually, for every point $v \in E \cap B(0, 1-5\psi)$, we can find $v_i \to v$ and each $v_i \in B(0, 1-5\psi)\backslash E$. Then $\sharp\left\{\vec{u}^{-1}(v_i)\right\} mod\ 2 = 1$ and we can find $y_i \in B(x_0, 1-3\psi)$ such that $\vec{u}(y_i) = v_i$. By taking subsequence if necessary, we can assume $y_i \to y_\infty \in B(x_0, 1-2\psi)$. The gradient estimate of $\vec{u}$ then implies that

$$\vec{u}(y_\infty) = \lim_{i\to\infty} \vec{u}(y_i) = \lim_{i\to\infty} v_i = v.$$

Consequently, $v \in \vec{u}(B(x_0, 1))$. By the arbitrary choice of $v \in E \cap B(0, 1-5\psi)$, we obtain (4.4). Then Proposition 4.2 follows from (4.3). □

Similar to Corollary 2.19 of Colding [13], we can deduce the following proposition from Proposition 4.2. The proof is almost the same. We write it down here for the convenience of the readers.

Proposition 4.9. *For each $\epsilon > 0$, there exists $\delta = \delta(n, \epsilon)$ with the following property.*

Suppose $(X, x_0, g) \in \widetilde{\mathscr{K}\mathscr{S}}(n, \kappa)$ satisfies $d_{GH}(B(x_0, 1), B(0, 1)) < \delta$, where $B(0, 1)$ is the standard unit ball in $\mathbb{R}^m$. Then we have

$$-\epsilon < |B(x_0, 1)| - \omega_m \leq 0. \tag{4.16}$$

Proof. It suffices to prove the first inequality in (4.16). Given each positive ϵ very small , we choose $L = L(\frac{\epsilon}{100}, n)$ by Proposition 4.2 and set $\nu = \frac{\epsilon}{100\omega_m L}$. Then we have

$$\omega_m(1 - L\nu)^m > \left(1 - \frac{\epsilon}{100\omega_m}\right)\omega_m.$$

By the definition of volume, we can find finitely disjoint balls $B(y_j, r_j) \subset B(0, 1 - L\nu)$ with $r_j < \nu$ such that

$$\sum_j \omega_m r_j^m > \left(1 - \frac{\epsilon}{100\omega_m}\right)\omega_m.$$

Define $\mu = \min\{r_j\}$. Suppose $d_{GH}(B(x_0, 1), B(0, 1)) < \frac{\mu\xi}{L^2}$ for some tiny ξ. By choosing ξ very small, we can pick finitely many disjoint balls $B(y_j^*, r_j) \subset B(x_0, 1)$ such that

$$d_{GH}\left(B(y_j^*, Lr_j), B(y_j, Lr_j)\right) < \frac{\mu}{L}.$$

Then we apply Proposition 4.2 to obtain

$$|B(y_j^*, r_j)| \geq \left(1 - \frac{\epsilon}{100\omega_m}\right)\omega_m r_j^m.$$

It follows that

$$|B(x_0,1)| \geq \sum_j |B(y_j^*, r_j)| \geq \left(1 - \frac{\epsilon}{100\omega_m}\right) \sum_j \omega_m r_j^m \geq \left(1 - \frac{\epsilon}{100\omega_m}\right)^2 \omega_m > \omega_m - \epsilon,$$

which is nothing but the first inequality of (4.16). The proof of Proposition 4.9 is complete. □

We now follow the proof of Theorem 0.1. of Colding [13] to prove Theorem 4.1. The difficulty caused by the existence of $\mathcal{S}$ can be overcome by a standard covering argument.

Proof of Theorem 4.1. It suffices to prove the following statement:

Suppose $(Y_i, y_i, h_i) \in \widetilde{\mathscr{KS}}(n,\kappa)$ converges to $(X, x_0, g) \in \widetilde{\mathscr{KS}}(n,\kappa)$ in the pointed-Gromov-Hausdorff topology, then we have

$$\lim_{i\to\infty} |B(y_i, 1)| = |B(x_0, 1)|. \tag{4.17}$$

Let $\mathcal{S}_r$ be the r-neighborhood of $\mathcal{S} \cap B(x_0,1)$. Then it follows from the definition of Minkowski dimension and the fact $\dim_{\mathcal{M}} \mathcal{S} < m-3$ that

$$|\mathcal{S}_r| \leq Cr^3 \tag{4.18}$$

for some C depending on (X, x_0, g). Fix $r > 0$ small. Up to a perturbation, we can assume that $\mathcal{R}_r = B(x_0,1)\backslash\mathcal{S}_r$ is a smooth manifold with boundary. Fix arbitrary $0 < \eta << r$, by the definition of volume, we can find finitely many disjoint geodesic balls $B(x_j, r_j) \subset \mathcal{R}_r$ such that

$$\sum_{j=1}^N \omega_m r_j^m \geq (1-\eta)|\mathcal{R}_r|, \quad |B(x_j, r_j)| \geq (1-\eta)\omega_m r_j^m.$$

Therefore, for large i, we can find balls $B(y_{j,i}, r_j)$ such that $\lim_{i\to\infty} d_{GH}(B(y_{j,i}, r_j), B(x_j, r_j)) = 0$. It follows from Proposition 4.9 that $\lim_{i\to\infty} |B(y_{j,i}, r_j)| = |B(x_j, r_j)|$. Consequently, for i large, we have

$$|B(y_i,1)| \geq \sum_{j=1}^N |B(y_{j,i}, r_j)| \geq (1-\eta)\sum_{j=1}^N |B(x_j, r_j)| \geq (1-\eta)^2 \sum_{j=1}^N \omega_m r_j^m \geq (1-\eta)^3|\mathcal{R}_r|.$$

Plugging (4.18) into the last inequality, we derive that

$$|B(y_i,1)| \geq (1-\eta)\left\{|B(x_0,1)| - Cr^3\right\}. \tag{4.19}$$

On the other hand side, by (4.18) and the definition of $\mathcal{S}_r$, we can find finitely many points $z_j \in \mathcal{S} \cap B(x_0,1)$ such that $B(z_j, r)$ are disjoint to each other and $\cup_{j=1}^K B(z_j, 5r)$ covers $\mathcal{S} \cap B(x_0,1)$. The κ-noncollapsing condition and the volume ratio upper bound guarantees that

$$K \leq Cr^{3-m} \tag{4.20}$$

where C is independent of r. The set $B(x_0,1)\backslash\cup_{j=1}^{K} B(z_j,5r)$ is a bounded measurable set in $\mathcal{R}$. Then we can cover it by finitely many sets $B(w_j,\rho_j)$ such that

$$\sum_{j=1}^{L}\omega_m\rho_j^m<\left|B(x_0,1)\backslash\cup_{j=1}^{K} B(z_j,5r)\right|+\eta \tag{4.21}$$

and $\rho_j<\eta$ for each $j\in\{1,2,\cdots,L\}$. Note that $B(x_0,1)$ is covered by

$$\left\{\bigcup_{j=1}^{L}B(w_j,\rho_j)\right\}\cup\left\{\bigcup_{j=1}^{K}B(z_j,5r)\right\}.$$

As (Y_i,y_i,h_i) converges to (X,x_0,g) in the pointed-Gromov-Hausdorff sense, for large i, we can cover $B(y_i,1)$ by

$$\left\{\bigcup_{j=1}^{L}B(w_{i,j}^*,\rho_j)\right\}\cup\left\{\bigcup_{j=1}^{K}B(z_{i,j}^*,5r)\right\}.$$

Consequently, we have

$$|B(y_i,1)|\leq\sum_{j=1}^{L}|B(w_{i,j}^*,\rho_j)|+\sum_{j=1}^{K}|B(z_{i,j}^*,5r)|$$

By Gromov-Bishop volume comparison, (4.20) and (4.21), we obtain

$$\begin{aligned}|B(y_i,1)|&\leq\sum_{j=1}^{L}\omega_m\rho_j^m+Cr^3\leq\left|B(x_0,1)\backslash\cup_{j=1}^{K} B(z_j,5r)\right|+\eta+Cr^3\\&\leq|B(x_0,1)|+\eta+Cr^3.\end{aligned} \tag{4.22}$$

It follows from (4.19) and (4.22) that

$$(1-\eta)\left\{|B(x_0,1)|-Cr^3\right\}\leq\liminf_{i\to\infty}|B(y_i,1)|\leq\limsup_{i\to\infty}|B(y_i,1)|\leq|B(x_0,1)|+\eta+Cr^3 \tag{4.23}$$

for each r and each $\eta<<r$. Let $\eta\to 0$ and then let $r\to 0$, we have

$$\liminf_{i\to\infty}|B(y_i,1)|=\limsup_{i\to\infty}|B(y_i,1)|=\lim_{i\to\infty}|B(y_i,1)|=|B(x_0,1)|,$$

whence we have (4.17). The proof of the theorem is complete. □

5 Proof of almost Kähler cone splitting

Theorem 5.1 (**Almost Kähler cone splitting**, Proposition 2.44 of [16]). *For each $\epsilon>0$, there exists $\xi=\xi(\epsilon|n)$ with the following properties.*

Suppose $X \in \widetilde{\mathscr{KS}}(n,\kappa)$, $x_0 \in X$, b is a smooth function on $B(x_0,2)\backslash\mathcal{S}$ satisfying

$$\sup_{B(x_0,2)\backslash\mathcal{S}} |\nabla b| \leq 2, \quad \fint_{B(x_0,2)\backslash\mathcal{S}} |Hess_b|^2 \leq \epsilon^2. \tag{5.1}$$

Suppose also $\dfrac{|B(x_0,2)|}{|B(x_0,1)|} \geq (1-\epsilon)2^{2n}$*, i.e., $B(x_0,1)$ is an almost volume cone. Then there exists a smooth function $\tilde{b}$ on $B(x_0,1)\backslash\mathcal{S}$ such that*

$$\sup_{B(x_0,1)\backslash\mathcal{S}} \left|\tilde{b}\right| \leq 3, \tag{5.2}$$

$$\fint_{B(x_0,1)\backslash\mathcal{S}} \left|\nabla\tilde{b} - J\nabla b\right|^2 \leq \xi. \tag{5.3}$$

The corresponding version of Theorem 5.1 in the smooth setting is the Lemma 9.14 of [10], with slight modifications. The key is to make use of the fact that $B(x_0,1)$ is an almost metric cone, which is obtained by solving Dirichlet problem and developing effective estimates for the approximation smooth functions(c.f. Proposition 4.50 and Proposition 4.81 of Cheeger-Colding [8]). Important ingredients for such estimates include quantitative maximum principle, integration by parts, Poincaré inequality and Bishop-Gromov volume comparison. All these ingredients hold in the current setting. The proof then follows verbatim from Cheeger-Colding-Tian [10] and Cheeger-Colding [8].

The following Proposition is an improvement of general maximum principle. The statement and proof is almost the same as Lemma 8.5 of Cheeger [6]. We write it down here with complete detailed proof for the convenience of the readers.

Proposition 5.2 (**Quantitative maximum principle**). *Suppose $\Delta f = \delta \geq 0$ in Ω and $\Omega \subset B(y_0,R_2)\backslash B(y_0,R_1)$. Then for all $t \geq 0$, $x \in \Omega$, we have*

$$f(x) \geq (\delta\underline{L}_{R_2} + t\underline{G}_{R_2})(R) + \min_{\partial\Omega}\left\{f - (\delta\underline{L}_{R_2} + t\underline{G}_{R_2})\right\} \tag{5.4}$$

where $\underline{L}$ and $\underline{G}$ are poled at the base point y_0, $R = d(x,y_0)$.

Proof. According to their constructions, we know both $\underline{L}$ and $\underline{G}$ vanish on $\partial B(y_0,R_2)$. Moreover, on $B(y_0,R_2)\backslash\{y_0\}$, we have $\Delta\underline{G} \geq 0$ and $\Delta\underline{L} \geq 1$(c.f. Lemma 2.36 of [16]). Therefore, we have

$$\Delta\left(f - \delta\underline{L}_{R_2} - t\underline{G}_{R_2}\right) \leq 0.$$

It follows from maximum principle that

$$f(x) - \delta\underline{L}_{R_2}(R) - t\underline{G}_{R_2}(R) \geq \min_{\partial\Omega}\left\{f - (\delta\underline{L}_{R_2} + t\underline{G}_{R_2})\right\},$$

whence we arrive at (5.4). □

A direct consequence of the Segment inequality is the following estimate of Dirichlet Poincaré constant. For example, see page 23 of Cheeger [6], or Lemma 4.16 of Cheeger-Colding [8]. Both proofs work here verbatim. For the convenience of the readers, we write down full details following the route of Cheeger [6].

Proposition 5.3 (Estimate of Dirichlet Poincaré constant). *For each $B(x_0, r) \subset X$, we have*

$$\lambda_1(B(x_0, r)) = \inf_{f \in N_c^{1,2}(B(x_0,r))} \frac{\int_{B(x_0,r)} |\nabla f|^2}{\int_{B(x_0,r)} f^2} \geq cr^{-2} \tag{5.5}$$

for some small positive constant $c = c(m)$.

Proof. Let set A_1 be $B(x_0, r)$, set A_2 be $B(x_0, 2r)\backslash B(x_0, r)$. Note that every geodesic connecting two points from A_1 and A_2 must locate in $B(x_0, 10r)$ by triangle inequality. Since $f \in N_c^{1,2}(B(x_0, r))$, we can extend f to be defined on $B(x_0, 10r)$ trivially. Namely, for each $x \in B(x_0, 10r)\backslash B(x_0, r)$, we set $f(x) = 0$. Now we apply the segment inequality(Proposition 2.6 of [16]) on the function $|\nabla f^2|$. We have

$$\int_{A_1 \times A_2} \mathcal{F}_{|\nabla f^2|}(x_1, x_2) \leq C(m)r(|A_1| + |A_2|) \int_{B(x_0,10r)} |\nabla f^2|. \tag{5.6}$$

Note that $x_1 \in A_1 = B(x_0, r)$ and $x_2 \in A_2 = B(x_0, 2r)\backslash B(x_0, r)$. It follows from our construction that $|f\nabla f|(x_2) = 0$. Consequently, we have

$$|f^2|(x_1) = \left||f^2|(x_1) - |f^2|(x_2)\right| \leq \mathcal{F}_{|\nabla f^2|}(x_1, x_2).$$

Plugging the above inequality into (5.6) yields that

$$\begin{aligned}
|A_2| \int_{A_1} |f^2|(x_1) dv_{x_1} &\leq C(m)r(|A_1| + |A_2|) \int_{B(x_0,10r)} |f\nabla f| \\
&\leq C(m)r(|A_1| + |A_2|) \left\{\int_{B(x_0,10r)} |f|^2\right\}^{\frac{1}{2}} \left\{\int_{B(x_0,10r)} |\nabla f|^2\right\}^{\frac{1}{2}} \\
&= C(m)r(|A_1| + |A_2|) \left\{\int_{A_1} |f|^2\right\}^{\frac{1}{2}} \left\{\int_{A_1} |\nabla f|^2\right\}^{\frac{1}{2}},
\end{aligned}$$

where we used the fact that f vanishes outside A_1 in the last step. Therefore, we have

$$\left\{\frac{\int_{A_1} |\nabla f|^2}{\int_{A_1} f^2}\right\}^{\frac{1}{2}} \geq \frac{1}{C(m)r} \frac{|A_2|}{|A_1| + |A_2|} = \frac{1}{C(m)r} \cdot \frac{|B(x_0, 2r)| - |B(x_0, r)|}{|B(x_0, 2r)|}. \tag{5.7}$$

Note that $\partial B(x_0, 10r) \neq \emptyset$. Let γ be a shortest unit speed geodesic connecting x_0 with some point $y_0 \in \partial B(x_0, 2r)$ such that $\gamma(0) = x_0$ and $\gamma(2r) = y_0$. Let $z_0 = \gamma(1.5r)$. By triangle inequality, it is clear that

$$B(z_0, 0.1r) \subset B(x_0, 2r)\backslash B(x_0, r), \qquad B(x_0, 2r) \subset B(z_0, 10r).$$

Plugging the above relationships into (5.7), we derive

$$\left\{\frac{\int_{A_1} |\nabla f|^2}{\int_{A_1} f^2}\right\}^{\frac{1}{2}} \geq \frac{1}{C(m)r} \cdot \frac{|B(z_0, 0.1r)|}{|B(z_0, 10r)|} \geq \frac{1}{100^m C(m)r},$$

whence we obtain (5.5), as $A_1 = B(x_0, r)$. □

The next proposition is the almost version of Lemma 2.34 of [16]. The proof is very similar to that of Lemma 2.39 of [16]. The smooth version of the next proposition is the combination of Proposition 4.50 and Proposition 4.81 of Cheeger-Colding [8].

Proposition 5.4. *For each small positive number ϵ, there exists small numbers $\xi_1(\epsilon|n)$ and $\xi_2(\epsilon|n)$ with the following properties.*

Suppose $X \in \widetilde{\mathscr{K}\mathscr{S}}(n,\kappa)$, $x_0 \in X$. Suppose $\frac{|B(x_0,2)|}{|B(x_0,1)|} \geq (1-\epsilon)2^{2n}$. Then there exists an $r_0 \in (1.5, 2)$ such that

$$\frac{r_0|\partial B(x_0,r_0)|}{|B(x_0,r_0)|} \geq 2n - 8\epsilon. \tag{5.8}$$

Fix r_0 and let w be the solution of the Dirichlet Poinssoin equation

$$\begin{cases} \Delta w = 2n, \quad in \quad B(x_0,r_0); \\ w = \frac{r_0^2}{2}, \quad on \quad \partial B(x_0,r_0). \end{cases} \tag{5.9}$$

Let r be the distance function to x_0. Then we have

$$\left|2w - r^2\right|_{L^\infty(B(x_0,1))} + \fint_{B(x_0,1)\backslash \mathcal{S}} ||\nabla w| - r| + \fint_{B(x_0,1)\backslash \mathcal{S}} |Hess_w - g|^2 < \xi_1. \tag{5.10}$$

Furthermore, on $B(x_0,1)\backslash\mathcal{S}$, we have

$$|\nabla w| - r < \xi_2. \tag{5.11}$$

Proof. Recall that $m = 2n$. Similar to the argument in Lemma 2.34, we set $A(r) = \frac{|\partial B(x_0,r)|}{r^{m-1}}$ and obtain

$$\frac{d}{dr}\log\frac{|B(x_0,r)|}{r^m} = r^{-1}\left\{\frac{r^m A(r)}{|B(x_0,r)|} - m\right\} \leq 0.$$

Integrating the above inequality on $[1,2]$, it follows from our condition $\frac{|B(x_0,2)|}{|B(x_0,1)|} \geq (1-\epsilon)2^{2n}$ that

$$0 \geq \int_1^2 r^{-1}\left\{\frac{r^m A(r)}{|B(x_0,r)|} - m\right\} dr = \log\left\{\frac{|B(x_0,2)|}{|B(x_0,1)|}\cdot\frac{1}{2^m}\right\} \geq \log(1-\epsilon) \geq -2\epsilon.$$

By the non-positivity of the integrand in the above inequalities, we can shrink the interval and obtain

$$0 \leq \int_{1.5}^2 r^{-1}\left\{m - \frac{r^m A(r)}{|B(x_0,r)|}\right\} dr \leq 2\epsilon.$$

By mean value theorem, it is clear that there is some $r_0 \in [1.5, 2]$ such that

$$m - \frac{r_0^m A(r_0)}{|B(x_0,r_0)|} \leq 8\epsilon, \tag{5.12}$$

which is equivalent to (5.8).

We proceed to prove (5.10). We first claim that w is bounded. The upper bound is relatively easy to obtain. Actually, it follows from comparison geometry(c.f. Proposition 2.30 of [16]) that $\Delta\frac{r^2}{2} \leq 2n$ in the distribution sense. Consequently, $w - \frac{r^2}{2}$ is a subharmonic function on $B(x_0, r_0)$ and vanishes on $\partial B(x_0, r_0)$. It is clear that $w - \frac{r^2}{2} \leq 0$ on $B(x_0, r_0)$. On the other hand, we can choose $y_0 \in \partial B(x_0, 5)$. Then $B(x_0, r_0) \subset B(y_0, 10)\backslash B(y_0, 3)$. By setting $\delta = m$ and $\Omega = B(x_0, r_0)$, we obtain $w \geq -C(m)$ on $B(x_0, r_0)$. In short, we have obtained

$$|w| \leq C, \quad \text{on } B(x_0, r_0) \tag{5.13}$$

for some $C = C(m)$.

For each small positive number ζ, define a "cutoff" function

$$\eta(x) \triangleq \begin{cases} 1, & \text{if } r(x) < r_0 - \zeta, \\ \frac{1-r(x)}{\zeta}, & \text{if } r_0 - \zeta \leq r(x) \leq r_0. \end{cases}$$

Then we calculate

$$0 \leq \int_{B(x_0,r_0)} \eta\Delta\left(w - \frac{r^2}{2}\right) = m\int_{B(x_0,r_0)} \eta - \frac{1}{\zeta}\int_{B(x_0,r_0)\backslash B(x_0,r_0-\zeta)} r.$$

Let $\zeta \to 0$, we obtain that

$$0 \leq \int_{B(x_0,r_0)} \Delta\left(w - \frac{r^2}{2}\right) = m|B(x_0, r_0)| - |A(r_0)|r_0^m = |B(x_0, r_0)|\left\{m - \frac{r_0^m A(r_0)}{|B(x_0, r_0)|}\right\} \leq 8\epsilon|B(x_0, r_0)|,$$

where we used (5.12) in the last step. Consequently, we have

$$⨍_{B(x_0,r_0)} \Delta\left(w - \frac{r^2}{2}\right) \leq 8\epsilon. \tag{5.14}$$

By (5.13), it is clear that $-C \leq w - \frac{r^2}{2} \leq 0$ on the ball $B(x_0, r_0)$. Consequently, we obtain

$$⨍_{B(x_0,r_0)} |\nabla w - r\nabla r|^2 = -⨍_{B(x_0,r_0)} \left(w - \frac{r^2}{2}\right)\Delta\left(w - \frac{r^2}{2}\right) \leq C⨍_{B(x_0,r_0)} \Delta\left(w - \frac{r^2}{2}\right) \leq C\epsilon. \tag{5.15}$$

Applying the uniform Dirichlet Poincaré constant(c.f. Proposition 5.3) to the above inequality, we arrive at

$$⨍_{B(x_0,r_0)} \left|w - \frac{r^2}{2}\right| \leq C\sqrt{\epsilon}. \tag{5.16}$$

Define

$$K \triangleq \sup_{B(x_0,1.2)} \left|w - \frac{r^2}{2}\right|.$$

Suppose K is achieved at $z_0 \in \overline{B(x_0,1.2)}$. Note that $|\nabla(w-\frac{r^2}{2})| < D$ on $B(x_0,1.2)$ by Cheng-Yau estimate. Without loss of generality, we may assume that $D > 10$. Then we obtain that in the ball $B(z_0,\frac{K}{2D})$, the value of $\left|w-\frac{r^2}{2}\right|$ is greater than $0.5K$. Consequently, we obtain

$$0.5K\left|B\left(z_0,\frac{K}{2D}\right)\right| \le \int_{B(z_0,\frac{K}{2D})}\left|w-\frac{r^2}{2}\right| \le \int_{B(x_0,r_0)}\left|w-\frac{r^2}{2}\right|.$$

It follows that

$$\begin{aligned} K &\le C\cdot\frac{|B(x_0,r_0)|}{|B(z_0,\frac{K}{2D})|}\cdot \fint_{B(x_0,r_0)}\left|w-\frac{r^2}{2}\right| \le C\cdot\frac{|B(z_0,3)|}{|B(z_0,\frac{K}{2D})|}\cdot \fint_{B(x_0,r_0)}\left|w-\frac{r^2}{2}\right| \\ &\le CK^{-m}\fint_{B(x_0,r_0)}\left|w-\frac{r^2}{2}\right|. \end{aligned}$$

Consequently, we have

$$K \le C(m)\left\{\fint_{B(x_0,r_0)}\left|w-\frac{r^2}{2}\right|\right\}^{\frac{1}{m+1}}.$$

Plugging (5.16) into the above inequality, we arrive at

$$K \le C(m)\epsilon^{\frac{1}{2(m+1)}},$$

which implies that

$$\left|2w-r^2\right|_{L^\infty(B(x_0,1.2))} < C\epsilon^{\frac{1}{2(m+1)}}. \tag{5.17}$$

The above inequality provides the estimate of the first term on the left hand side of (5.10). We continue to estimate the remainder two terms. It follows from (5.15) that

$$\begin{aligned} \fint_{B(x_0,1)}||\nabla w|-r|^2 &\le \fint_{B(x_0,1)}|\nabla w-r\nabla r|^2 \le \frac{\int_{B(x_0,r_0)}|\nabla w-r\nabla r|^2}{|B(x_0,1)|} \\ &\le r_0^m\fint_{B(x_0,r_0)}|\nabla w-r\nabla r|^2 \le C\epsilon. \end{aligned} \tag{5.18}$$

Using Weinzenböck formula, we have

$$|Hess_w-g|^2 = |Hess_w|^2-2\Delta w+2n = |Hess_w|^2-2n = \frac{1}{2}\Delta\left\{|\nabla w|^2-2w\right\}.$$

Let φ be a nonnegative cutoff function such that $\varphi\equiv 1$ on $B(x_0,1)$ and vanishes outside $B(x_0,1.2)$. Moreover, $|\nabla\varphi|\le 10$ and $|\Delta\varphi|\le C$ on the annulus $B(x_0,1.2)\backslash B(x_0,1)$. Then we have

$$\begin{aligned} \int_{B(x_0,1)}|Hess_w-g|^2 &\le \int_{B(x_0,1.2)}\varphi|Hess_w-g|^2 = \frac{1}{2}\int_{B(x_0,1.2)}\varphi\Delta\left\{|\nabla w|^2-2w\right\} \\ &= -\frac{1}{2}\int_{B(x_0,1.2)}\left\{|\nabla w|^2-2w\right\}\Delta\varphi \\ &\le C\int_{B(x_0,1.2)}\left||\nabla w|^2-2w\right| \le C\int_{B(x_0,1.2)}\left\{\left||\nabla w|^2-r^2\right|+|r^2-2w|\right\}. \end{aligned}$$

Note that $-C \leq w \leq \frac{r_0^2}{2}$ on $B(x_0, r_0)$ by maximum principle and $|\nabla w| \leq C$ on $B(x_0, 1.2)$ by Cheng-Yau estimate. Then from the above inequality we derive

$$\int_{B(x_0,1)} |Hess_w - g|^2 \leq C \int_{B(x_0,1.2)} \left| |\nabla w|^2 - 2w \right| \leq C \int_{B(x_0,1.2)} \left\{ ||\nabla w| - r| + |r^2 - 2w| \right\}.$$

Therefore, by (5.17), (5.18) and volume comparison, we have

$$\fint_{B(x_0,1)} |Hess_w - g|^2 \leq C \left\{ \fint_{B(x_0,r_0)} ||\nabla w| - r| + \fint_{B(x_0,1.2)} |r^2 - w| \right\} \leq C(\epsilon + \epsilon^{\frac{1}{2(m+1)}}) \leq C\epsilon^{\frac{1}{2(m+1)}}. \tag{5.19}$$

Then (5.10) follows from the combination of (5.17), (5.18) and (5.19).

We move on to prove (5.11). Direct calculation shows that

$$\Delta \left\{ |\nabla w|^2 - 2w \right\} = |Hess_w - g|^2 \geq 0.$$

Let $f = \max\{|\nabla w|^2 - 2w, 0\} \geq 0$. Then we have $\Delta f \geq 0$ in the distribution sense. Therefore, the De giorgi-Nash-Moser iteration implies that

$$\|f\|_{L^\infty(B(x_0,1)\backslash\mathcal{S})} \leq C\|f\|_{L^2(B(x_0,1.2)\backslash\mathcal{S})} \leq C\left\| |\nabla w|^2 - w \right\|_{L^2(B(x_0,1.2)\backslash\mathcal{S})}.$$

Note that both $|\nabla w| + r$ and $r^2 + 2|w|$ are uniformly bounded on $B(x_0, 1.2)\backslash\mathcal{S}$, it follows from (5.17) and the modification of (5.18) that

$$\begin{aligned} \left\| |\nabla w|^2 - 2w \right\|^2_{L^2(B(x_0,1.2)\backslash\mathcal{S})} &= \left\| |\nabla w|^2 - r^2 \right\|^2_{L^2(B(x_0,1.2)\backslash\mathcal{S})} + \left\| r^2 - 2w \right\|^2_{L^2(B(x_0,1.2)\backslash\mathcal{S})} \\ &\leq C \int_{B(x_0,1.2)\backslash\mathcal{S}} \left\{ ||\nabla w| - r|^2 + \left| r^2 - 2w \right|^2 \right\} < C\epsilon^{\frac{1}{m+1}}. \end{aligned}$$

Combining the previous two steps, we obtain

$$\max\{|\nabla w|^2 - 2w, 0\} = f \leq C\epsilon^{\frac{1}{2(m+1)}}$$

on $B(x_0, 1)\backslash\mathcal{S}$. Plugging (5.17) into the above inequality, we arrive at

$$|\nabla w|^2 \leq 2w + C\epsilon^{\frac{1}{2(m+1)}} \leq r^2 + C\epsilon^{\frac{1}{2(m+1)}}$$

whence we arrive (5.11). □

Proof of Theroem 5.1. Let u be the function w constructed in Proposition 5.4. On $B(x_0, 1)\backslash\mathcal{S}$, we define

$$\tilde{b} \triangleq \langle J\nabla b, \nabla u \rangle \tag{5.20}$$

where J is the complex structure on $\mathcal{R}$. Clearly, $\tilde{b}$ is a smooth function on $B(x_0, 2)\backslash\mathcal{S}$. We only need to prove the estimates (5.2) and (5.3).

We first prove (5.2). Note that J preserves metric. It follows from the first inequality of (5.1) and (5.11) that

$$\begin{aligned}|J\nabla b| &\le 2,\\ |\nabla u| &< r + \xi' < 1.5,\end{aligned}$$

on $B(x_0, 1)\backslash\mathcal{S}$. Therefore, (5.2) follows from combining the definition equation (5.20) with the above two inequalities.

Then we move on to prove (5.3). By the Kähler condition on $\mathcal{R}$, the complex structure J satisfies $J^2 = -Id$, $g(J\cdot, J\cdot) = g(\cdot,\cdot)$ and $\nabla J \equiv 0$ on $\mathcal{R}$. Then it is clear that

$$\begin{aligned}\nabla_Y \tilde{b} &= \langle J\nabla_Y \nabla b, \nabla u\rangle + \langle J\nabla b, \nabla_Y \nabla u\rangle = -\langle \nabla_Y \nabla b, J\nabla u\rangle + Hess_u(Y, J\nabla b)\\ &= -Hess_b(Y, J\nabla u) + Hess_u(Y, J\nabla b)\end{aligned}$$

for every smooth vector field Y on $\mathcal{R}$. Therefore, we have

$$\begin{aligned}\nabla\tilde{b} &= -Hess_b(J\nabla u, \cdot) + Hess_u(J\nabla b, \cdot),\\ \nabla\tilde{b} - J\nabla b &= -Hess_b(J\nabla u, \cdot) + (Hess_u - g)(J\nabla b, \cdot).\end{aligned}$$

By condition (5.1), $|\nabla b| \le 2$. Also, we know $|\nabla u| \le C$ by Cheng-Yau estimate. Therefore we obtain

$$⨍_{B(x_0,1)\backslash\mathcal{S}} |\nabla\tilde{b} - J\nabla b|^2 \le C ⨍_{B(x_0,1)\backslash\mathcal{S}} \left\{|Hess_b|^2 + |Hess_u - g|^2\right\}.$$

It follows from the second inequality in (5.1) and (5.10) in Proposition 5.4 that

$$⨍_{B(x_0,1)\backslash\mathcal{S}} \left\{|Hess_b|^2 + |Hess_u - g|^2\right\} < \epsilon^2 + \xi'(\epsilon|n).$$

Then (5.3) follows from the combination of the previous two inequalities. □

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