(n, 2n)-Dominating Numbers of Undirected Toroidal Mesh C(3, 3, . . . , 3)

XIE Xin1,2, XU Jun Ming2
(1. Department of Mathematics, Huangshan University, Anhui 245021, China; 2. Department of Mathematics, University of Science and Technology of China, Anhui 230026, China)
(E-mail: xie-xin188@sohu.com)

Abstract The (d, k)-dominating number is a new measure to characterize reliability of resource-sharing in fault tolerant networks. This paper obtains that the (n, 2n)-dominating number of the n-dimensional undirected toroidal mesh C(3, 3, . . . , 3) is equal to 3 (n ≥ 3).

Keywords reliability; wide-diameter; undirected toroidal mesh; (d, k)-dominating number.

1. Introduction

In this paper, we quote from [1] the terminology and notations and use the graphs to represent networks. Let G = (V, E) be a k-connected graph (simple undirected). By Menger’s theorem, we know graph G contains at least k internally disjoint (x, y)-paths for any two distinct vertices x and y in G.

In order to characterize the reliability of transmission delay in a network, Hsu and Lyuu[2], Fladrin and Li[3] independently introduced wide-diameter as follows:

Definition 1 Let G be a k-connected graph. The distance with width k from vertices x to y, denoted by dk(G; x, y), is the minimum number d for which there are k internally disjoint (x, y)-paths in G of length at most d. The diameter with width k, denoted by dk(G), is the minimum number d for which there are k internally disjoint (x, y)-paths in G of length at most d for any distinct vertices x and y in G.

In a real-time processing network, Li and Xu[4] defined a new parameter (d, k)-dominating number, which, in some sense, can more accurately characterize the reliability of resource-sharing in a fault-tolerant network.

Definition 2 Let d (≥ 1) be an integer, G be a k(≥ 1)-connected graph, ∅ ≠ S ⊂ V(G), and y ∈ V(G − S). A path from y to some vertex in S is called a (y, S)-path. For a given integer d, if there are k internally disjoint (y, S)-paths in G of length at most d, then we say...
that $S$ can $(d,k)$-dominate $y$. If $S$ can $(d,k)$-dominate every vertex in $G - S$, then $S$ is called a $(d,k)$-dominating set of $G$. We use the symbol $S_{d,k}(G)$ to denote a set of all $(d,k)$-dominating sets in $G$. The parameter

$$\gamma_{d,k}(G) = \min\{|S| : S \in S_{d,k}(G)|$$

is called the $(d,k)$-dominating number of $G$. A $(d,k)$-dominating set $S$ of $G$ is called minimum if $|S| = \gamma_{d,k}(G)$.

Clearly, $(1,1)$-dominating set and $(1,1)$-dominating number are usually dominating set and dominating number. Thus, $(d,k)$-dominating set and $(1,1)$-dominating number are direct generalizations of the dominating set and dominating number. However, the problem determining the dominating number of a graph is NP-complete [5], and, hence, the problem finding $(d,k)$-dominating number is NP-complete too. Thus, it is very important to find the $(d,k)$-dominating number of some well-known networks for their wide applications. Of course, we have $\gamma_{d,k}(G) = 1$ for $d \geq d_k(G)$ and $\gamma_{d,k}(G) \geq 2$ for $d < d_k(G)$. For such graphs $G$, it is of interest to determine vaules of $\gamma_{d,k}(G)$ for $d < d_k(G)$.

**Definition 3** We define the $n$-dimensional toroidal mesh $C(d_1, d_2, \ldots, d_n)$ with $V = \{(x_1, x_2, \ldots, x_n)| x_i \in \{0, 1, \ldots, d_i - 1\}, i = 1, 2, \ldots, n\}$ as the set of vertices. The vertex $(x_1, x_2, \ldots, x_n)$ is adjacent to the vertex $(y_1, y_2, \ldots, y_n)$ if and only if there exists $i \in \{1, \ldots, n\}$ such that

$$\begin{align*}
  x_j &= y_j, & j \neq i, \\
  x_i - y_i &= 1 \text{ or } d_i - 1 \text{ (mod) } d_i, & j = i.
\end{align*}$$

The $n$-dimensional undirected toroidal mesh $C(d_1, d_2, \ldots, d_n)$ is $2n$-regular and is vertex-transitive. Its connectivity is $2n$. It was proved in [6] that the diameter of $d(C(d_1, d_2, \ldots, d_n))$ is $\sum_{i=1}^{n}(\frac{d_i}{2})$, and the wide diameter $d_{2n}(C(d_1, d_2, \ldots, d_n))$ is $\sum_{i=1}^{n}(\frac{d_i}{2}) + 1$. The toroidal mesh is widely used in network’s theory (see [7, 8, 9]). Lü Changhong and Zhang Keming[10] proved the $(d, 2n)$-dominating number of $n$-dimensional undirected toroidal mesh $C(d_1, d_2, \ldots, d_n)$ ($\neq C(3, 3, \ldots, 3)$) is $2$ ($n \geq 3, d_i \geq 3, i \in \{1, \ldots, n\}$ for $d = d(C(d_1, d_2, \ldots, d_n))$).

In this paper, we denote by $C_n(3)$ the $n$-dimensional undirected toroidal mesh $C(3, 3, \ldots, 3)$, the diameter $d(C_n(3))$ of which is $n$. We will prove that the $(n, 2n)$-dominating number of $C_n(3)$ is $3$.

2. Main results

**Theorem** Let $G = C_n(3)$. Then we have $\gamma_{n,2n}(G) = 3$.

**Proof** First we prove $\gamma_{n,2n}(G) > 2$. It is easy to verify $\gamma_{n,2n}(G) \geq 2$ for $d(G) = n$ and $d_{2n}(G) = n + 1$. If $\gamma_{n,2n}(G) = 2$, we suppose $S = \{u, v\}$ is one of the $(n, 2n)$-dominating sets by vertex-translity, where $u = (0, 0, \ldots, 0)$, $v = (x_1, x_2, \ldots, x_n)$, $x_i \in \{0, 1, 2\}$, $i = 1, 2, \ldots, n$, and $v \neq u$. By vertex-translity, we only consider three cases as follows:

**Case 1** $v = (x_1, \ldots, x_k, 0, \ldots, 0)$, where $x_1 \neq 0, \ldots, x_k \neq 0, 1 \leq k \leq n - 2$.

Let $y = (3 - x_1, \ldots, 3 - x_k, 1, \ldots, 1) \in V(G) - S$. Then we have $3 - x_i \neq x_i$ and $x_i \neq 0$ when $i = 1, 2, \ldots, n - 2$. Since $G$ is $2n$-regular, there is one path which goes through vertex
(3 - x_1, \ldots, 3 - x_k, 2, 1, \ldots, 1) of 2n internally disjoint \((y, S)\)-paths. Let

\[ P_1 : y = (3 - x_1, \ldots, 3 - x_k, 1, \ldots, 1) \rightarrow (3 - x_1, \ldots, 3 - x_k, 2, 1, \ldots, 1) \rightarrow \cdots \rightarrow u = (0, \ldots, 0) \]

or \( v = (x_1, \ldots, x_k, 0, \ldots, 0) \).

Of course we have \(|P_1| \geq n + 1\), so \(S\) cannot \((n, 2n)\)-dominate the vertex \(y\).

**Case 2** \( v = (x_1, \ldots, x_{n-1}, 0) \), where \(x_1 \neq 0, \ldots, x_{n-1} \neq 0\).

Let \( y = (3 - x_1, \ldots, 3 - x_{n-1}, 1) \in V(G) - S \). Then we have \(3 - x_i \neq x_i\) and \(x_i \neq 0\) when \(i = 1, 2, \ldots, n - 1\). And there is one path which goes through vertex \((3 - x_1, \ldots, 3 - x_{n-1}, 2)\) of 2n internally disjoint \((y, S)\)-paths. Let

\[ P_2 : y = (3 - x_1, \ldots, 3 - x_{n-1}, 1) \rightarrow (3 - x_1, \ldots, 3 - x_{n-1}, 2) \rightarrow \cdots \rightarrow u = (0, \ldots, 0) \]

or \( v = (x_1, \ldots, x_{n-1}, 0) \).

We have \(|P_2| \geq n + 1\), so \(S\) cannot \((n, 2n)\)-dominate the vertex \(y\).

**Case 3** \( v = (x_1, \ldots, x_n) \), where \(x_1 \neq 0, \ldots, x_n \neq 0\).

Let \( y = (x_1, 3 - x_2, \ldots, 3 - x_n) \in V(G) - S \). Then we have \(3 - x_i \neq x_i\) and \(x_i \neq 0\) when \(i = 1, 2, \ldots, n\). And there is one path which goes through vertex \((3 - x_1, \ldots, 3 - x_n)\) of 2n internally disjoint \((y, S)\)-paths. Let

\[ P_3 : y = (x_1, 3 - x_2, \ldots, 3 - x_n) \rightarrow (3 - x_1, \ldots, 3 - x_n) \rightarrow \cdots \rightarrow u = (0, \ldots, 0) \]

or \( v = (x_1, \ldots, x_n) \).

We have \(|P_3| \geq n + 1\), so \(S\) cannot \((n, 2n)\)-dominate the vertex \(y\).

Thus \(S\) is not an \((n, 2n)\)-dominating set of \(G\), so we have \(\gamma_{n,2n}(G) \geq 2\) by vertex-transitivity.

Next we prove \(\gamma_{n,2n}(G) \leq 3\). Let \( S = \{u, v, w\} \subset V(G) \), where \( u = (0, \ldots, 0) \), \( v = (1, \ldots, 1) \), \( w = (2, \ldots, 2) \). Now we prove \(S\) can \((n, 2n)\)-dominate any vertex \(x \in V(G) - S\).

We only consider the following cases:

**Case 1** \( x = (0, \ldots, 0, 1, \ldots, 1) \), where \(k \geq 1\) and \(n - k \geq 1\).

**Subcase 1a** \( k > 1 \), and \(n - k > 1\).

We can construct the 2n internally disjoint \((x, S)\)-paths, denoted by \( P_i \) (\(1 \leq i \leq 2n\)):
$(n, 2n)$-dominating numbers of undirected toroidal mesh $C(3, 3, \ldots, 3)$

\[ P_{n-k+1} : x = (0, \ldots, 0, 1, \ldots, 1) \rightarrow (1, 0, \ldots, 0, 1, \ldots, 1) \rightarrow (1, 1, 0, \ldots, 0, 1, \ldots, 1) \rightarrow \cdots \rightarrow (1, \ldots, 1, 0, 1, \ldots, 1) \rightarrow v = (1, \ldots, 1); \]
\[ P_{n-k+2} : x = (0, \ldots, 0, 1, \ldots, 1) \rightarrow (0, 1, 0, \ldots, 0, 1, \ldots, 1) \rightarrow (0, 1, 1, 0, \ldots, 0, 1, \ldots, 1) \rightarrow \cdots \rightarrow (0, 1, \ldots, 1) \rightarrow v = (1, \ldots, 1); \]
\[ \vdots \]
\[ P_n : x = (0, \ldots, 0, 1, \ldots, 1) \rightarrow (0, \ldots, 0, 1, 0, 1, \ldots, 1) \rightarrow (1, 0, \ldots, 0, 1, 1, \ldots, 1) \rightarrow \cdots \rightarrow (1, \ldots, 1, 1, 0, 1, \ldots, 1) \rightarrow v = (1, \ldots, 1); \]
\[ P_{n+1} : x = (0, \ldots, 0, 1, \ldots, 1) \rightarrow (0, \ldots, 0, 2, 1, \ldots, 1) \rightarrow (0, \ldots, 0, 2, 2, 1, \ldots, 1) \rightarrow \cdots \rightarrow (0, \ldots, 0, 2, \ldots, 2) \rightarrow (2, 0, \ldots, 0, 2, \ldots, 2) \rightarrow (2, 2, 0, \ldots, 0, 2, \ldots, 2) \rightarrow \cdots \rightarrow w = (2, \ldots, 2); \]
\[ P_{n+2} : x = (0, \ldots, 0, 1, \ldots, 1) \rightarrow (0, \ldots, 0, 1, 2, 1, \ldots, 1) \rightarrow \cdots \rightarrow (0, \ldots, 0, 1, 2, \ldots, 2) \rightarrow (0, 2, 0, \ldots, 0, 1, 2, \ldots, 2) \rightarrow \cdots \rightarrow (2, 0, \ldots, 0, 1, 2, \ldots, 2) \rightarrow \cdots \rightarrow (2, \ldots, 2, 1, 2, \ldots, 2) \rightarrow w = (2, \ldots, 2); \]
\[ \vdots \]
\[ P_{2n-k} : x = (0, \ldots, 0, 1, \ldots, 1) \rightarrow (0, \ldots, 0, 1, \ldots, 1, 2) \rightarrow (2, 0, \ldots, 0, 1, \ldots, 1, 2) \rightarrow (2, 2, 0, \ldots, 0, 1, \ldots, 1, 2) \rightarrow \cdots \rightarrow (2, \ldots, 2, 1, \ldots, 1, 2) \rightarrow \cdots \rightarrow (2, \ldots, 2, 1, \ldots, 1) \rightarrow w = (2, \ldots, 2); \]
\[ P_{2n-k+1} : x = (0, \ldots, 0, 1, \ldots, 1) \rightarrow (2, 0, \ldots, 0, 1, \ldots, 1) \rightarrow (2, 2, 0, \ldots, 0, 1, \ldots, 1) \rightarrow \cdots \rightarrow (2, \ldots, 2, 1, \ldots, 1) \rightarrow (2, \ldots, 2, 2, \ldots, 1) \rightarrow \cdots \rightarrow (2, \ldots, 2, 2, \ldots, 2) \rightarrow w = (2, \ldots, 2); \]
\[ P_{2n-k+2} : x = (0, \ldots, 0, 1, \ldots, 1) \rightarrow (0, 2, 0, \ldots, 0, 1, \ldots, 1) \]

\[ \rightarrow (0, 2, 2, 0, \ldots, 0, 1, \ldots, 1) \rightarrow \cdots \rightarrow (0, 2, \ldots, 2, 1, \ldots, 1) \]

\[ \rightarrow (0, 2, \ldots, 2, 2, 1, \ldots, 1) \rightarrow \cdots \rightarrow (0, 2, \ldots, 2, \ldots, 2) \rightarrow w = (2, \ldots, 2); \]

\[ : \]

\[ P_{2n} : x = (0, \ldots, 0, 1, \ldots, 1) \rightarrow (0, \ldots, 0, 2, 1, \ldots, 1) \rightarrow (0, \ldots, 0, 2, 2, 1, \ldots, 1) \]

\[ \rightarrow \cdots \rightarrow (0, \ldots, 0, 2, \ldots, 2) \rightarrow (2, 0, \ldots, 0, 2, \ldots, 2) \]

\[ \rightarrow (2, 2, 0, \ldots, 0, 2, \ldots, 2) \rightarrow \cdots \rightarrow w = (2, \ldots, 2). \]

We can easily get \(|P_i| \leq n, \ i = 1, \ldots, 2n.\)

**Subcase 1b** \(x = (0, 1, \ldots, 1)\) for \(k = 1\), or \(x = (0, \ldots, 0, 1)\) for \(n-k = 1\).

One can carry out the proof in the same way as in Subcase 1a.

**Case 2** \(x = (0, \ldots, 0, 1, \ldots, 1, 2, \ldots, 2)\), where \(k_1 \geq 1, k_2 \geq 1, n - k_1 - k_2 \geq 1.\)

**Subcase 2a** \(k_1 > 1, k_2 > 1, \text{ and } n - k_1 - k_2 > 1.\)

We can construct the \(2n\) internally disjoint \((x, S')\)-paths, denoted by \(P_i\) \((1 \leq i \leq 2n)\):

\[ P_1 : x \rightarrow (1, 0, \ldots, 0, 1, \ldots, 1, 2, \ldots, 2) \rightarrow (1, 1, 0, \ldots, 0, 1, \ldots, 1, 2, \ldots, 2) \]

\[ \rightarrow \cdots \rightarrow (1, \ldots, 1, 2, \ldots, 2) \rightarrow (1, \ldots, 1, 2, \ldots, 2) \rightarrow \cdots \]

\[ \rightarrow (1, \ldots, 1, 2) \rightarrow v = (1, \ldots, 1); \]

\[ P_2 : x \rightarrow (0, 1, 0, \ldots, 0, 1, \ldots, 1, 2, \ldots, 2) \rightarrow (0, 1, 1, 0, \ldots, 0, 1, \ldots, 1, 2, \ldots, 2) \]

\[ \rightarrow \cdots \rightarrow (0, 1, \ldots, 1, 2, \ldots, 2) \rightarrow (0, 1, \ldots, 1, 2, \ldots, 2) \rightarrow \cdots \]

\[ \rightarrow (0, 1, \ldots, 1, 2) \rightarrow (0, 1, \ldots, 1) \rightarrow v = (1, \ldots, 1); \]

\[ : \]

\[ P_{k_1} : x \rightarrow (0, \ldots, 0, 1, \ldots, 1, 2, \ldots, 2) \rightarrow (0, \ldots, 0, 1, \ldots, 1, 2, \ldots, 2) \]

\[ \rightarrow \cdots \rightarrow (0, \ldots, 0, 1, \ldots, 1) \rightarrow (1, 0, \ldots, 0, 1, \ldots, 1) \rightarrow \cdots \]

\[ \rightarrow (1, 1, 0, \ldots, 0, 1, \ldots, 1) \rightarrow v = (1, \ldots, 1); \]
\[ P_{k_1+1} : x \rightarrow (0, \ldots , 0, 2, 1, \ldots , 1, 2, \ldots , 2) \rightarrow (0, \ldots , 0, 2, 2, 1, \ldots , 1, 2, \ldots , 2) \]
\[ \quad \rightarrow \ldots \rightarrow (0, \ldots , 0, 2, \ldots , 2) \rightarrow (2, 0, \ldots , 0, 2, \ldots , 2) \]
\[ \quad \rightarrow (2, 2, 0, \ldots , 0, 2, \ldots , 2) \rightarrow \ldots \rightarrow w = (2, \ldots , 2); \]

\[ P_{k_1+k_2} : x \rightarrow (0, \ldots , 0, 1, \ldots , 1, 2, \ldots , 2) \rightarrow (2, 0, \ldots , 0, 1, \ldots , 1, 2, \ldots , 2) \]
\[ \quad \rightarrow (2, 2, 0, \ldots , 0, 1, \ldots , 1, 2, \ldots , 2) \rightarrow \ldots \rightarrow (2, \ldots , 2, 1, \ldots , 1, 2, \ldots , 2) \]
\[ \quad \rightarrow (2, \ldots , 2, 1, \ldots , 1, 2, \ldots , 2) \rightarrow \ldots \rightarrow w = (2, \ldots , 2); \]

\[ P_{k_1+k_2+1} : x \rightarrow (0, \ldots , 0, 1, \ldots , 1, 2, \ldots , 2) \rightarrow (0, \ldots , 0, 1, \ldots , 1, 0, 2, \ldots , 2) \]
\[ \quad \rightarrow \ldots \rightarrow (0, \ldots , 0, 1, \ldots , 1, 0, 0, \ldots , 0) \rightarrow (0, \ldots , 0, 1, \ldots , 1, 0, \ldots , 0) \]
\[ \quad \rightarrow \ldots \rightarrow u = (0, \ldots , 0); \]

\[ P_n : x \rightarrow (0, \ldots , 0, 1, \ldots , 1, 2, \ldots , 2, 0) \rightarrow (0, \ldots , 0, 0, 1, \ldots , 1, 2, \ldots , 2, 0) \]
\[ \quad \rightarrow \ldots \rightarrow (0, \ldots , 0, 2, \ldots , 2, 0) \rightarrow (0, \ldots , 0, 2, \ldots , 2, 0) \rightarrow \ldots \]
\[ \quad \rightarrow u = (0, \ldots , 0); \]

\[ P_{n+1} : x \rightarrow (2, 0, \ldots , 0, 1, \ldots , 1, 2, \ldots , 2) \rightarrow (2, 2, 0, \ldots , 0, 1, \ldots , 1, 2, \ldots , 2) \]
\[ \quad \rightarrow \ldots \rightarrow (2, 2, 1, \ldots , 1, 2, \ldots , 2) \rightarrow (2, 2, 1, \ldots , 1, 2, \ldots , 2) \]
\[ \quad \rightarrow (2, \ldots , 2, 1, 2, \ldots , 2) \rightarrow \ldots \rightarrow w = (2, \ldots , 2); \]

\[ P_{n+k_1} : x \rightarrow (0, \ldots , 0, 2, 1, \ldots , 1, 2, \ldots , 2) \rightarrow (0, \ldots , 0, 2, 2, 1, \ldots , 1, 2, \ldots , 2) \]
\[ \quad \rightarrow (0, \ldots , 0, 2, \ldots , 2) \rightarrow (2, 0, \ldots , 0, 2, \ldots , 2) \rightarrow \ldots \rightarrow w = (2, \ldots , 2); \]

\[ P_{n+k_1+1} : x \rightarrow (0, \ldots , 0, 0, 1, \ldots , 1, 2, \ldots , 2) \rightarrow (0, \ldots , 0, 0, 0, 1, \ldots , 1, 2, \ldots , 2) \]
\[ \quad \rightarrow \ldots \rightarrow (0, \ldots , 0, 2, \ldots , 2) \rightarrow (0, \ldots , 0, 2, \ldots , 2) \rightarrow \ldots \]
\[ \quad \rightarrow u = (0, \ldots , 0); \]
\[ P_{n+k_1+k_2} : x \rightarrow (0, \ldots, 0, 1, \ldots, 1, 0, 2, \ldots, 2) \rightarrow (0, \ldots, 0, 1, \ldots, 1, 0, 0, 2, \ldots, 2) \]
\[ \rightarrow \cdots \rightarrow (0, \ldots, 0, 1, \ldots, 1, 0, 0, \ldots, 0) \rightarrow (0, \ldots, 0, 1, \ldots, 1, 0, 0, \ldots, 0) \]
\[ \rightarrow \cdots \rightarrow u = (0, \ldots, 0); \]
\[ P_{n+k_1+k_2+1} : x \rightarrow (0, \ldots, 0, 1, \ldots, 1, 2, \ldots, 2) \rightarrow \cdots \rightarrow (0, \ldots, 0, 1, \ldots, 1) \]
\[ \rightarrow (1, 0, \ldots, 0, 1, \ldots, 1) \rightarrow \cdots \rightarrow v = (1, \ldots, 1); \]
\[ \vdots \]
\[ P_{2n} : x \rightarrow (0, \ldots, 0, 1, \ldots, 1, 2, \ldots, 2, 1) \]
\[ \rightarrow (1, 0, \ldots, 0, 1, \ldots, 1, 2, \ldots, 2, 1) \]
\[ \rightarrow \cdots \rightarrow (1, \ldots, 1, 2, \ldots, 2, 1) \rightarrow (1, \ldots, 1, 2, \ldots, 2, 1) \]
\[ \rightarrow \cdots \rightarrow v = (1, \ldots, 1). \]

We have \(|P_i| \leq n, i = 1, \ldots, 2n.\)

**Subcase 2b** \(k_1 = 1, k_2 = 1\) or \(n - k_1 - k_2 = 1.\)

One can carry out the proof in the same way as in Subcase 2a.

Therefore we obtain \(\gamma_{n,2n}(G) \leq 3\) from above.

Thus we can see \(\gamma_{n,2n}(G) = 3.\) The proof of the Theorem is completed. \(\square\)

**References**