Chapter 2

Sobolev Spaces

Sobolev spaces turn out often to be the proper setting in which to apply ideas of functional analysis to get information concerning partial differential equations. Here, we collect a few basic results about Sobolev spaces. A general reference to this topic is Adams [1], Gilbarg-Trudinger [29], or Evans [26].

2.1 Hölder spaces

Assume Ω ⊂ R^n is open and 0 < γ ≤ 1.

Definition 2.1.1
(i) If u : Ω → R is bounded and continuous, we write
\[ \|u\|_{C(\overline{\Omega})} := \sup_{x \in \Omega} |u(x)|. \]

(ii) The γth-Hölder semi-norm of u : Ω → R is
\[ [u]_{C^{0,\gamma}(\overline{\Omega})} := \sup_{x,y \in \Omega, x \neq y} \left\{ \frac{|u(x) - u(y)|}{|x - y|^\gamma} \right\}, \]
and the γth-Hölder norm is
\[ \|u\|_{C^{0,\gamma}(\overline{\Omega})} := \|u\|_{C(\overline{\Omega})} + [u]_{C^{0,\gamma}(\overline{\Omega})}. \]

(iii) The Hölder space
\[ C^{k,\gamma}(\overline{\Omega}) \]
consists of all functions $u \in C^k(\Omega)$ for which the norm
\[ \|u\|_{C^k,\gamma}(\Omega) := \sum_{|\alpha| \leq k} \|D^\alpha u\|_{C(\Omega)} + \sum_{|\alpha| = k} [D^\alpha u]_{C^0,\gamma}(\Omega) \] (2.1.1)
is finite.

**Theorem 2.1.2** The space of functions $C^{k,\gamma}(\Omega)$ is a Banach space.

**Proof.** It is easy to verify that $\| \cdot \|_{C^{k,\gamma}(\Omega)}$ is a norm (Exercise). It suffices to check the completeness of $C^{k,\gamma}(\Omega)$.

1. **Construction of limit functions.**

   Let $\{u_m\}_{m=1}^\infty$ be a Cauchy sequence in $C^{k,\gamma}(\Omega)$, i.e.,
   \[ \|u_m - u_n\|_{C^{k,\gamma}(\Omega)} \to 0, \text{ as } m, n \to +\infty. \]

   Since any Cauchy sequence is a bounded sequence, so $\|u_m\|_{C^{k,\gamma}(\Omega)}$ is uniformly bounded. Particularly, $\|u_m\|_{C(\Omega)}$ is uniformly bounded, that is, $\{u_m(x)\}$ is a uniformly bounded sequence. Furthermore, the boundedness of $\|Du_m\|_{C(\Omega)}$ or $[D^\alpha u_m]_{C^{0,\gamma}(\Omega)}$ implies the equicontinuity of the sequence $\{u_m(x)\}$. The Arzela-Ascoli theorem implies there exists a subsequence, still denoted by $\{u_m\}_{m=1}^\infty$, which converges to a bounded continuous function $u(x)$ uniformly in $\Omega$. The same argument can be repeated for any derivative of $D^\alpha u_m$ of order up to and including $k$. That is, there exist functions $u_\alpha \in C(\Omega)$, such that
   \[ D^\alpha u_m \to u_\alpha, \text{ uniformly in } \Omega, \forall |\alpha| \leq k. \]

2. **$D^\alpha u = u_\alpha$.**

   Deduce by induction, first let $\alpha = (1, 0, \cdots, 0)$, since the series
   \[ u_1(x) + \sum_{j=1}^{+\infty} (u_{j+1} - u_j) = \lim_{m \to \infty} u_{m+1}(x) = u(x), \]

   uniformly in $\Omega$ and the series
   \[ D^\alpha u_1(x) + \sum_{j=1}^{+\infty} (D^\alpha u_{j+1} - D^\alpha u_j) = \lim_{m \to \infty} D^\alpha u_{m+1}(x) = u_\alpha(x), \]
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uniformly in $\Omega$. The theory of series implies that $u(x)$ is differentiable in $\Omega$ and $D^\alpha u = u_\alpha$. The induction method shows that $D^\alpha u = u_\alpha$ is valid for all multi-index $\alpha$ such that $|\alpha| \leq k$.

3. $u \in C^{k,\gamma}(\Omega)$.

It suffices to show that for any multi-index $\alpha$ such that $|\alpha| = k$, there holds $[D^\alpha u]_{C^{0,\gamma}(\Omega)} < +\infty$.

For any $x, y \in \Omega$, $x \neq y$, since $D^\alpha u_m \to D^\alpha u$ uniformly in $\bar{\Omega}$, there exists a $N$ large enough, such that:

$$|D^\alpha u_N(z) - D^\alpha u(z)| \leq \frac{1}{2}|x - y|^\gamma, \; \forall \; z \in \Omega.$$ Calculate:

$$|D^\alpha u(x) - D^\alpha u(y)| \leq |D^\alpha u(x) - D^\alpha u_N(x)| + |D^\alpha u_N(x) - D^\alpha u_N(y)| + |D^\alpha u_N(y) - D^\alpha u(y)|$$

$$\leq |x - y|^\gamma + |x - y|^\gamma \|D^\alpha u_N\|_{C^{0,\gamma}(\Omega)}$$

$$\leq (1 + \sup_m \|D^\alpha u_m\|_{C^{0,\gamma}(\Omega)})|x - y|^\gamma,$$

Thus the boundedness of $\|D^\alpha u_m\|_{C^{0,\gamma}(\Omega)}$ implies $[D^\alpha u]_{C^{0,\gamma}(\Omega)} < +\infty$.

4. $u_m \to u$ in $C^{k,\gamma}(\bar{\Omega})$.

It also suffices to show that for any multi-index $\alpha$ such that $|\alpha| = k$, there holds

$$[D^\alpha u_m - D^\alpha u]_{C^{0,\gamma}(\bar{\Omega})} \to 0, \; \text{as} \; m \to \infty.$$ In fact, for any $x, y \in \Omega$, $x \neq y$, there holds

$$|D^\alpha u_m(x) - D^\alpha u(x) - (D^\alpha u_m(y) - D^\alpha u(y))|$$

$$\leq \limsup_{n \to \infty} |D^\alpha u_m(x) - D^\alpha u_n(x) - (D^\alpha u_m(y) - D^\alpha u_n(y))|$$

$$\leq |x - y|^\gamma \lim_{n \to \infty} \|D^\alpha u_m - D^\alpha u_n\|_{C^{0,\gamma}(\bar{\Omega})}.$$

Since $\{u_m\}_{m=1}^\infty$ is a Cauchy sequence in $C^{k,\gamma}(\bar{\Omega})$, $[D^\alpha u_m - D^\alpha u]_{C^{0,\gamma}(\bar{\Omega})} \leq \lim_{n \to \infty} [D^\alpha u_m - D^\alpha u_n]_{C^{0,\gamma}(\bar{\Omega})} \to 0$ as $m \to 0$. 

\[\Box\]
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2.2 Sobolev spaces

2.2.1 $L^p$ spaces

The $L^p(\Omega)$ space consists of all the function $u : \Omega \to \mathbb{R}$ such that
$$\int_{\Omega} |u|^p \, dx < \infty,$$
where $1 \leq p \leq \infty$; with norm
$$\|u\|_{L^p(\Omega)} = \begin{cases} \left( \int_{\Omega} |u|^p \, dx \right)^{1/p}, & \text{if } 1 \leq p < \infty, \\ \text{esssup}_{\Omega} |u|, & \text{if } p = \infty. \end{cases}$$

In order to deduce some properties of $L^p$ space, let us start from some basic inequalities, which will be used latter.

Lemma 2.2.1 (Young inequality) Let $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, then
$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Proof. Method 1. The mapping $x \mapsto e^x$ is convex, and consequently,
$$ab = e^{\log a + \log b} = e^{\frac{1}{p} \log a^p + \frac{1}{q} \log b^q} \leq \frac{1}{p} e^{\log a^p} + \frac{1}{q} e^{\log b^q} = \frac{a^p}{p} + \frac{b^q}{q}.$$

Method 2. Let $y = x^{p-1}, x \geq 0$, then $x = y^{\frac{1}{p-1}} = y^{q-1}$, Consider the area of the region restricted by $y = 0$, $x = a$, and curve $y = x^{p-1}$ is
$$I = \int_0^a x^{p-1} \, dx = \frac{a^p}{p};$$
and the area of the region restricted by $x = 0$, $y = b$, and curve $x = y^{q-1}$ is
$$II = \int_0^b y^{q-1} \, dy = \frac{b^q}{q};$$
The area of the rectangle $[0, a] \times [0, b]$ is $ab$, Thus $ab \leq I + II \leq \frac{a^p}{p} + \frac{b^q}{q}$, and the equality holds if and only if $b = a^{p-1}$. 

It is easy to obtain the following Young inequality with $\varepsilon$ (Exercise):
Lemma 2.2.2 (Young inequality with $\varepsilon$) Let $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, then
\[
ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \quad a, b > 0.
\]

Lemma 2.2.3 (Hölder inequality) Let $1 \leq p, q \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, then if $u \in L^p(\Omega)$, $v \in L^q(\Omega)$, we have
\[
\int_{\Omega} |uv| \, dx \leq \|u\|_{L^p(\Omega)} \|v\|_{L^q(\Omega)}.
\]

Proof. 1. In the cases where $p = \infty$ or $q = \infty$, it is easy, because there exists a subset $\Omega' \subset \Omega$, with $|\Omega'| = |\Omega|$, such that $\sup_{\Omega'} |u| = \|u\|_{L^\infty(\Omega)}$ or $\sup_{\Omega'} |v| = \|v\|_{L^\infty(\Omega)}$.

2. In the cases where $1 < p, q < \infty$. By the homogeneity of the inequality, we may assume that $\|u\|_{L^p(\Omega)} = \|v\|_{L^q(\Omega)} = 1$. Then the Young inequality implies that
\[
\int_{\Omega} |uv| \, dx \leq \frac{1}{p} \int_{\Omega} |u|^p \, dx + \frac{1}{q} \int_{\Omega} |u|^q \, dx = 1 = \|u\|_{L^p(\Omega)} \|v\|_{L^q(\Omega)}.
\]

By induction argument, it is easy to show the following Generalized Hölder inequality (Exercise):

Lemma 2.2.4 (Generalized Hölder inequality) Let $1 \leq p_1, \cdots, p_m \leq \infty$, $\frac{1}{p_1} + \cdots + \frac{1}{p_m} = 1$, then if $u_k \in L^{p_k}(\Omega)$ for $k = 1, \cdots, m$, we have
\[
\int_{\Omega} |u_1 \cdots u_m| \, dx \leq \prod_{k=1}^{m} \|u_k\|_{L^{p_k}(\Omega)}.
\]

Lemma 2.2.5 (Minkowski inequality) Assume $1 \leq p \leq \infty$, $u, v \in L^p(\Omega)$, then
\[
\|u + v\|_{L^p(\Omega)} \leq \|u\|_{L^p(\Omega)} + \|v\|_{L^p(\Omega)}.
\]
**Proof.** From the Hölder inequality, we have

\[
\|u + v\|_{L^p(\Omega)}^p = \int_\Omega |u + v|^p \, dx \\
\leq \int_\Omega |u + v|^{p-1} (|u| + |v|) \, dx \\
\leq \left( \int_\Omega |u + v|^{(p-1)\frac{p}{p-1}} \, dx \right)^\frac{p-1}{p} \left( \left( \int_\Omega |u|^p \, dx \right)^{1/p} + \left( \int_\Omega |v|^p \, dx \right)^{1/p} \right) \\
= \|u + v\|_{L^p(\Omega)}^{p-1} \left( \|u\|_{L^p(\Omega)} + \|v\|_{L^p(\Omega)} \right).
\]

\[\blacksquare\]

**Lemma 2.2.6 (Interpolation inequality for \(L^p\)-norms)** Assume \(1 \leq s \leq r \leq t \leq \infty\), and \(\frac{1}{r} = \frac{s}{\theta} + \frac{(1-\theta)}{t}\). Suppose also \(u \in L^s(\Omega) \cap L^t(\Omega)\). Then \(u \in L^r(\Omega)\), and

\[
\|u\|_{L^r(\Omega)} \leq \|u\|_{L^s(\Omega)}^\theta \|u\|_{L^t(\Omega)}^{1-\theta}.
\]

**Proof.** From the Hölder inequality, we have

\[
\int_\Omega |u|^r \, dx = \int\int_\Omega |u|^{\theta r} |u|^{(1-\theta)r} \, dx \\
\leq \left( \int_\Omega |u|^{\theta r\frac{s}{\theta}} \, dx \right)^{\frac{\theta}{r}} \left( \int_\Omega |u|^{(1-\theta)r\frac{t}{1-\theta}} \, dx \right)^{\frac{1-\theta}{r}}.
\]

\[\blacksquare\]

**Theorem 2.2.7 (Completeness of \(L^p(\Omega)\))** \(L^p(\Omega)\) is a Banach space if \(1 \leq p \leq \infty\).

**Proof.** It is easy to check that \(\|\cdot\|_{L^p(\Omega)}\) is a norm (Exercise). It suffices to verify the completeness of \(L^p(\Omega)\)

1. Construction of the limit function.
   Assume that \(1 \leq p < \infty\) and let \(\{u_m\}\) be a Cauchy sequence in \(L^p(\Omega)\). There is a subsequence \(\{u_{m_j}\}\) of \(\{u_m\}\) such that

\[
\|u_{m_{j+1}} - u_{m_j}\|_p \leq 1/2^j, \ j = 1, 2 \ldots.
\]
Let \( v_m(x) = \sum_{j=1}^{m} |u_{m+1}(x) - u_m(x)| \), then

\[
\|v_m\|_p \leq \sum_{j=1}^{m} \|u_{m+1} - u_m\|_p \leq \sum_{j=1}^{m} 1/2^j < 1.
\]

Putting \( v(x) = \lim_{m \to \infty} v_m(x) \), which may be infinite for some \( x \), we obtain by the Fatou Lemma that

\[
\int_{\Omega} |v(x)|^p \, dx \leq \liminf_{m \to \infty} \int_{\Omega} |v_m(x)|^p \, dx \leq 1.
\]

Hence \( v(x) < \infty \) a.e. in \( \Omega \) and its absolutely convergence implies the series

\[
u_{m_1} + \sum_{j=1}^{\infty} (u_{m+j+1}(x) - u_m(x)) \tag{2.2.1}
\]

converges to a limit \( u(x) \) a.e. in \( \Omega \). Let \( u(x) = 0 \) whenever it is undefined as the limit of series (2.2.1). Thus

\[
\lim_{j \to \infty} u_{m_j}(x) = u(x) \text{ a.e. in } \Omega.
\]

2. \( u \in L^p(\Omega) \) and \( u_m \to u \) in \( L^p(\Omega) \).

For any \( \varepsilon > 0 \), there exists \( N \) such that if \( m, n \geq N \), then \( \|u_m - u_n\|_p < \varepsilon \). Hence by the Fatou Lemma again, we have

\[
\int_{\Omega} |u_m(x) - u(x)|^p \, dx = \int_{\Omega} \lim_{j \to \infty} |u_m(x) - u_{m_j}(x)|^p \, dx
\]

\[
\leq \liminf_{j \to \infty} \int_{\Omega} |u_m(x) - u_m| \, dx
\]

\[
\leq \varepsilon^p.
\]

If \( m \geq N \), thus \( u = (u - u_m) + u_m \in L^p(\Omega) \), and \( \|u_m - u\|_p \to 0 \) as \( m \to \infty \).

3. In the case where \( p = \infty \), let \( \{u_m\} \) be a Cauchy sequence in \( L^\infty(\Omega) \), then there exists a subset \( A \subset \Omega \), with \( |A| = 0 \), such that if \( x \in \Omega \setminus A \), then for \( n, m = 1, \ldots \), there holds

\[
|u_m(x)| \leq \|u_m\|_\infty, \quad |u_m(x) - u_n(x)| \leq \|u_m - u_n\|_\infty.
\]

That is, \( u_m(x) \) is uniformly bounded in \( \Omega \setminus A \), and \( u_m(x) \) is a Cauchy sequence for any \( x \in \Omega \setminus A \), thus \( u_m(x) \) converges uniformly on \( \Omega \setminus A \) to
a bounded function \( u(x) \). Set \( u(x) = 0 \) if \( x \in A \). We have \( u \in L^\infty(\Omega) \) and \( \|u_m - u\|_\infty \to 0 \) as \( m \to \infty \). \( \square \)

**Theorem 2.2.8 (Riesz Representation Theorem for \( L^p(\Omega) \))** Let \( 1 < p < \infty \) and \( L \in [L^p(\Omega)]' \). Then there exists \( v \in L^{p'}(\Omega) \), \( \frac{1}{p} + \frac{1}{p'} = 1 \) such that for all \( u \in L^p(\Omega) \), there holds

\[
L(u) = \int_\Omega u(x)v(x) \, dx.
\]

Moreover, \( \|v\|_{L^{p'}(\Omega)} = \|L\|_{[L^p(\Omega)]'} \), that is,

\[
[L^p(\Omega)]' \cong L^{p'}(\Omega).
\]

**Theorem 2.2.9 (Riesz Representation Theorem for \( L^1(\Omega) \))** Let \( L \in [L^1(\Omega)]' \). Then there exists \( v \in L^\infty(\Omega) \) such that for all \( u \in L^1(\Omega) \), there holds

\[
L(u) = \int_\Omega u(x)v(x) \, dx.
\]

Moreover, \( \|v\|_{L^\infty(\Omega)} = \|L\|_{[L^1(\Omega)]'} \), that is,

\[
[L^1(\Omega)]' \cong L^\infty(\Omega).
\]

**Theorem 2.2.10 (Reflexivity of \( L^p \))** \( L^p(\Omega) \) is reflexive if and only if \( 1 < p < \infty \).

**Theorem 2.2.11** \( C_0^\infty(\Omega) \) is dense in \( L^p(\Omega) \) if \( 1 \leq p < \infty \).

**Theorem 2.2.12 (Separability of \( L^p(\Omega) \))** \( L^p(\Omega) \) is separable \( 1 \leq p < \infty \).

### 2.2.2 Weak derivatives

Let \( C_0^\infty(\Omega) \) denotes the space of infinitely differentiable functions \( \phi : \Omega \to \mathbb{R} \), with compact support in \( \Omega \). We will call a function \( \phi \) belonging to \( C_0^\infty(\Omega) \) a test function.
Motivation for definition of weak derivatives. Assume we are given a function \( u \in C^1(\Omega) \). Then if \( \varphi \in C_0^\infty(\Omega) \), we see from the integration by parts that
\[
\int_\Omega u \phi_x \, dx = -\int_\Omega u x_i \phi \, dx \quad (i = 1, \ldots, n).
\] (2.2.2)

More general now, if \( k \) is a positive integer, \( u \in C^k(\Omega) \), and \( \alpha = (\alpha_1, \cdots, \alpha_n) \) is a multi-index of order \( |\alpha| = \alpha_1 + \cdots + \alpha_n = k \), then
\[
\int_\Omega u(D^\alpha \phi) \, dx = (-1)^{|\alpha|} \int_\Omega (D^\alpha u) \phi \, dx.
\] (2.2.3)

We next examine that formula (2.2.3) is valid for \( u \in C^k(\Omega) \), and ask whether some variant of it might still be true even if \( u \) is not \( k \) times continuously differentiable. Note that the left-hand side of (2.2.3) make sense if \( u \) is only locally summable: the problem is rather that if \( u \) is not \( C^k \), then the expression \( D^\alpha u \) on the right-hand side of (2.2.3) has no obvious meaning. We overcome this difficulty by asking if there exists a locally summable function \( v \) for which formula (2.2.3) is valid, with \( v \) replacing \( D^\alpha u \):

**Definition 2.2.13** Suppose \( u, v \in L^1_{\text{loc}}(\Omega) \), and \( \alpha \) is a multi-index. We say that \( v \) is the \( \alpha \)-th weak partial derivative of \( u \), written
\[
D^\alpha u = v,
\]
provided that
\[
\int_\Omega u D^\alpha \phi \, dx = (-1)^{|\alpha|} \int_\Omega (D^\alpha u) \phi \, dx
\] (2.2.4)
for all test functions \( \varphi \in C_0^\infty(\Omega) \).

**Lemma 2.2.14 (Uniqueness of weak derivative)** The weak partial derivative of \( u \), if it exists, is uniquely defined up to a set of measure zero.

**Example 2.2.15** Let \( n = 1 \), \( \Omega = (0, 2) \), and
\[
u(x) = \begin{cases} x, & \text{if } 0 < x \leq 1, \\ 1, & \text{if } 1 \leq x < 2. \end{cases}
\]
Define
\[ v(x) = \begin{cases} 
1, & \text{if } 0 < x \leq 1, \\
0, & \text{if } 1 \leq x < 2.
\end{cases} \]

From the definition 2.2.13, \( u' = v \). (Exercise)

**Example 2.2.16** Let \( n = 1, \ \Omega = (0, 2), \) and
\[ u(x) = \begin{cases} 
x, & \text{if } 0 < x \leq 1, \\
2, & \text{if } 1 \leq x < 2.
\end{cases} \]

We assert \( u' \) does not exist in the weak sense. (Exercise)

### 2.2.3 Definition of Sobolev spaces

Fix \( 1 \leq p \leq \infty \) and let \( k \) be a nonnegative integer. We define now certain function spaces, whose members have weak derivatives of various orders lying in various \( L^p \) spaces.

**Definition 2.2.17**

i) The Sobolev space
\[ W^{k,p}(\Omega) \]
consists of all locally summable functions \( u : \Omega \to \mathbb{R} \) such that for each multi-index \( \alpha \) with \( |\alpha| \leq k \), \( D^\alpha u \) exists in the weak sense and belongs to \( L^p(\Omega) \).

ii) If \( u \in W^{k,p}(\Omega) \), we define its norm to be
\[
\|u\|_{W^{k,p}(\Omega)} := \begin{cases} 
\left( \sum_{|\alpha| \leq k} \int_{\Omega} |D^\alpha u|^p \, dx \right)^{1/p}, & (1 \leq p < \infty), \\
\sum_{|\alpha| \leq k} \text{ess sup}_{\Omega} |D^\alpha u|, & (p = \infty).
\end{cases}
\]

**Definition 2.2.18 (Convergence in \( W^{k,p}(\Omega) \))**

i) Let \( \{u_m\}_{m=1}^\infty, \ u \in W^{k,p}(\Omega) \). We say \( u_m \) converges (strongly) to \( u \) in \( W^{k,p}(\Omega) \), written
\[
u_m \rightarrow u \text{ in } W^{k,p}(\Omega),\]
provided that
\[
\lim_{m \to \infty} \|u_m - u\|_{W^{k,p}(\Omega)} = 0.
\]
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ii) We write

\[ u_m \rightharpoonup u \text{ in } W^{k,p}(\Omega) \]

to mean

\[ u_m \rightharpoonup u \text{ in } W^{k,p}(U), \]

for each \( U \subset \subset \Omega \).

Definition 2.2.19 We denote by

\[ W_0^{k,p}(\Omega) \]

the closure of \( C_0^\infty(\Omega) \) in \( W^{k,p}(\Omega) \).

Remark 2.2.20 (i) If \( p = 2 \), we usually write

\[ H^k(\Omega) = W^{k,2}(\Omega), \quad H_0^k(\Omega) = W_0^{k,2}(\Omega), \quad (k = 0,1,\ldots). \]

The letter \( H \) is used, since - as we will see - \( H^k(\Omega) \) is a Hilbert space. Note that \( H^0(\Omega) = L^2(\Omega) \).

(ii) We henceforth identity functions in \( W^{k,p}(\Omega) \) which agree almost everywhere.

Example 2.2.21 Let \( \Omega = B(0,1) \), the open unit ball in \( \mathbb{R}^n \), and

\[ u(x) = |x|^{-\alpha} \quad (x \in \Omega, x \neq 0), \quad \alpha > 0. \]

Then \( u \in W^{1,p}(\Omega) \) if and only if \( 0 < (\alpha + 1)p < n \). In particular \( u \notin W^{1,p}(\Omega) \) for each \( p \geq n \). (Exercise)

Example 2.2.22 Let \( \{r_k\}_{k=1}^\infty \) be a countable, dense subset of \( \Omega = B(0,1) \in \mathbb{R}^n \). Set

\[ u(x) = \sum_{k=1}^\infty \frac{1}{2^k} |x - r_k|^{-\alpha} \quad (x \in \Omega). \]

Then \( u \in W^{1,p}(\Omega) \) if and only if \( (\alpha + 1)p < n \). If \( 0 < (\alpha + 1)p < n \), we see that \( u \) belongs to \( W^{1,p}(\Omega) \) and yet is unbounded on each open subset of \( \Omega \). (Exercise)

Theorem 2.2.23 (Properties of weak derivatives) Assume \( u, v \in W^{k,p}(\Omega), \; |\alpha| \leq k \). Then
(i) $D^\alpha u \in W^{k-|\alpha|,p}(\Omega)$ and $D^\beta(D^\alpha u) = D^\alpha(D^\beta u) = D^{\alpha+\beta} u$ for all multi-indices $\alpha$, $\beta$ with $|\alpha + \beta| \leq k$;

(ii) For each $\lambda$, $\mu \in \mathbb{R}$, $\lambda u + \mu v \in W^{k,p}(\Omega)$ and $D^\alpha(\lambda u + \mu v) = \lambda D^\alpha u + \mu D^\alpha v$, $|\alpha| \leq k$;

(iii) If $\Omega'$ is an open subset of $\Omega$, then $u \in W^{k,p}(\Omega')$;

(iv) If $\xi \in C_0^\infty(\Omega)$, then $\xi u \in W^{k,p}(\Omega)$ and $D^\alpha(\xi u) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^\beta \xi D^{\alpha-\beta} u$ (Leibniz formula),

where $\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha - \beta)!}$.

**Theorem 2.2.24 (Sobolev spaces as function spaces)** For each $k = 0, 1, \cdots$ and $1 \leq p \leq \infty$, Sobolev space $W^{k,p}(\Omega)$ is a Banach space; $W^{k,p}(\Omega)$ is reflexive if and only if $1 < p < \infty$; For $1 \leq p < \infty$, the subspace $W^{k,p} \cap C^\infty(\Omega)$ is dense in $W^{k,p}(\Omega)$, thus $W^{k,p}(\Omega)$ is separable; Moreover, $W^{k,2}(\Omega)$ is a Hilbert space with scalar product

$$(u, v)_{W^{k,2}} = \sum_{\alpha \leq k} \int_{\Omega} D^\alpha u D^\alpha v \, dx.$$ 

**Proof.**

1. It is easy to check that $\| \cdot \|_{W^{k,p}(\Omega)}$ is a norm (Exercise). Next to show that $W^{k,p}(\Omega)$ is complete.

2. Construction of limit functions.

Assume $\{u_m\}_{m=1}^\infty$ is a Cauchy sequence in $W^{k,p}(\Omega)$. Then for each $|\alpha| \leq k$, $\{D^\alpha u_m\}_{m=1}^\infty$ is a Cauchy sequence in $L^p(\Omega)$. Since $L^p(\Omega)$ is complete, there exist functions $u_\alpha \in L^p(\Omega)$, such that

$$D^\alpha u_m \to u_\alpha, \text{ in } L^p(\Omega),$$

for each $|\alpha| \leq k$. In particular,

$$u_m \to u_{(0,\cdots,0)} =: u \text{ in } L^p(\Omega).$$

3. $u \in W^{k,p}(\Omega)$ and $D^\alpha u = u_\alpha$, $|\alpha| \leq k$. 


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In fact, let $\psi \in C^\infty_0(\Omega)$, then

$$\int_\Omega u D^\alpha \psi \, dx = \lim_{m \to \infty} \int_\Omega u_m D^\alpha \psi \, dx$$

$$= \lim_{m \to \infty} (-1)^{|\alpha|} \int_\Omega (D^\alpha u_m) \psi \, dx$$

$$= \int_\Omega u_\alpha \psi \, dx.$$  

4. $u_m \to u$ in $W^{k,p}(\Omega)$.

Since $D^\alpha u_m \to D^\alpha u$ in $L^p(\Omega)$ for all $|\alpha| \leq k$, it follows that $u_m \to u$ in $W^{k,p}(\Omega)$.

5. Regard $W^{k,p}(\Omega)$ as a closed subspace of a Cartesian product space of spaces $L^p(\Omega)$: Let $N = N(n, k) = \sum_{0 \leq |\alpha| \leq k} 1$ be the number of multi-indexes $\alpha$ satisfying $0 \leq |\alpha| \leq k$. For $1 \leq p \leq \infty$, let $L^p_N = \prod_{j=1}^N L^p(\Omega)$, the product norm of $u = (u_1, \cdots, u_N)$ in $L^p_N$ being given by

$$\|u\|_{L^p_N} = \begin{cases} \left( \sum_{j=1}^N \|u_j\|_p^p \right)^{1/p}, & \text{if } 1 \leq p < \infty, \\ \max_{1 \leq j \leq N} \|u_j\|_\infty, & \text{if } p = \infty. \end{cases}$$

Since a closed subspace of a reflexive (Resp. separable, uniformly convex) Banach space is also reflexive (Resp. separable, uniformly convex), thus $W^{k,p}(\Omega)$ is reflexive if and only if $1 < p < \infty$; and $W^{k,p}(\Omega)$ is separable if $1 \leq p < \infty$. Furthermore, $(C^\infty(\Omega))^N$ is dense in $L^p_N$ and thus in $W^{k,p}(\Omega)$ if $1 \leq p < \infty$.

6. If $p = 2$, define the scalar product as

$$(u, v)_{W^{k,2}} = \sum_{|\alpha| \leq k} \int_\Omega D^\alpha u D^\alpha v \, dx.$$  

It is easy to verify $W^{k,2} = H^k$ is a Hilbert space.

2.2.4 Inequalities

Theorem 2.2.25 (Gagliardo-Nirenberg-Sobolev inequality) Let $1 \leq p < n$, $p^* = \frac{np}{n-p}$. There exists a constant $C$, depending only on
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$p$ and $n$, such that

$$||u||_{L^p(R^n)} \leq C ||Du||_{L^p(R^n)}, \quad (2.2.5)$$

for all $u \in C^1_0(R^n)$.

**Proof.** 1. First assume $p = 1$.

Since $u$ has compact support, for each $i = 1, 2 \cdots, n$ and $x \in R^n$ we have

$$u(x) = \int_{-\infty}^{x_i} u_x(x_1, \cdots, x_{i-1}, y_i, x_{i+1}, \cdots, x_n) dy_i,$$

and so

$$|u(x)| \leq \int_{-\infty}^{+\infty} |Du(x_1, \cdots, x_{i-1}, y_i, x_{i+1}, \cdots, x_n)| dy_i, \ i = 1, 2 \cdots, n.$$

Consequently

$$|u(x)|^{\frac{n}{n-1}} \leq \prod_{i=1}^{n} \left( \int_{-\infty}^{+\infty} |Du(x_1, \cdots, x_{i-1}, y_i, x_{i+1}, \cdots, x_n)| dy_i \right)^{\frac{1}{n-1}}.$$

Integrate this inequality with respect to $x_1$:

$$\int_{-\infty}^{+\infty} |u(x)|^{\frac{n}{n-1}} dx_1$$

$$\leq \int_{-\infty}^{+\infty} \prod_{i=1}^{n} \left( \int_{-\infty}^{+\infty} |Du(x_1, \cdots, x_{i-1}, y_i, x_{i+1}, \cdots, x_n)| dy_i \right)^{\frac{1}{n-1}} dx_1$$

$$= \left( \int_{-\infty}^{+\infty} |Du(y_1, x_2, \cdots, x_n)| dy_1 \right)^{\frac{1}{n-1}}$$

$$\cdot \prod_{i=2}^{n} \left( \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |Du(x_1, \cdots, x_{i-1}, y_i, x_{i+1}, \cdots, x_n)| dy_i \right)^{\frac{1}{n-1}} dx_1$$

$$\leq \left( \int_{-\infty}^{+\infty} |Du(y)| dy_1 \right)^{\frac{1}{n-1}} \left( \prod_{i=2}^{n} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |Du(x_1, y_1)| dy_1 \right)^{\frac{1}{n-1}}, \quad (2.2.6)$$

the last inequality follows from the Generalized Hölder inequality.

Now integrate (2.2.6) with respect to $x_2$:

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |u(x)|^{\frac{n}{n-1}} dx_1 dx_2$$

$$\leq \left( \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |Du(x_1)| dx_1 dx_2 \right)^{\frac{1}{n-1}} \int_{-\infty}^{+\infty} \prod_{i=1}^{n} I_i^{\frac{1}{n-1}} dx_2.$$
where
\[ I_1 = \int_{-\infty}^{+\infty} |Du| dy_1, \quad I_i = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |Du| dx_1 dy_i, \quad i = 3, \ldots, n. \]

Applying once more the Generalized Hölder inequality, we find
\[
\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left| u(x) \right|^{\frac{n}{n-1}} dx_1 dx_2 \\
\leq \left( \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |Du| dx_1 dy_2 \right)^{\frac{1}{n-1}} \left( \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |Du| dx_1 dx_2 \right)^{\frac{1}{n-1}} \\
\cdot \prod_{i=3}^{n} \left( \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |Du| dx_1 dx_2 dy_i \right)^{\frac{1}{n-1}}.
\]

We continue by integrating with respect to \( x_3, \ldots, x_n \), eventually to find
\[
\int_{R^n} |u(x)|^{\frac{n}{n-1}} dx \\
\leq \prod_{i=1}^{n} \left( \int_{-\infty}^{+\infty} \left| \int_{-\infty}^{+\infty} |Du| dx_1 \cdots dx_n \right|^{\frac{1}{n-1}} \right) \\
= \left( \int_{R^n} |Du| dx \right)^{\frac{n}{n-1}}.
\] (2.2.7)

This is conclusion for \( p = 1 \).

2. Consider now the case that \( 1 < p < n \). We apply (2.2.7) to \( v := |u|^\gamma \), where \( \gamma > 1 \) is to be selected. Then
\[
\left( \int_{R^n} |u(x)|^{\frac{n}{n-1}} dx \right)^{\frac{n-1}{n}} \leq \int_{R^n} |D|u|^\gamma| \ dx = \gamma \int_{R^n} |u|^{\gamma-1}|Du| \ dx \\
\leq \gamma \left( \int_{R^n} |u|^{(\gamma-1)\frac{p}{p-1}} dx \right)^{\frac{p-1}{p}} \left( \int_{R^n} |Du|^p \right)^{\frac{1}{p}}.
\] (2.2.8)

We choose \( \gamma \) so that \( \gamma \frac{n}{n-1} = (\gamma - 1) \frac{p}{p-1} \). That is, we get
\[
\gamma = \frac{p(n-1)}{n-p} > 1,
\]
in which case \( \gamma \frac{n}{n-1} = (\gamma - 1) \frac{p}{p-1} = \frac{np}{n-p} = p^* \). Thus, (2.2.8) becomes (2.2.5).
Remark 2.2.26 The exponent \( p^* = \frac{np}{n-p} \) is called critical Sobolev exponent, because it is the unique exponent \( q = p^* \) to make the following inequality:
\[
\|u\|_{L^q(\mathbb{R}^n)} \leq C\|Du\|_{L^p(\mathbb{R}^n)}
\]
be true for all \( u \in C^1_0(\mathbb{R}^n) \).

Proof. For any \( u \in C^1_0(\mathbb{R}^n) \) and \( \lambda > 0 \), define the scaled function \( u_\lambda(x) = u(\lambda x) \). Direct calculations show that
\[
\|u_\lambda\|_{L^q(\mathbb{R}^n)} = \lambda^{-\frac{n}{q}}\|u\|_{L^q(\mathbb{R}^n)}
\]
and
\[
\|Du_\lambda\|_{L^p(\mathbb{R}^n)} = \lambda^{1-\frac{n}{p}}\|Du\|_{L^p(\mathbb{R}^n)}.
\]
If the inequality in remark is true for all \( u \in C^1_0(\mathbb{R}^n) \), replacing \( u \) by \( u_\lambda \), there holds
\[
\|u\|_{L^q(\mathbb{R}^n)} \leq C\lambda^{1-\frac{n}{p} + \frac{n}{q}}\|Du\|_{L^p(\mathbb{R}^n)}.
\]
If \( 1 - \frac{n}{p} + \frac{n}{q} \neq 0 \), upon sending \( \lambda \) to either 0 or \( \infty \), we will get \( C = \infty \).

Theorem 2.2.27 (Poincare inequality for \( W^{1,p}_0(\Omega) \)) Assume \( \Omega \) is a bounded, open subset of \( \mathbb{R}^n \). Suppose \( u \in W^{1,p}_0(\Omega) \) for some \( 1 \leq p < n \). Then there exists a constant \( C \), depending only on \( p, n \) and \( \Omega \), such that
\[
\|u\|_{L^q(\Omega)} \leq C\|Du\|_{L^p(\Omega)}, \quad (2.2.9)
\]
for each \( q \in [1, p^*] \), the constant \( C \) depending only on \( p, q, n \) and \( \Omega \).

Proof. Since \( u \in W^{1,p}_0(\Omega) \), then exists a sequence of functions \( \{u_m\}_{m=1}^\infty \subset C^\infty(\Omega) \) converging to \( u \) in \( W^{1,p}_0(\Omega) \). We extend each function \( u_m(x) \) to be 0 on \( \mathbb{R}^n \setminus \Omega \) and apply Theorem 2.2.25 to discover \( \|u\|_{L^p(\Omega)} \leq C\|Du\|_{L^p(\Omega)} \). Since \( |\Omega| < \infty \), the Hölder inequality implies \( \|u\|_{L^q(\Omega)} \leq C\|u\|_{L^{p^*}(\Omega)} \) if \( 1 \leq q \leq p^* \).

Remark 2.2.28 In view of Theorem 2.2.27, on \( W^{1,p}_0(\Omega) \), the norm \( \|Du\|_{L^p(\Omega)} \) is equivalent to \( \|u\|_{W^{1,p}_0(\Omega)} \) if \( \Omega \) is bounded.
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Theorem 2.2.29 (Morrey inequality) Assume $n \leq p < \infty$, $\gamma := 1 - \frac{n}{p}$. Then there exists a constant $C$, depending only on $p$ and $n$, such that

$$
\|u\|_{C^{0,\gamma}({\mathbb R}^n)} \leq C \|u\|_{W^{1,p}({\mathbb R}^n)},
$$

(2.2.10)

for all $u \in C^1({\mathbb R}^n)$.

Proof. 1. First choose any ball $B(x, r) \subset {\mathbb R}^n$. We claim there exists a constant $C$, depending only on $n$, such that

$$
\int_{B(x, r)} |u(x) - u(y)| dy \leq C \int_{B(x, r)} \frac{|Du(y)|}{|y - x|^{n-1}} dy.
$$

(2.2.11)

To prove this, fix any point $w \in \partial B(0, 1)$. Then if $0 < s < r$, there holds

$$
|u(x + sw) - u(x)| = \left| \int_0^s \frac{d}{dt} u(x + tw) dt \right|
$$

$$
= \left| \int_0^s Du(x + tw) \cdot w dt \right|
$$

$$
\leq \int_0^s |Du(x + tw)| dt.
$$

Hence

$$
\int_{\partial B(0,1)} |u(x + sw) - u(x)| dS \leq \int_0^s \int_{\partial B(0,1)} |Du(x + tw)| dS dt
$$

$$
= \int_0^s \int_{\partial B(0,1)} |Du(x + tw)| \frac{t^{n-1}}{t^{n-1}} dS dt.
$$

Let $y = x + tw$, so that $t = |x - y|$. Then converting from polar coordinates, we have

$$
\int_{\partial B(0,1)} |u(x + sw) - u(x)| dS \leq \int_{B(x, s)} \frac{|Du(y)|}{|x - y|^{n-1}} dy
$$

$$
\leq \int_{B(x, r)} \frac{|Du(y)|}{|y - x|^{n-1}} dy.
$$

Multiply by $s^{n-1}$ and integrate from 0 to $r$ with respect to $s$:

$$
\int_{B(x, r)} |u(x) - u(y)| dy \leq \frac{r^n}{n} \int_{B(x, r)} \frac{|Du(y)|}{|y - x|^{n-1}} dy.
$$
It follows (2.2.11).

2. \[ \sup_{\mathbb{R}^n} |u| \leq C \|u\|_{W^{1,p}(\mathbb{R}^n)}. \]

Now fix \( x \in \mathbb{R}^n \). We apply (2.2.11) as follows:

\[
|u(x)| \leq \int_{B(x,1)} |u(x) - u(y)| \, dy + \int_{B(x,1)} |u(y)| \, dy \\
\leq C \int_{B(x,1)} \frac{|Du(y)|}{|y-x|^{n-p}} \, dy + C\|u\|_{L^p(B(x,1))} \\
\leq C\|Du\|_{L^p(\mathbb{R}^n)} \left( \int_{B(x,1)} \frac{dy}{|x-y|^{(n-1)p/p}} \right)^{\frac{p-1}{p}} + C\|u\|_{L^p(\mathbb{R}^n)} \\
\leq C\|u\|_{W^{1,p}(\mathbb{R}^n)}. \quad (2.2.12)
\]

The last estimate holds since \( p > n \) implies \((n-1)p/p - 1 < n\); so that

\[
\int_{B(x,1)} \frac{dy}{|x-y|^{(n-1)p/p}} < \infty.
\]

As \( x \in \mathbb{R}^n \) is arbitrary, (2.2.12) implies

\[ \sup_{\mathbb{R}^n} |u| \leq C \|u\|_{W^{1,p}(\mathbb{R}^n)}. \quad (2.2.13) \]

3. \[ [u]_{C^{0,1-n/p}(\mathbb{R}^n)} \leq C\|Du\|_{L^p(\mathbb{R}^n)}. \]

Choose any two points \( x, y \in \mathbb{R}^n \) and write \( r := |x-y| \). Let \( W := B(x,r) \cap B(y,r) \). Then

\[
|u(x) - u(y)| \leq \int_W |u(x) - u(z)| \, dz + \int_W |u(y) - u(z)| \, dz. \quad (2.2.14)
\]

On the other hand, (2.2.11) allows us to estimate

\[
\int_W |u(x) - u(z)| \, dz \leq C \int_{B(x,r)} |u(x) - u(z)| \, dz \\
\leq C\|Du\|_{L^p(\mathbb{R}^n)} \left( \int_{B(x,1)} \frac{dz}{|x-z|^{(n-1)p/p}} \right)^{\frac{p-1}{p}} \\
\leq C \left( r^{n-(n-1)p/p} \right)^{\frac{p-1}{p}} \|Du\|_{L^p(\mathbb{R}^n)} \\
= C r^{1-\frac{n}{p}} \|Du\|_{L^p(\mathbb{R}^n)}. \quad (2.2.15)
\]
Similarly, there holds
\[ \int_W |u(y) - u(z)| \, dz \leq Cr^{1-\frac{n}{p}} \|Du\|_{L^p(R^n)}. \]
Substituting this estimate and (2.2.15) into (2.2.14) yields
\[ |u(x) - u(y)| \leq C|x - y|^{1-n/p} \|Du\|_{L^p(R^n)}. \]
Thus
\[ [u]_{C^{0,1-n/p}(R^n)} = \sup_{x \neq y} \left\{ \frac{|u(x) - u(y)|}{|x - y|^{1-n/p}} \right\} \leq C \|Du\|_{L^p(R^n)}. \]
This inequality and (2.2.13) complete the proof.

2.2.5 Embedding theorems and Trace theorems

**Definition 2.2.30** Let \((X, \| \cdot \|_X), (Y, \| \cdot \|_Y)\) be Banach spaces. \(X\) is (continuously) embedded into \(Y\), denoted \(X \hookrightarrow Y\), if there exists an injective linear map \(i : X \rightarrow Y\) and a constant \(C\) such that
\[ \|i(x)\|_Y \leq C\|x\|_X, \forall x \in X. \]
\(X\) is compactly embedded into \(Y\), denoted \(X \hookrightarrow\hookrightarrow Y\), if \(i\) maps bounded subsets of \(X\) into precompact subsets of \(Y\).

**Theorem 2.2.31** (Exercise)
(i) For \(\Omega \subset R^n\) with Lebesgue measure \(L^n(\Omega) < \infty\), \(1 \leq p < q \leq \infty\), we have \(L^q(\Omega) \hookrightarrow L^p(\Omega)\). This ceases to be true if \(L^n(\Omega) = \infty\);
(ii) Suppose \(\Omega\) is a precompact domain in \(R^n\), let \(m = 0, 1, \ldots\), and \(0 \leq \alpha < \beta \leq 1\). Then \(C^{m,\beta}(\overline{\Omega}) \hookrightarrow C^{m,\alpha}(\overline{\Omega})\) compactly.

**Theorem 2.2.32** (Sobolev embedding theorem) Let \(\Omega \subset R^n\) be a bounded domain with Lipschitz boundary, \(k \in N, 1 \leq p \leq \infty\). Then the following hold:
(i) If \(kp < n\), then \(W^{k,p}(\Omega) \hookrightarrow L^q(\Omega)\) for \(1 \leq q \leq p^* = \frac{np}{n-kp}\); the embedding is compact, if \(q < \frac{np}{n-kp}\);
(ii) If $0 \leq m < k - \frac{n}{p} < m + 1$, then $W^{k,p}(\Omega) \hookrightarrow C^{m,\alpha}(\bar{\Omega})$, for $0 \leq \alpha \leq k - m - \frac{n}{p}$; the embedding is compact, if $\alpha < k - m - \frac{n}{p}$.

**Proof.** 1. The embedding of the case where $kp < n$.

Assume that $kp < n$ and $u \in W^{k,p}(\Omega)$. Then $D^{\alpha}u \in L^p(\Omega)$ for all $|\alpha| = k$. The Gagliardo-Nirenberg-Sobolev inequality implies

$$\|D^\beta u\|_{L^p(\Omega)} \leq C\|u\|_{W^{k,p}(\Omega)}, \text{ for } |\beta| = k - 1, \quad p_1 = \frac{np}{n - p},$$

so $u \in W^{k-1,p_1}(\Omega)$. Similarly, we find $u \in W^{k-2,p_2}(\Omega)$, where $p_2 = \frac{np}{n - 2p}$. Continuing, we eventually discover after $k$ steps that $W^{0,p_k}(\Omega) = L^{p_k}(\Omega)$, for $p_k = \frac{np}{n - k - p}$, and

$$\|u\|_{L^{p_k}(\Omega)} \leq C\|u\|_{W^{k,p}(\Omega)}.$$

Since $\Omega$ is bounded, thus for any $1 \leq q \leq p^* = \frac{np}{n - kp}$, $W^{k,p}(\Omega) \hookrightarrow L^q(\Omega)$.

2. The compactness of embedding.

That is to show that if $\{u_m\}_{m=1}^\infty$ is a bounded sequence in $W^{k,p}(\Omega)$, there exists a subsequence $\{u_{m_j}\}_{j=1}^\infty$ which converges in $L^q(\Omega)$. For simplicity, we just prove the case $k = 1$. By induction argument, one get the proof of general case.

By extension argument, without loss of generality assume that $\Omega = \mathbb{R}^n$ and the functions $\{u_m\}_{m=1}^\infty$ all have compact support in some bounded open set $U \subset \mathbb{R}^n$, moreover,

$$\sup_m \|u_m\|_{W^{k,p}(U)} < \infty. \quad (2.2.16)$$

Let us first study the smooth functions:

$$u_m^\varepsilon := \eta_{\varepsilon} * u_m \quad (\varepsilon > 0, m = 1, 2, \cdots),$$

where $\eta_{\varepsilon} := \frac{1}{\varepsilon^n} \eta(x/\varepsilon)$ denotes the usual mollifier and $\eta$ satisfies

$$\eta \in C_0^\infty(\mathbb{R}^n), \quad \eta \geq 0, \quad \int_{\mathbb{R}^n} \eta(x) \, dx = 1.$$

A concrete example of $\eta$ is

$$\eta(x) := \begin{cases} \text{Cexp}(\frac{1}{|x|^2 - 1}), & \text{if } |x| < 1, \\ 0, & \text{if } |x| \geq 1, \end{cases}$$
and constant $C > 0$ is selected so that $\int_{\mathbb{R}^n} \eta(x) \, dx = 1$. We may suppose that the functions $\{u_m^\varepsilon\}_{m=1}^\infty$ all have support in $U$ as well.

3. Claim

\[ u_m^\varepsilon \rightarrow u_m \quad \text{in} \quad L^q(U) \quad \text{as} \quad \varepsilon \rightarrow 0, \quad \text{uniformly in} \quad m. \quad \tag{2.2.17} \]

To prove this claim, we first note that if $u_m$ is smooth, then

\[
\begin{align*}
    u_m^\varepsilon(x) - u_m(x) &= \int_{B(0,1)} \eta(y) (u_m(x - \varepsilon y) - u_m(x)) \, dy \\
    &= \int_{B(0,1)} \eta(y) \int_0^1 \frac{d}{dt} u_m(x - \varepsilon ty) \, dt \, dy \\
    &= -\varepsilon \int_{B(0,1)} \eta(y) \int_0^1 D u_m(x - \varepsilon ty) \cdot y \, dt \, dy.
\end{align*}
\]

Thus

\[
\int_U |u_m^\varepsilon(x) - u_m(x)| \, dx \leq \varepsilon \int_{B(0,1)} \eta(y) \int_0^1 \int_U |D u_m(x - \varepsilon ty)| \, dx \, dt \, dy \\
\leq \varepsilon \int_U |D u_m(z)| \, dz.
\]

By approximation this estimate holds if $u_m \in W^{1,p}(U)$. Hence

\[
\|u_m^\varepsilon - u_m\|_{L^1(U)} \leq \varepsilon \|D u_m\|_{L^1(U)} \leq \varepsilon \|D u_m\|_{L^p(U)},
\]

the latter inequality holds since $U$ is bounded. Owing to (2.2.16) we thereby discover

\[ u_m^\varepsilon \rightarrow u_m \quad \text{in} \quad L^1(U) \quad \text{as} \quad \varepsilon \rightarrow 0, \quad \text{uniformly in} \quad m. \quad \tag{2.2.18} \]

But then since $1 \leq q < p^*$, we using the Interpolation inequality for $L^p$-norm that

\[
\|u_m^\varepsilon - u_m\|_{L^q(U)} \leq \|u_m^\varepsilon - u_m\|_{L^1(U)}^{\theta} \|u_m^\varepsilon - u_m\|_{L^{p^*}(U)}^{1-\theta},
\]

where $\frac{1}{q} = \theta + \frac{(1-\theta)}{p^*}$, $0 < \theta \leq 1$. Consequently, (2.2.16) and the Gagliardo-Nirenberg-Sobolev inequality imply

\[
\|u_m^\varepsilon - u_m\|_{L^q(U)} \leq C \|u_m^\varepsilon - u_m\|_{L^1(U)}^{\theta}.
\]
whence assertion (2.2.17) follows from (2.2.18).

4. Next we claim

\[
\begin{cases}
\text{for each fixed } \varepsilon > 0, \text{ the sequence } \{u_m^\varepsilon\}_{m=1}^\infty \\
\text{is uniformly bounded and equi-continuous.}
\end{cases}
\] (2.2.19)

Indeed, if \( x \in \mathbb{R}^n \), then

\[
|u_m^\varepsilon(x)| \leq \int_{B(x,\varepsilon)} \eta_\varepsilon(x-y)|u_m(y)|dy \\
\leq \|\eta_\varepsilon\|_{L^\infty(\mathbb{R}^n)}\|u_m\|_{L^1(U)} \leq \frac{C}{\varepsilon^n} < \infty,
\]

for \( m = 1, 2, \ldots \). Similarly

\[
|D u_m^\varepsilon(x)| \leq \int_{B(x,\varepsilon)} |D\eta_\varepsilon(x-y)||u_m(y)|dy \\
\leq \|D\eta_\varepsilon\|_{L^\infty(\mathbb{R}^n)}\|u_m\|_{L^1(U)} \leq \frac{C}{\varepsilon^{n+1}} < \infty,
\]

for \( m = 1, 2, \ldots \). Thus the claim (2.2.19) follows these two estimates.

5. Now fix \( \delta > 0 \). We will show that there exists a subsequence \( \{u_{mj}\}_{j=1}^\infty \) of \( \{u_m\}_{m=1}^\infty \) such that

\[
\limsup_{j,k} \|u_{mj} - u_{mk}\|_{L^q(U)} \leq \delta. 
\] (2.2.20)

To see this, let us first employ assertion (2.2.17) to select \( \varepsilon > 0 \) so small that

\[
\|u_m^\varepsilon - u_m\|_{L^q(U)} \leq \delta/2, 
\] (2.2.21)

for \( m = 1, 2, \ldots \).

We now observe that since functions \( \{u_m\}_{m=1}^\infty \), and thus functions \( \{u_m^\varepsilon\}_{m=1}^\infty \), have support in some fixed bounded set \( U \subset \mathbb{R}^n \), we may use (2.2.19) and the Arzela-Ascoli theorem, to obtain a subsequence \( \{u_{mj}\}_{j=1}^\infty \subset \{u_m^\varepsilon\}_{m=1}^\infty \) which converges uniformly on \( U \). In particular therefore

\[
\limsup_{j,k} \|u_{mj}^\varepsilon - u_{mk}^\varepsilon\|_{L^q(U)} = 0. 
\] (2.2.22)

Then (2.2.21) and (2.2.22) imply (2.2.20).
6. We next employ assertion (2.2.20) with $\delta = 1, \frac{1}{2}, \frac{1}{3}, \cdots$ and use the standard diagonal argument to extract a subsequence $\{u_{m_l}\}_{l=1}^\infty$ of $\{u_m\}_{m=1}^\infty$ such that

$$\limsup_{l,k \to \infty} \|u_{m_l} - u_{m_k}\|_{L^q(U)} = 0.$$ 

Conclusion (i) of theorem is proved by steps 1 - 6. Next to prove conclusion (ii) of theorem.

7. Assume that $0 \leq m < k - \frac{n}{p} < m + 1$. Then as above we see

$$u \in W^{k-l,r}(\Omega), \quad (2.2.23)$$

for $r = \frac{np}{n-lp}$ provided that $lp < n$. We choose the integer $l_0 = \lfloor \frac{n}{p} \rfloor$. Consequently, (2.2.23) holds for $r = \frac{np}{n-pl_0} > n$. Hence (2.2.23) and Morrey inequality imply that $D^\alpha u \in C^{0,1-\frac{n}{p}}(\Omega)$ for all $|\alpha| \leq k - l_0 - 1$.

Observe also that $1 - \frac{n}{r} = 1 - \frac{n}{p} + l_0 = \lfloor \frac{n}{p} \rfloor + 1 - \frac{n}{p}$. Thus $u \in C^{m,\gamma}(\bar{\Omega})$ with $m = k - l_0 - 1 = k - \lfloor \frac{n}{p} \rfloor - 1$, $\gamma = 1 - \frac{n}{r} = \lfloor \frac{n}{p} \rfloor + 1 - \frac{n}{p}$, and the stated estimate follows easily.

8. The Arzela-Ascoli theorem implies the compactness of embedding $W^{k,p}(\Omega) \hookrightarrow C^{m,\gamma}(\bar{\Omega})$ provided that $\alpha < \gamma = k - m - \frac{n}{p}$.

Denote $(u)_{\Omega} = \int_{\Omega} u \, dx = \text{average of } u \text{ over } \Omega.$

**Theorem 2.2.33 (Poincare inequality for $W^{1,p}(\Omega)$)** Let $\Omega \subset \mathbb{R}^n$ be bounded and connected, with $C^1$ boundary $\partial \Omega$. Assume $1 \leq p \leq \infty$. Then there exists a constant $C$, depending only on $p$, $n$ and $\Omega$, such that

$$\|u - (u)_{\Omega}\|_{L^p(\Omega)} \leq C \|Du\|_{L^p(\Omega)}, \quad (2.2.24)$$

for all $u \in W^{1,p}(\mathbb{R}^n)$.

**Proof.** Argue by contradiction. If the stated estimate is false, that is, for each integer $k = 1, 2, \cdots$, there would exist a function $u_k \in W^{1,p}(\Omega)$ such that

$$\|u_k - (u_k)_{\Omega}\|_{L^p(\Omega)} > k \|Du_k\|_{L^p(\Omega)}. \quad (2.2.25)$$
Normalize by defining
\[ v_k := \frac{u_k - (u_k)_\Omega}{\|u_k - (u_k)_\Omega\|_{L^p(\Omega)}}, \quad k = 1, 2, \ldots. \]

Then, there holds
\[ (v_k)_\Omega = 0, \quad \|v_k\|_{L^p(\Omega)} = 1, \quad (2.2.26) \]
and from (2.2.25), we have
\[ \|Dv_k\|_{L^p(\Omega)} < \frac{1}{k}, \quad k = 1, 2, \ldots. \quad (2.2.27) \]
Particularly, \( \{v_k\}_{k=1}^\infty \) is a bounded in \( W^{1,p}(\Omega) \). From the compact embedding, there exists a subsequence \( \{v_{k_j}\}_{j=1}^\infty \subset \{v_k\}_{k=1}^\infty \), and a function \( v \in L^p(\Omega) \) such that
\[ v_{k_j} \to v \text{ in } L^p(\Omega). \]
From (2.2.26), it follows that
\[ (v)_\Omega = 0, \quad \|v\|_{L^p(\Omega)} = 1. \]
On the other hand, (2.2.27) implies that for every \( \varphi \in C_0^\infty(\Omega) \), there holds
\[ \int_\Omega v \varphi x_i \, dx = \lim_{j \to \infty} \int_\Omega v_{k_j} \varphi x_i \, dx = -\lim_{j \to \infty} \int_\Omega (v_{k_j})_x \varphi \, dx = 0, \]
consequently, \( v \in W^{1,p}(\Omega) \) and \( Dv = 0 \text{ a.e. in } \Omega \). Thus \( v \) is constant, since \( \Omega \) is connected, furthermore, \( (v)_\Omega = 0 \), hence \( v \equiv c = 0 \), which contradicts the fact that \( \|v\|_{L^p(\Omega)} = 1. \]

\textbf{Theorem 2.2.34 (Poincare inequality for a ball)} Suppose \( 1 \leq p < n \). Then there exists a constant \( C \), depending only on \( p \) and \( n \), such that
\[ \|u - (u)_{x,r}\|_{L^p(B(x,r))} \leq Cr\|Du\|_{L^p(B(x,r))}, \quad (2.2.28) \]
holds for each ball \( B(x,r) \subset R^n \) and each function \( u \in W^{1,p}_0(B(x,r)) \).

\textbf{Proof.} The case \( \Omega = B(0,1) \) follows from Theorem 2.2.33. In general, if \( u \in W^{1,p}_0(B(x,r)) \), set
\[ v(y) := u(x + ry), \quad (y \in B(0,1)). \]
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Then \( v \in W^{1,p}_0(B(0,1)) \), and there holds
\[
\|v - (v)_{0,1}\|_{L^p(B(0,1))} \leq C \|Dv\|_{L^p(B(0,1))},
\]
Changing variables, we recover estimate (2.2.28).

Lemma 2.2.35 (Poincare-Sobolev inequality) For any \( \varepsilon > 0 \) there exists a \( C = C(\varepsilon, n) \) such that for \( u \in H^1(B_1) \) with
\[
|\{x \in B_1; u = 0\}| \geq \varepsilon |B_1|,
\]
there holds
\[
\int_{B_1} u^2 \leq C \int_{B_1} |Du|^2.
\]

Proof. Suppose not. Then there exists a sequence \( \{u_m\} \subset H^1(B_1) \) such that
\[
|\{x \in B_1; u_m = 0\}| \geq \varepsilon |B_1|,
\]
and
\[
\int_{B_1} u_m^2 = 1, \quad \int_{B_1} |Du_m|^2 \to 0 \quad \text{as} \quad m \to \infty.
\]
Hence we may assume \( u_m \to u_0 \in H^1(B_1) \) strongly in \( L^2(B_1) \) and weakly in \( H^1(B_1) \). Then
\[
\lim_{m \to \infty} (|Du_m|^2 - |D(u_m - u_0)|^2) = |Du_0|^2
\]
and
\[
|Du_0|^2 \leq \lim_{m \to \infty} |Du_m|^2 = 0.
\]
Thus \( u_0 \) is a nonzero constant. So
\[
0 = \lim_{m \to \infty} \int_{B_1} |u_m - u_0|^2 \geq \lim_{m \to \infty} \int_{\{u_m = 0\}} |u_m - u_0|^2\]
\[
\geq |u_0|^2 \inf_m |\{u_m = 0\}| > 0.
\]
This contradiction completes the proof.

Definition 2.2.36 For a domain \( \Omega \) with \( C^k \)-boundary \( \partial\Omega = \Gamma \), \( k \in N \), \( 1 < p < \infty \), denote \( W^{k-\frac{1}{p}}(\Gamma) \) as the set of equivalent classes \( \{\{u\} + W^{k,p}_0(\Omega); \ u \in W^{k,p}(\Omega)\} \), endowed with norm
\[
\|u\|_{W^{k-\frac{1}{p}}(\Gamma)} = \inf \{\|v\|_{W^{k,p}(\Omega)} : \ u - v \in W^{k,p}_0(\Omega)\}.
\]
The elements of \( W^{k-\frac{1}{p}}(\Gamma) \) is call the traces \( u|_\Gamma \) of functions \( u \in W^{k,p}(\Omega) \).
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**Theorem 2.2.37 (Trace Theorem)** Assume $\Omega$ is a domain with $C^k$-boundary $\partial \Omega = \Gamma$, $k \in \mathbb{N}, 1 < p < \infty$. Then $W^{k-\frac{1}{p},p}(\Gamma)$ is a Banach space.

**Theorem 2.2.38 (Extension Theorem)** For any domain $\Omega$ with $C^k$-boundary $\partial \Omega = \Gamma$, $k \in \mathbb{N}, 1 < p < \infty$, there exists a continuous linear extension operator $\text{ext} : W^{k-\frac{1}{p},p}(\Gamma) \rightarrow W^{k,p}(\Omega)$ such that $(\text{ext}(u)) \big|_{\Gamma} = u$, for all $u \in W^{k-\frac{1}{p},p}(\Gamma)$.

**Theorem 2.2.39** Suppose $\Omega$ is a domain with $C^k$-boundary $\partial \Omega = \Gamma$, $k \in \mathbb{N}, 1 < p < \infty$. Then $W^{k,p}(\Gamma) \hookrightarrow W^{k-\frac{1}{p},p}(\Gamma) \hookrightarrow W^{k-1,p}(\Gamma)$ and both embeddings are compact.

**EXERCISES**

1. Prove The Young inequality with $\varepsilon$ - Lemma 2.2.2.
2. Prove the generalized H"older inequality - Lemma 2.2.4.
3. Complete Example 2.2.15.
4. Complete Example 2.2.16.
5. Complete Example 2.2.21.
6. Complete Example 2.2.22.
7. Verify $\| \cdot \|_{C^k,\gamma(\overline{\Omega})}$, $\| \cdot \|_{L^p(\Omega)}$ and $\| \cdot \|_{W^{k,p}(\Omega)}$ are norms.
8. Prove Theorem 2.2.31.