Existence results for Brezis–Nirenberg problems with Hardy potential and singular coefficients

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Abstract

In this paper, we consider the existence of non-trivial solutions to semi-linear Brezis–Nirenberg type problems with Hardy potential and singular coefficients. First, we shall study the corresponding eigenvalue problem, and obtain some basic properties of eigenvalues and asymptotic estimates of the eigenfunctions and approximating eigenfunctions. Secondly, we consider the extremal functions of the best embedding constant, and get some crucial estimates for the cut-off function of the extremal functions. Thirdly, applying different variational theorems for distinct cases of those parameters appearing in the equation, we obtain two existence results for non-trivial solutions to semi-linear Brezis–Nirenberg type problems. Our existence results are divided into non-resonant and resonant cases.

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1. Introduction

In this paper, we consider the existence of non-trivial solutions to the following semi-linear Brezis–Nirenberg type problems with Hardy potential and singular coefficients:

\[
\begin{cases}
-\text{div} \left( \frac{Du}{|x|^{2a}} \right) - \mu \frac{u}{|x|^{2(a+1)}} = \lambda \frac{u}{|x|^{2(a+1)-c}} + |u|^{2^*-2}u, & \text{in } \Omega \\
\frac{u}{|x|^{d-2}}, & \text{on } \partial \Omega,
\end{cases}
\]

where \( \Omega \subset \mathbb{R}^N \) is an open bounded domain with \( C^1 \) boundary and \( 0 \in \Omega, -\infty < a < \frac{N-2}{2}, a \leq b < (a+1), c > 0, 2^* = \frac{2N}{N-2}, d = a + 1 - b, \lambda, \mu \) are two real parameters.

The starting point of the variational approach to these problems is the Caffarelli–Kohn–Nirenberg inequality (see [5]): There is a constant \( C_{a,b} > 0 \) such that

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\[
\left( \int_{\mathbb{R}^N} |x|^{-2b} |u|^2 \, dx \right)^{2/2^*} \leq C_{a,b} \int_{\mathbb{R}^N} |x|^{-2a} |Du|^2 \, dx,
\]
for all \( u \in C_0^\infty(\mathbb{R}^N) \), where
\[
-\infty < a < \frac{N - 2}{2}, \quad a \leq b \leq a + 1, \quad 2_* = \frac{2N}{N - 2d}, \quad d = a + 1 - b.
\]

Let \( \mathcal{D}^{1,2}_a(\Omega) \) be the completion of \( C_0^\infty(\mathbb{R}^N) \), with respect to the weighted norm \( \| \cdot \| \) defined by
\[
\|u\| = \left( \int_{\Omega} |x|^{-2a} |Du|^2 \, dx \right)^{1/2}.
\]

From the boundedness of \( \Omega \) and the standard approximation arguments, it is easy to see that (1.2) holds for any \( u \in \mathcal{D}^{1,2}_a(\Omega) \) in the sense:
\[
\left( \int_{\Omega} |x|^{-2q} |u|^q \, dx \right)^{2/q} \leq C \int_{\Omega} |x|^{-2a} |Du|^2 \, dx,
\]
for \( 1 \leq q \leq 2_* \). Let \( L^q(\Omega, |x|^{-bq}) \) be the weighted \( L^q \) space with weighted norm defined as
\[
\|u\|_{b,q} := \left( \int_{\Omega} |x|^{-bq} |u|^q \, dx \right)^{1/q}.
\]

Then (1.3) reads as
\[
\|u\|_{b,q} \leq C\|u\|
\]
for any \( u \in \mathcal{D}^{1,2}_a(\Omega) \), where here and hereafter \( C \) denotes a universal positive constant, which may change its value from line to line. This inequality shows that the embedding \( \mathcal{D}^{1,2}_a(\Omega) \hookrightarrow L^q(\Omega, |x|^{-bq}) \) is continuous. Furthermore, if \( 1 \leq q < 2_* \), the embedding is indeed compact (see [8,24] for more general cases).

On \( \mathcal{D}^{1,2}_a(\Omega) \), we can define the energy functional
\[
I_{\lambda,\mu}(u) = \frac{1}{2} \int_{\Omega} |Du|^2 \frac{\mu}{|x|^{2a}} \, dx - \frac{1}{2} \mu \int_{\Omega} \frac{u^2}{|x|^{2(a + 1)}} \, dx - \frac{\lambda}{2} \int_{\partial \Omega} \frac{u^2}{|x|^{2(a + 1) - c}} \, d\omega - \frac{1}{2} \int_{\Omega} |u|^{2*} \, dx.
\]

From (1.3), \( I_{\lambda,\mu}(u) \) is well defined in \( \mathcal{D}^{1,2}_a(\Omega) \), and \( I_{\lambda,\mu}(u) \in C^1(\mathcal{D}^{1,2}_a(\Omega), \mathbb{R}) \). Furthermore, the critical points of \( I_{\lambda,\mu} \) are weak solutions of problem (1.1).

We note that for \( \mu = 0, a = b = 0 \) and \( c = 2 \), problem (1.1) becomes
\[
\begin{cases}
-\Delta u = \lambda u + |u|^{2* - 2}u, & \text{in } \Omega \\
u = 0, & \text{on } \partial \Omega,
\end{cases}
\]
where \( 2^* = \frac{2N}{N - 2} \) is the critical Sobolev exponent. Problem (1.4) has been studied in a more general context in the famous paper by Brezis and Nirenberg [4] in 1983. Since the embedding \( H^1_0(\Omega) \hookrightarrow L^q(\Omega) \) is not compact for \( q = 2^* \), the corresponding energy functional does not satisfy the (PS) condition globally, which caused serious difficulty when trying to find critical points by standard variational methods. By carefully analyzing the energy level of a cut-off function related to the extremal functions of the Sobolev inequality in \( \mathbb{R}^n \), Brezis and Nirenberg obtained that the energy functional does satisfy (PS) for energy levels \( c < \frac{1}{N} S^{N/2} \), where \( S \) is the best constant of the Sobolev inequality. The results of [4] show a somewhat surprising phenomenon: the existence of positive solutions to problem (1.4) depends not only on \( \lambda \) but also on the couple \( (N, \lambda) \). In their case, the results were divided into two cases \( N \geq 4 \) and \( N = 3 \). From then on, the dimension \( N = 3 \) was called the critical dimension, since problem (1.4) has a positive solution if and only if \( \lambda \in (\lambda^*, \lambda_1) \) for some \( \lambda^* \in (0, \lambda_1) \), when \( N = 3 \); while when \( N \geq 4 \), (1.4) has a positive solution for all \( \lambda \in (0, \lambda_1) \), where \( \lambda_1 \) is the principal eigenvalue of \(-\Delta\) with the Dirichlet condition. The Brezis–Nirenberg type problems have been generalized to many situations (see [9–11,16,18–20,24,23,26] and references therein).
In 1985, Capozzi et al. [7] considered the existence of a non-trivial solution to problem (1.4) for the case \( \lambda \geq \lambda_1 \) (see also [2,15,25]), and obtained the following results:

(i) If \( N = 4, \lambda > 0 \) and \( \lambda \not\in \sigma \), where \( \sigma \) is the spectrum of the Laplacian \(-\Delta\) on \( \Omega \) with the zero Dirichlet condition, then (1.4) has a non-trivial solution.

(ii) If \( N \geq 5 \), (1.4) has a non-trivial solution for all \( \lambda > 0 \).

There exists an apparent difference as regards the existence of non-trivial solution between \( N = 4 \) and \( N \geq 5 \), in agreement with [14,13]; \( N = 4 \) is called the non-resonant dimension, since the existence result is only obtained for \( \lambda \not\in \sigma \).

For \( \mu > 0 \), \( a = b = 0 \), \( c = 2 \), problem (1.1) becomes

\[
\begin{cases}
-\Delta u - \frac{\mu}{|x|^2} u = \lambda u + |u|^{2^*-2} u, & \text{in } \Omega \\
u = 0, & \text{on } \partial \Omega,
\end{cases}
\]

In 1999, Jannelli [17] generalized the main result of [4] to problem (1.5), and found that \( \bar{\mu} - 1 < \mu < \bar{\mu} \) is critical in the sense that (1.5) has a positive solution if only if \( \lambda \in (\lambda^*, \lambda_1) \) for some \( \lambda^* \in (0, \lambda_1) \), when \( \bar{\mu} - 1 < \mu < \bar{\mu} \) and \( \Omega \) is the unit ball; while when \( \mu \leq \bar{\mu} - 1 \), (1.5) has a positive solution for all \( \lambda \in (0, \lambda_1) \).

In 2001, Ferrero and Gazzola [13] considered the existence of non-trivial solutions in the case where \( \lambda \) may be larger than \( \lambda_1 \). They distinguished two distinct cases: the resonant and non-resonant cases of the Brezis–Nirenberg type problem (1.5). For the resonant case, they only studied a special case: \( \Omega \) is the unit ball and \( \lambda = \lambda_1 \). The general case was left as an open problem. In 2004, Cao and Han [6] studied the general case. The results of [13,6] show that \( \bar{\mu} - (\frac{N+2}{N})^2 \leq \mu \leq \bar{\mu} - 1 \) is non-resonant, in the sense that the existence of a non-trivial solution to (1.5) is only obtained for \( \lambda \not\in \sigma_\mu \) when \( \bar{\mu} - (\frac{N+2}{N})^2 \leq \mu \leq \bar{\mu} - 1 \), where \( \sigma_\mu \) is the spectrum of the operator \(-\Delta - \mu/|x|^2\) on \( \Omega \) with the zero Dirichlet condition.

In this paper, we study the existence results for the Brezis–Nirenberg type problem (1.1) in a more general setting: \(-\infty < a < \frac{N-2}{N-2}, a \leq b < (a+1), c > 0 \). Note that here we have three more parameters, \( a, b, c \). We will try to distinguish the resonant and non-resonant cases of problem (1.1), and hence extend the results of [13,6] to this general case. In particular, our results show that the numbers \( \bar{\mu} - 1 \) and \( \bar{\mu} - (\frac{N+2}{N})^2 \) in [13,6] will become \( \bar{\mu} - M \) and \( \bar{\mu} - (\frac{N+2d}{2N} c + \frac{2b-d}{N})^2 \) in our case, where \( M = \max\{b^2, \frac{c^2}{4}\} \) and \( b^- = \max\{-b, 0\} \). The existence of a non-trivial solution to problem (1.1) is only obtained for \( \lambda \) not an eigenvalue of the corresponding linear operator when \( \bar{\mu} - (\frac{N+2d}{2N} c + \frac{2b-d}{N})^2 \leq \mu < \bar{\mu} - M \), which means that \( \bar{\mu} - (\frac{N+2d}{2N} c + \frac{2b-d}{N})^2 \leq \mu < \bar{\mu} - M \) is non-resonant.

This paper is organized as follows. Section 2 is devoted to the corresponding eigenvalue problem. Since the appearance of the Hardy potential causes the eigenfunctions to blow up near \( x = 0 \), we cut off the eigenfunctions near \( x = 0 \), and carefully analyze their asymptotic behavior near \( x = 0 \), which will play a crucial role in the proofs of the existence results. In Section 3, we consider the extremal functions of the best embedding constant, which are solutions to a “limit” equation in \( \mathbb{R}^N \setminus \{0\} \) associated with (1.1). Applying the Bliss lemma [3], we will derive the explicit formula for the extremal functions. Due to the appearance of the Hardy potential, the extremal functions also blow up at \( x = 0 \). For use in next section, we derive some estimates of the cut-off functions of extremal functions, which show the concentration of extremal functions near \( x = 0 \). In Section 4, we first prove that \( I_{\lambda, \mu} \) satisfies the (PS)\(_c\) condition if the energy level \( \beta \) is under a threshold. Secondly, we recall some variational principles, and show that \( I_{\lambda, \mu} \) satisfies the geometric conditions of those variational principles in each case, and hence get the (PS)\(_\beta\) sequence for some minimax values \( \beta \). Thirdly, via a fine balance between the blow-up of eigenfunctions and the concentration of extremal functions near \( x = 0 \), we try to show that the minimax values \( \beta \) are under the threshold, and thus obtain the existence results for the resonant and non-resonant cases.

2. Eigenvalue problem

In this section, we consider the following eigenvalue problem with Hardy potential and singular coefficients:

\[
\begin{cases}
-\text{div}(|x|^{-2a} Du) - \mu |x|^{-2(a+1)} u = \lambda |x|^{-2(a+1)+c} u, & \text{in } \Omega \\
u = 0, & \text{on } \partial \Omega,
\end{cases}
\]

(2.1)
Let $L_\mu u := -\text{div}(|x|^{-2a} Du) - \mu |x|^{-2(a+1)} u$; then the operator is formally symmetric. We define the associated bilinear form $B[\cdot, \cdot]$ as

$$B[u, v] = \int_\Omega |x|^{-2a} Du \cdot Dv \, dx - \mu \int_\Omega |x|^{-2(a+1)} uv \, dx.$$ 

From [8], it follows that when $b = a + 1$, the best constant in (1.2) is $C_{a, a+1} = \left( \frac{n-2(a+1)}{2} \right)^2$, denoted by $\bar{\mu}$ later, and never achieved. Hence, $B[\cdot, \cdot]$ is symmetric and uniformly positive definite when $\mu < \bar{\mu}$. Let $\mathcal{H}_\mu$ denote the space $\mathcal{D}_{a,2}^1(\Omega)$ with the norm deduced from $B[\cdot, \cdot]$. One can adapt the usual eigenvalue theory of the symmetric elliptic operator (see [12]) to our case, except the regularity of eigenfunctions due to the appearance of the Hardy potential and singular coefficients. Next, we present results concerning the eigenvalues and eigenfunctions for problem (2.1) and omit the proof. The interested reader can refer to Theorems 1 and 2 in Section 6.5 of [12].

**Theorem 2.1** (Eigenvalues and Eigenfunctions). Assume that $\mu < \bar{\mu}$. We have:

(i) Each eigenvalue of problem (2.1) is real and of finite multiplicity.

(ii) If we repeat each eigenvalue according to its multiplicity, we have

$$\sigma_\mu = \{\lambda_i\}_{i=1}^{\infty},$$

where

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots,$$

and $\lambda_i \to \infty$ as $i \to \infty$.

(iii) There exists an orthonormal basis $\{e_i\}_{i=1}^{\infty}$ in $L^2(\Omega, |x|^{-2(a+1)+c})$, where $L^2(\Omega, |x|^{-2(a+1)+c})$ is the weighted $L^2$ space with norm

$$\|u\|_2 := \|u\|_{L^2(\Omega, |x|^{-2(a+1)+c})} = \left( \int_\Omega |x|^{-2(a+1)+c} u^2 \, dx \right)^{1/2},$$

deduced by the weighted inner product

$$(u, v)_2 := \int_\Omega |x|^{-2(a+1)+c} uv \, dx,$$

and $e_i \in \mathcal{D}_{a,2}^1(\Omega)$ is an eigenfunction corresponding to $\lambda_i$:

$$\begin{cases} 
L_\mu e_i = \lambda_i |x|^{-2(a+1)+c} e_i, & \text{in } \Omega \\
0 = 0, & \text{on } \partial \Omega,
\end{cases} \quad i = 1, 2, \ldots.$$  

(iv) Eigenvalue

$$\lambda_1 = \min\{B[u, u] : \|u\|_{L^2(\Omega, |x|^{-2(a+1)+c})} = 1\} > 0$$

is simple.

In the rest of this section, we will study the asymptotic behavior of eigenfunctions $e_i$ near $x = 0$, which is the only singular point of $e_i$. First, following an idea from [6], we have the following estimate:

**Lemma 2.2.** If $\sqrt{\bar{\mu} - \sqrt{\bar{\mu} - \mu} + a} < (n - 2)/2, c > 0$, then any solution $e_i \in \mathcal{D}_{a,2}^1(\Omega)$ of (2.2) satisfies

$$|e_i(x)| \leq C|x|^{-\sqrt{\bar{\mu} - \mu} + c}, \quad x \in \Omega \setminus \{0\}.$$ 

**Proof.** Let $v(x) = |x|^{\sqrt{\bar{\mu} - \sqrt{\bar{\mu} - \mu} + a}} e_i(x)$. A direct calculation yields

$$\begin{cases} 
-\text{div}(|x|^{-2(\sqrt{\bar{\mu} - \sqrt{\bar{\mu} - \mu}} + a)} Dv) = \lambda_i |x|^{-2(\sqrt{\bar{\mu} - \sqrt{\bar{\mu} - \mu} + a}) + c} v, & \text{in } \Omega \\
v = 0, & \text{on } \partial \Omega,
\end{cases} \quad (2.3)$$
From the regularity result of solutions to (2.3) (see [24]), we know that $|v(x)| \leq C$ for any $x \in \Omega \setminus \{0\}$, that is,

$$|e_i(x)| \leq C|x|^{-\sqrt{\mu} + \sqrt{\mu - \mu}}. \quad x \in \Omega \setminus \{0\}. \quad \blacksquare$$

To study further the asymptotic behavior of eigenfunctions near $x = 0$, we introduce the approximating eigenfunctions, like [13]. Let $m$ be large enough such that $B_{2/m} \subset \Omega$, where $B_{2/m}$ denotes the ball centered at 0 with radius $2/m$. Define function $\xi_m : \Omega \to \mathbb{R}$ as

$$\xi_m(x) = \begin{cases} 
0, & x \in B_{1/m} \\
|m|x| - 1, & x \in A_m := B_{2/m} \setminus B_{1/m} \\
1, & x \in \Omega \setminus B_{2/m}.
\end{cases}$$

We call $e_i^m = \xi_m e_i$ the approximating eigenfunction for $i \in \mathbb{N}$. We will show that $e_i^m$ converges to $e_i$ as $m \to \infty$ and estimate the error of convergence. For $k \in \mathbb{N}$ fixed, let $\mathcal{H}^- : = \text{span}\{e_1, e_2, \ldots, e_k\}$, $\mathcal{H}_{m}^+ := \text{span}\{e_1^m, e_2^m, \ldots, e_k^m\}$.

**Lemma 2.3 (Approximating Eigenfunction).** Assuming that $\mu < \bar{\mu}$, we have:

(i) For any $i \in \mathbb{N}$, $\{e_i^m\}$ converges to $e_i$ in $\mathcal{H}_{\mu}$.

(ii) Moreover, we have the following error estimates

$$\|e_i^m\|_{\mathcal{H}_{\mu}}^2 \leq \lambda_i + C m^{-2\sqrt{\mu - \mu}} \quad (2.4)$$

$$|(e_i^m, e_j^m)_{\mathcal{H}_{\mu}}| \leq C m^{-2\sqrt{\mu - \mu}}, \quad i \neq j \quad (2.5)$$

$$|(e_i^m, e_j^m)_{\mathcal{H}_{\mu}}| \leq C m^{-c - 2\sqrt{\mu - \mu}}, \quad i \neq j \quad (2.6)$$

$$\|e_i^m\|_{\mathcal{H}_{\mu}}^2 \geq 1 - C m^{-c - 2\sqrt{\mu - \mu}} \quad (2.7)$$

(iii) $\max_{u \in \mathcal{H}_{m}} \|u\|_{\mathcal{H}_{\mu}}^2 \leq \lambda_k + C m^{-2\sqrt{\mu - \mu}}$.

**Proof.** 1. To prove the convergence, from (1.2) with $b = a + 1$, it suffices to show that

$$\int_{\Omega} |x|^{-2a} |D(e_i^m - e_i)|^2 \, dx \to 0, \quad (2.8)$$

as $m \to \infty$.

In fact, we have

$$\int_{\Omega} \frac{|D(e_i^m - e_i)|^2}{|x|^{2a}} \, dx \leq 2 \int_{A_m} \frac{|D\xi_m|^2}{|x|^{2a}} e_i^2 \, dx + 2 \int_{B_{2/m}} (\xi_m - 1)^2 |De_i|^2 \, dx. \quad (2.9)$$

From the weighted Hölder inequality, it follows that

$$\int_{A_m} |x|^{-2a} |D\xi_m|^2 e_i^2 \, dx = m^2 \int_{A_m} |x|^{-2a} e_i^2 \, dx < m^2 \int_{B_{2/m}} |x|^{-2a} e_i^2 \, dx \leq m^2 \left( \int_{B_{2/m}} \frac{|x|^{-2a}}{|x|^{2a - 2a}} \, dx \right)^{\frac{n - 2a - 2}{n - 2a}} \left( \int_{B_{2/m}} |x|^{-2a} \, dx \right)^{\frac{2}{n - 2a}},$$
and we note that
\[
\int_{B_{2/m}} \frac{1}{|x|^{2a}} \, dx = C \int_0^{2/m} r^{n-1-2a} \, dr \\
= C m^{-(n-2a)}.
\]
Thus we have
\[
2 \int_{A_m} |x|^{-2a} |D \xi_m|^2 e_i^2 \, dx \lesssim C \left( \int_{B_{2/m}} |x|^{-2a} |e_i|^{\frac{2(n-2a)}{n-2a+m}} \, dx \right)^{\frac{n-2a-2}{n-2a+m}} \to 0,
\] (2.10)
as \( m \to \infty \). On the other hand, we have
\[
2 \int_{B_{2/m}} |x|^{-2a} (\xi_m - 1)^2 |D e_i|^2 \, dx \lesssim 2 \int_{B_{2/m}} |x|^{-2a} |D e_i|^2 \, dx \to 0,
\] (2.11)
as \( m \to \infty \). Hence, (2.9)–(2.11) imply (2.8).

2. We only prove (2.5) and omit the proofs of (2.4), (2.6) and (2.7).

For \( i \neq j \), we have
\[
(e_i^m, e_j^m)_{H_\mu} = \int_\Omega \left| |x|^{-2a} D e_i^m \cdot D e_j^m - \mu |x|^{-2(a+1)} e_i e_j \right| \, dx \\
= \int_\Omega \left( (\xi_m^2 - 1) D e_i D e_j + \xi_m e_i D \xi_m D e_j + \xi_m e_j D \xi_m D e_i + |D \xi_m|^2 e_i e_j \\
- \mu (\xi_m^2 - 1) e_i e_j \right) \frac{|x|}{|x|^{2(a+1)}} \, dx.
\] (2.12)
Multiplying (2.2) by \((\xi_m^2 - 1) e_j^2\) and integrating by parts yields
\[
\int_\Omega \left( (\xi_m^2 - 1) D e_i D e_j + 2 \xi_m e_j D \xi_m D e_i \\
- \mu (\xi_m^2 - 1) e_i e_j \right) \frac{|x|}{|x|^{2(a+1)-c}} \, dx = \lambda_i \int_\Omega (\xi_m^2 - 1) e_i e_j \frac{|x|}{|x|^{2(a+1)-c}} \, dx.
\] (2.13)
Similarly, we have
\[
\int_\Omega \left( (\xi_m^2 - 1) D e_i D e_j + 2 \xi_m e_j D \xi_m D e_j \\
- \mu (\xi_m^2 - 1) e_i e_j \right) \frac{|x|}{|x|^{2(a+1)-c}} \, dx \\
= \lambda_j \int_\Omega (\xi_m^2 - 1) e_i e_j \frac{|x|}{|x|^{2(a+1)-c}} \, dx.
\] (2.14)
Thus, substituting (2.13) and (2.14) into (2.12), we have
\[
(e_i^m, e_j^m)_{H_\mu} = \int_\Omega |D \xi_m|^2 \frac{e_i e_j}{|x|^{2a}} \, dx + \frac{(\lambda_j + \lambda_i)}{2} \int_\Omega (\xi_m^2 - 1) e_i e_j \frac{|x|}{|x|^{2(a+1)-c}} \, dx.
\]
It follows from Lemma 2.2 that
\[
|e_i^m, e_j^m)_{H_\mu} | \lesssim \int_{B_{2/m}(0)} |D \xi_m|^2 \frac{|e_i||e_j|}{|x|^{2a}} \, dx + \frac{(\lambda_j + \lambda_i)}{2} \int_{B'_{2/m}(0)} (1 - \xi_m^2) |e_i||e_j| \frac{|x|}{|x|^{2(a+1)-c}} \, dx \\
\lesssim C m^2 \int_0^{2/m} r^{n-1-2(\sqrt{\mu} - \sqrt{\mu - \mu})-2a} \, dr + C \int_0^{2/m} r^{n-1-2(\sqrt{\mu} - \sqrt{\mu - \mu})-2a+c} \, dr \\
\lesssim C m^{2\sqrt{\mu} - \mu}.
\]
3. Conclusion (iii) is true.
   Since $\mathcal{H}_m$ is of finite dimension, there exists $u_m \in \mathcal{H}_m$, $\|u_m\|_2 = 1$ such that
   \[
   \max_{\{u \in \mathcal{H}_m : \|u\|_2 = 1\}} \|u\|_{\mathcal{H}_\mu}^2 = \|u_m\|_{\mathcal{H}_\mu}^2. \tag{2.15}
   \]
   It follows from the definition of $\mathcal{H}_m$ that there exist constants $\alpha_1^m, \alpha_2^m, \ldots, \alpha_k^m$ such that $u_m = \sum_{i=1}^k \alpha_i^m e_i^m$. It is easy to see that
   \[
   \|u_m\|_{\mathcal{H}_\mu}^2 = \sum_{i=1}^k (\alpha_i^m)^2 \|e_i^m\|_{\mathcal{H}_\mu}^2 + 2 \sum_{1 \leq i < j \leq k} \alpha_i^m \alpha_j^m \langle e_i^m, e_j^m \rangle_{\mathcal{H}_\mu}, \tag{2.16}
   \]
   and
   \[
   1 = \|u_m\|_2^2 = \sum_{i=1}^k (\alpha_i^m)^2 \|e_i^m\|_2^2 + 2 \sum_{1 \leq i < j \leq k} \alpha_i^m \alpha_j^m \langle e_i^m, e_j^m \rangle. \tag{2.17}
   \]
   From (2.6), there exists $m_0 > 0$ large enough such that for any $m \geq m_0$, we have
   \[
   |\langle e_i^m, e_j^m \rangle| < \frac{1}{4}.
   \]
   Thus, from (2.7) and (2.17), for any $m \geq m_0$, it follows that
   \[
   1 \geq \sum_{i=1}^k (\alpha_i^m)^2 \|e_i^m\|_2^2 - 2 \sum_{1 \leq i < j \leq k} |\alpha_i^m| |\alpha_j^m| |\langle e_i^m, e_j^m \rangle| \\
   \geq \sum_{i=1}^k (\alpha_i^m)^2 - Cm^{-c-2\sqrt{\mu-\mu}} - \frac{1}{4} \sum_{1 \leq i < j \leq k} [(\alpha_i^m)^2 + (\alpha_j^m)^2] \\
   \geq \frac{1}{2} \sum_{i=1}^k (\alpha_i^m)^2 - Cm^{-c-2\sqrt{\mu-\mu}},
   \]
   which implies that
   \[
   |\alpha_i^m| \leq C, \quad i = 1, 2, \ldots, k. \tag{2.18}
   \]
   Furthermore, (2.6), (2.7) and (2.17) imply that
   \[
   1 \geq \sum_{i=1}^k (\alpha_i^m)^2 \|e_i^m\|_2^2 - 2 \sum_{1 \leq i < j \leq k} |\alpha_i^m| |\alpha_j^m| |\langle e_i^m, e_j^m \rangle| \\
   \geq \sum_{i=1}^k (\alpha_i^m)^2 - Cm^{-c-2\sqrt{\mu-\mu}},
   \]
   and this implies that
   \[
   \sum_{i=1}^k (\alpha_i^m)^2 \leq 1 + Cm^{-c-2\sqrt{\mu-\mu}}. \tag{2.19}
   \]
   Substituting (2.4), (2.5), (2.18) and (2.19) into (2.16), we obtain
   \[
   \|u_m\|_{\mathcal{H}_\mu}^2 \leq \lambda_k + C m^{-c-2\sqrt{\mu-\mu}} + C m^{-2\sqrt{\mu-\mu}} \\
   \leq \lambda_k + C m^{-2\sqrt{\mu-\mu}},
   \]
   for $m$ large enough.  \[\blacksquare\]
3. Extremal functions

In this section, applying the Bliss lemma, we first obtain the explicit form of the extremal functions of the best constant of embedding: \( \mathcal{H}_\mu \hookrightarrow L^2(\Omega, |x|^{-2a}b) \), and then get some crucial estimates of the cut-off functions of the extremal functions, which describe the concentration of the extremal functions at \( x = 0 \).

Define
\[
S_\mu(\Omega) := \inf_{u \in H^1(\Omega, |x|^{-2a})} \frac{\int_{\Omega} |x|^{-2a} |Du|^2 \, dx - \mu \int_{\Omega} |x|^{-2(a+1)} u^2 \, dx}{(\int_{\Omega} |x|^{-2a}b|u|^2 \, dx)^{\frac{2}{2a}}}
\]
and
\[
S_\mu := \inf_{u \in H^1(\mathbb{R}^N, |x|^{-2a})} \frac{\int_{\mathbb{R}^N} |x|^{-2a} |Du|^2 \, dx - \mu \int_{\mathbb{R}^N} |x|^{-2(a+1)} u^2 \, dx}{(\int_{\mathbb{R}^N} |x|^{-2a}b|u|^2 \, dx)^{\frac{2}{2a}}}.
\]

By the scaling argument and the homogeneity in the definition of \( S_\mu(\Omega) \) and \( S_\mu \), it is easy to see that \( S_\mu(\Omega) = S_\mu \) for any smooth domain \( \Omega \subset \mathbb{R}^N \) with \( 0 \in \Omega \). By the Lagrange multiplier method, the extremal functions of \( S_\mu \) satisfies the following Euler–Lagrange equation
\[
-\text{div}(|x|^{-2a} Du) - \mu |x|^{-2(a+1)} u = \delta |x|^{-2a}b|u|^{2-2}u, \quad x \in \mathbb{R}^n \setminus \{0\}. \tag{3.1}
\]
Since the left hand side of (3.1) is homogeneous, of first order, while the right hand side is of \( 2a - 1 > 1 \) order, without loss of generality, we assume that \( \delta = 1 \). Eq. (3.1) is a special case of (1.1) with \( \lambda = 0 \). Usually, (3.1) is called the “limit” equation of (1.1).

On \( D^{1,2}_a(\mathbb{R}^N) \), we define functionals
\[
I(u) = \int_{\mathbb{R}^N} |x|^{-b_2} |u|^2 \, dx
\]
and
\[
J(u) = \int_{\mathbb{R}^N} |x|^{-2a} \nabla u^2 \, dx - \mu \int_{\mathbb{R}^N} |x|^{-2(a+1)} u^2 \, dx.
\]
Thus,
\[
S_{\mu}^{-2a/2} = \sup\{I(u) : u \in D^{1,2}_a(\mathbb{R}^N), J(u) = 1\}. \tag{3.2}
\]
To find the extremal functions of \( S_\mu \), we try to solve the maximal problem (3.2). First, we recall the Bliss lemma [3] as our first lemma of this section:

**Lemma 3.1 (Bliss Lemma [3]).** Let \( h(x) \) be a non-negative measurable function defined on \( \mathbb{R} \) such that \( J = \int_0^\infty h^q \, dx \) is finite and given. Set \( g(x) = \int_0^x h(t) \, dt \). Then \( I = \int_0^\infty g^p(x) x^{n-p} \, dx \) attains its maximum value for the functions \( h(x) = (1 + \lambda x^\alpha)^{-(\alpha+1)/\alpha} \), with \( p > q > 1 \), \( \alpha = (p/q) - 1 \), and \( \lambda > 0 \).

Applying the Bliss lemma [3] cited above, we will get the following explicit form of the extremal functions of \( S_\mu \).

**Theorem 3.2 (Explicit Form of Extremal Functions).** If \( 0 < \sqrt{\mu - \sqrt{\mu - \mu_a + a < (n - 2)/2, \mu < \mu_a - b^2} \), then the family of solutions to (3.2) is defined as
\[
u^*_{\mu}(x) = C_0 \epsilon^{\frac{2}{2a-2}} \left[ \frac{2a \mu_a - \mu}{\epsilon^{\frac{2}{2a-2}} |x|^{-\frac{2a-2}{2}} (\sqrt{\mu_a - \mu} - \sqrt{\mu - \mu_a}) + |x|^{-\frac{2a-2}{2}} (\sqrt{\mu_a - \mu} + \sqrt{\mu - \mu_a})} \right]^{-\frac{2}{2a-2}},
\]
where \( C_0 \) is such a positive constant that \( \nu^*_{\mu}(x) \) is a weak solution to (3.1) with \( \delta = 1 \). Furthermore, \( \|\nu^*_{\mu}\|_{2a}^* = \|\nu^*_{\mu}\|_{\mathcal{H}_\mu}^N, \) and \( S_{\mu} = \|\nu^*_{\mu}\|_{2a}^* \).
**Proof.** 1. To apply the Bliss lemma, for any \( u \in D^{1,2}_a(\mathbb{R}^N) \), set \( v(x) = |x|^{\sqrt{\mu - \bar{\mu}}} u(x) \); then \( I \) and \( J \) become

\[
I(v) = \int_{\mathbb{R}^N} \frac{|v|^2}{|x|^{2(b+\sqrt{\mu - \bar{\mu})}}} \, dx \quad \text{and} \quad J(v) = \int_{\mathbb{R}^N} \frac{|
abla v|^2}{|x|^{2(a+\sqrt{\mu - \bar{\mu})}}} \, dx.
\]

For \( 0 < \sqrt{\mu - \bar{\mu}} + a < (n-2)/2 \), it was shown in [8] that solutions to the maximal problem

\[
\max_{\{v \in D^{1,2}_{a+\sqrt{\mu - \bar{\mu}}}(\mathbb{R}^N) \mid J(v) = 1 \}} I(v)
\]

are radial functions. Let \( D^{1,2}_{R,a}(\mathbb{R}^N) = \{ u \in D^{1,2}_a(\mathbb{R}^N) \mid u(x) = u(|x|) \} \) denote the radial functions in \( D^{1,2}_a(\mathbb{R}^N) \). Thus, we only consider the case where \( v \in D^{1,2}_{R,a+\sqrt{\mu - \bar{\mu}}}(\mathbb{R}^N) \). Rewriting \( I \) and \( J \) in polar coordinates, we obtain

\[
I(v) = C_N \int_0^{+\infty} |v|^2 r^{-2(b+\sqrt{\mu - \bar{\mu})} + n-1} \, dr
\]

and

\[
J(v) = C_N \int_0^{+\infty} \frac{d^2}{d\rho^2} r^{-2(a+\sqrt{\mu - \bar{\mu})} + n-1} \, d\rho.
\]

where \( C_N \) is a positive constant only depending on \( N \). To write \( J \) in the standard form for the Bliss lemma, where the integrand does not depend on \( r \) explicitly, let \( \rho = r^{1/\beta} \) for some \( \beta \) to be determined later. Substituting into (3.4) yields

\[
J(v) = C_N \int_0^{+\infty} \left| \frac{d^2 v}{d\rho^2} \right|^2 r^{\left(\frac{1}{\beta} - 1\right) - 2(a + \sqrt{\mu - \bar{\mu})} + n-1} \, d\rho.
\]

Setting \( \frac{1}{\beta} = -\frac{2\sqrt{\mu - \bar{\mu}}}{n-1} \), we have

\[
\left(\frac{1}{\beta} - 1\right) - 2(a + \sqrt{\mu - \bar{\mu})} + n-1 = 0,
\]

which means that

\[
J(v) = C_1 \int_0^{+\infty} \left| \frac{d^2 v}{d\rho^2} \right|^2 d\rho
\]

and

\[
I(v) = C_2 \int_0^{+\infty} |v|^2 |\rho|^{-\frac{2}{n-2}} \, d\rho.
\]

Apply the Bliss lemma; noting that \( h(\rho) = \frac{dv}{d\rho} \), we know that

\[
\max_{\{v \in D^{1,2}_{R,a+\sqrt{\mu - \bar{\mu}}}(\mathbb{R}^N) \mid J(v) = 1 \}} I(v)
\]

is attained at

\[
h(\rho) = \frac{dv}{d\rho} = (\eta \rho^\alpha + 1)^{-\frac{\alpha+1}{\alpha}},
\]

where \( \alpha = \frac{2}{n-2} - 1, \rho = r^{-2\sqrt{\mu - \bar{\mu}}}, \eta \) is any positive real number. Integrating the above formula, we get

\[
v(\rho) = (\eta + \rho^{-\alpha})^{-\frac{1}{2}}.\]

Writing this with variable \( r \), we get

\[
v(r) = \left[ \eta + r^{(2\sqrt{\mu - \bar{\mu}) - \frac{2}{n-2}} \right]^{-\frac{1}{2}}.
\]

Hence, we have

\[
u(r) = \left[ \eta |x|^{\frac{2}{n-2}(\sqrt{\mu - \bar{\mu})} + |x|^{\frac{2}{n-2}(\sqrt{\mu + \sqrt{\mu - \bar{\mu}}) - \frac{2}{n-2}}.
\]
Lemma 3.3

The functions \( u^\ast_m \) are called the extremal functions of \( S_\mu \). For the applications in the proof of existence results, we need to analyze carefully the concentration of the extremal functions at \( x = 0 \) as \( \varepsilon \to 0 \). For any \( m \in \mathbb{N} \), \( \varepsilon > 0 \) such that \( B_{\frac{1}{m}} \subset \Omega \), define

\[
u^m_\varepsilon(x) = \begin{cases} u^\ast_m(x) - u^\ast_1 \left( \frac{1}{m} \right), & \text{if } x \in B_{\frac{1}{m}} \setminus \{0\}, \\ 0, & \text{if } x \in \Omega \setminus B_{\frac{1}{m}}. \end{cases}
\]

Lemma 3.3 (Concentration of Extremal Functions). For \( m \) large enough and \( \varepsilon \) small enough, we have

\[
\|u^m_\varepsilon\|_{L^2_\mu}^2 \leq S_\mu^{\frac{n}{2(n+1-\theta)}} + C_\varepsilon \frac{1}{2^2} m^2 \sqrt{\mu - \mu^\ast}, \tag{3.5}
\]

and

\[
\|u^m_\varepsilon\|_{L^2_\mu}^2 \geq S_\mu^{\frac{n}{2(n+1-\theta)}} - C_\varepsilon \frac{1}{2^2} m^2 \sqrt{\mu - \mu^\ast} - n, \tag{3.6}
\]

where here and hereafter \( C \) is a universal positive constant independent of \( m \) and \( \varepsilon \). Furthermore, let \( \varepsilon = \left( \frac{1}{m} \right)^h \) and \( h \geq \frac{2^2}{2} (1 + l) (\sqrt{\mu - \mu^\ast}) \), \( l \geq 0 \); we have the following estimate

\[
\|u^m_\varepsilon\|_{L^2} \geq C_\varepsilon \frac{1}{2^2} m^{(1+l)(2\sqrt{\mu - \mu^\ast} - \delta)} \quad \text{as } m \to \infty. \tag{3.7}
\]
Proof. We only prove (3.7) and omit the proofs of (3.5) and (3.6).

\[ \| u_m \|_{L^2}^2 = \int_{\Omega} |x|^{-2(a+1)+c} \left( u^*_m(x) - u^*_m \left( \frac{1}{m} \right) \right)^2 \, dx \]

\[ \geq \int_{B\left( \frac{1}{m} \right)} |x|^{-2(a+1)+c} \left( u^*_m(x) - u^*_m \left( \frac{1}{m} \right) \right)^2 \, dx \]

\[ \geq \int_{B\left( \frac{1}{m} \right)} |x|^{-2(a+1)+c} \left( u^*_m \left( \frac{1}{m} \right)^{1+l} \right) - u^*_m \left( \frac{1}{m} \right) \right)^2 \, dx \]

\[ \geq \left( u^*_m \left( \frac{1}{m} \right)^{1+l} \right) - u^*_m \left( \frac{1}{m} \right) \int_{B\left( \frac{1}{m} \right)} |x|^{-2(a+1)+c} \, dx \]

\[ = \frac{\omega_N}{N} \int_0^{(\frac{1}{m})^{1+l}} r^{N-1-2(a+1)+c} \, dr \, C_0^2 \, \varepsilon^{\frac{4}{2}} \]

\[ \times \left\{ \left( \frac{1}{m} \right)^{\frac{2b}{\sqrt{\mu-b}}} \left( \frac{1}{m} \right)^{(1+l) \frac{2b}{\sqrt{\mu-b}}} (\sqrt{\mu}+\sqrt{\mu-b}) \right\}^2 \]

\[ \left[ \left( \frac{1}{m} \right)^{\frac{2b}{\sqrt{\mu-b}}} \left( \frac{1}{m} \right)^{(1+l) \frac{2b}{\sqrt{\mu-b}}} (\sqrt{\mu}+\sqrt{\mu-b}) \right]^{-\frac{2}{2}} \]

\[ = C \varepsilon^{\frac{4}{2}} m^{(1+l)(2\sqrt{\mu-b})} (1+o(1)) \quad m \to \infty, \]

and in the last equality, we have used the fact that

\[ \frac{1}{r^a + r^b} = r^{-a} (1 + o(1)) \quad r \to 0, \]

if \( a < b \).

\[\square\]

4. Existence results

In this section, we shall prove two existence results for weak solutions to (1.1) in different ranges of parameters \( \lambda, \mu \). In particular, these results show that \( \mu - \frac{(N+2d)c}{2N} \leq \mu < \mu - M \) is non-resonant. First, we will give the threshold \( \frac{d}{N} S_{\mu}^{\frac{N}{2}} \), under which the energy functional \( I_{\lambda, \mu} \) satisfies the (PS)\( _c \) condition. Secondly, we apply various variational principles to ensure the existence of a (PS)\( _c \) sequence at some minimax values. Lastly, to obtain the existence of weak solutions to (1.1), we need to show that the minimax values obtained are under the threshold \( \frac{d}{N} S_{\mu}^{\frac{N}{2}} \).

Lemma 4.1. (Threshold of Existence) Suppose that \( \{ u_n \} \subset \mathcal{H}_\mu \) is a (PS)\( _c \) sequence of \( I_{\lambda, \mu} \) at energy level \( \beta \), i.e., we have

\[ I_{\lambda, \mu}(u_n) \rightarrow \beta, \quad \text{and} \quad I_{\lambda, \mu}'(u_n) \rightarrow 0 \quad \text{in} \ (\mathcal{H}_\mu)', \]

Then there exists \( u \in \mathcal{H}_\mu \) such that up to a subsequence we have

\[ u_m \rightharpoonup u, \quad \text{and} \quad I_{\lambda, \mu}'(u) = 0. \]

Furthermore, if \( \beta \in \left( 0, \frac{d}{N} S_{\mu}^{\frac{N}{2}} \right) \), then \( u \neq 0 \), and \( u \) is a non-trivial weak solution to (1.1).
**Proof.** 1. Any (PS)\textsubscript{c} sequence of $I_{\lambda, \mu}$ is bounded in $\mathcal{H}_\mu$.

Define
\[ f(x, u) = \lambda |x|^{-2(a+1)+\varepsilon} u + |x|^{-2(b+\varepsilon)} |u|^{2_\ast - 2} u \]
\[ F(x, u) = \frac{1}{2} \lambda |x|^{-2(a+1)+\varepsilon} u^2 + \frac{1}{2_\ast} |x|^{-2(b+\varepsilon)} |u|^{2_\ast} u. \]

It is easy to see that $f(x, u)$ and $F(x, u)$ satisfy the Ambrosetti–Rabinowitz condition [1], i.e., there exists $\theta \in \left(\frac{1}{2}, \frac{1}{2} \right)$, $\tilde{u} > 0$ such that
\[ F(x, u) \leq \theta f(x, u)u \quad \text{for a.e.} \ x \in \Omega, \forall |u| \geq \tilde{u}. \] (4.1)

Thus, the standard argument yields the boundedness of the (PS)\textsubscript{c} sequence in $\mathcal{H}_\mu$ (cf. [4,22,21]).

2. $I_{\lambda, \mu} : \mathcal{H}_\mu \rightarrow (\mathcal{H}_\mu)' = \mathcal{H}_\mu$ is weakly continuous.

Suppose that $u_m \rightarrow u$ in $\mathcal{H}_\mu$ as $m \rightarrow \infty$. For any $v \in \mathcal{H}_\mu$, we have
\[ (I'(u_m), v) = (u_m, v)_{\mathcal{H}_\mu} - \lambda \int_{\Omega} |x|^{-2(a+1)+\varepsilon} u_m v dx - \int_{\Omega} |x|^{-2(b+\varepsilon)} |u_m|^{2_\ast - 2} u_m v dx. \]

Since $\mathcal{H}_\mu \hookrightarrow L^2(\Omega, |x|^{-b_2})$ is compact, it suffices to show that
\[ \int_{\Omega} |x|^{-2(b+\varepsilon)} |u_m|^{2_\ast - 2} u_m v dx \rightarrow \int_{\Omega} |x|^{-2(b+\varepsilon)} |u|^{2_\ast - 2} u v dx. \]

Note that $\{u_m\}$ is weakly convergent, and hence bounded in $\mathcal{H}_\mu$ and also in $L^{2_\ast}(\Omega, |x|^{-b_2})$ by the continuous embedding $\mathcal{H}_\mu \hookrightarrow L^{2_\ast}(\Omega, |x|^{-b_2})$, and thus $|x|^{-(2_\ast - b_2)} |u_m|^{2_\ast - 1}$ is bounded in $L^{\frac{2_\ast}{2_\ast - b_2}}(\Omega)$, up to a subsequence, and converges weakly to $|x|^{-(2_\ast - b_2)} |u|^{2_\ast - 2} u$ in $L^{\frac{2_\ast}{2_\ast - b_2}}(\Omega)$.

3. $\{u_m\}$ converges strongly to $u$ in $\mathcal{H}_\mu$, which is a weak solution to (1.1). Furthermore, $u \neq 0$ if $\beta \in (0, \frac{d}{N} S_{\mu}^N)$.

In fact, from the above two steps, we know that there exists $u \in \mathcal{H}_\mu$ such that up to a subsequence $u_m \rightarrow u$ in $\mathcal{H}_\mu$ and $I_{\lambda, \mu}(u) = 0$. To prove $u \neq 0$, let $v_m = u_m - u$. If $\beta \in \left(0, \frac{d}{N} S_{\mu}^N\right)$, we assume that $u = 0$ by contradiction. From the compact embedding $\mathcal{H}_\mu \hookrightarrow L^2(\Omega, |x|^{-2(a+1)+\varepsilon})$, we know that $u_m \rightarrow 0$ in $L^2(\Omega, |x|^{-2(a+1)+\varepsilon})$, i.e., $\|u_m\|_{L^2} \rightarrow 0$. Thus, $(I'(u_m), v_m) = o(1)\|u_m\|_{\mathcal{H}_\mu} = o(1)$ implies that
\[ o(1) \geq \|u_m\|_{\mathcal{H}_\mu}^2 \left(1 - S_{\mu}^{-\frac{2}{2_\ast}} \|u_m\|_{L^2}^{2_\ast - 2}\right), \]
by the definition of $S_{\mu}$, i.e., $\|u_m\|_{\mathcal{H}_\mu} \geq S_{\mu} \|u_m\|_{L^2}^{2_\ast}$. Thus as $m \rightarrow \infty$, there are only two possibilities: $\|u\|_{\mathcal{H}_\mu} \rightarrow 0$ and $\|u_m\|_{\mathcal{H}_\mu}^2 \geq S_{\mu}^{\frac{2_\ast}{2_\ast - 2}} + o(1)$.

If $\|u\|_{\mathcal{H}_\mu} \rightarrow 0$, then
\[ I_{\lambda, \mu}(u_m) = \frac{1}{2} \|u_m\|_{\mathcal{H}_\mu}^2 - \frac{1}{2} \lambda \|u_m\|_{L^2}^2 - \frac{1}{2_\ast} \lambda \|u_m\|_{L^{2_\ast}}^{2_\ast} \rightarrow 0 = c, \]
which contradicts $c > 0$. On the other hand, if $\|u_m\|_{\mathcal{H}_\mu}^2 \geq S_{\mu}^{\frac{2_\ast}{2_\ast - 2}} + o(1)$, then
\[ I_{\lambda, \mu}(u_m) = \frac{1}{2} \|u_m\|_{\mathcal{H}_\mu}^2 - \frac{1}{2_\ast} \|u_m\|_{L^{2_\ast}}^{2_\ast} + o(1) \]
\[ = \frac{2_\ast - 2}{2} \|u_m\|_{\mathcal{H}_\mu}^2 + \frac{1}{2_\ast} (\|u_m\|_{\mathcal{H}_\mu}^2 - \|u_m\|_{L^{2_\ast}}^2) + o(1) \]
\[ \geq \frac{2_\ast - 2}{2} S_{\mu}^{\frac{2_\ast}{2_\ast - 2}} + o(1) \]
\[ = \frac{d}{N} S_{\mu}^N + o(1), \]
and thus, $c \geq \frac{d}{N} S_{\mu}^N$, which contradicts $c < \frac{d}{N} S_{\mu}^N$. \(\blacksquare\)
Lemma 3.3

(1.1)

Lemma 4.1 has a non-trivial solution, if parameters $N, a, b, c, \lambda, \mu$ satisfy the following conditions

$$
\begin{align*}
N > 2(\sqrt{M} + 1), & \quad M = \max \left\{ b^2, \frac{c^2}{4} \right\} \\
-\infty < a < \frac{N-2}{2}, & \quad a \leq b < a+1, \ c > 0 \\
\left( \frac{N-2(a+1)}{2} \right)^2 - \left( \frac{N-2}{2} \right)^2 \leq \mu < \left( \frac{N-2(a+1)}{2} \right)^2 - M \\
\lambda > 0, \ & \lambda \notin \sigma_\mu.
\end{align*}
$$

Proof. Our proof is divided into two different cases: $\lambda \in (0, \lambda_1)$ and $\lambda > \lambda_1$. In the process of the proof, we always take $\varepsilon = m^{-h}$, $h > 0$, and denote $u^m_\varepsilon$ by $u_m$.

1. $\lambda \in (0, \lambda_1)$ case.

It is easy to verify that the energy functional satisfies the geometrical conditions of the Mountain Pass lemma (cf. [1]); this implies that $I_{\lambda, \mu}$ has a (PS)$_c$ sequence at level

$$
\beta = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_{\lambda, \mu}(\gamma(t)) > 0,
$$

where

$$
\Gamma = \{ \gamma \in C([0,1], \mathcal{H}_\mu) \mid \gamma(0) = 0, \ \gamma(1) = e, \ I_{\lambda, \mu}(e) < 0 \}.
$$

From Lemma 4.1, it suffices to show that $\beta < \frac{d}{N} S_{\mu}^{\frac{N}{d}}$. To do this, we only need to prove that there exists $m$ large enough such that

$$
\max_{t \geq 0} I_{\lambda, \mu}(tu^m_\varepsilon) = \max_{t \geq 0} I_{\lambda, \mu}(tu_m) \geq \frac{d}{N} S_{\mu}^{\frac{N}{d}}.
$$

By contradiction, assume that for any $m > 0$, there exists $t_m \geq 0$ such that $I_{\lambda, \mu}(t_m u_m) \geq \frac{d}{N} S_{\mu}^{\frac{N}{d}}$.

Claim A. Up to a subsequence, $t_m \to t_0 > 0$, where $t_0$ is a finite number.

Assuming that $t_m$ is unbounded, then up to a subsequence, $t_m \to +\infty$, and thus

$$
I_{\lambda, \mu}(t_m u_m) = \frac{1}{2} t_m^2 \| u_m \|_{H_\mu}^2 - \frac{1}{2} t_m^2 \lambda \| u_m \|_{L^2}^2 - \frac{1}{2} t_m^2 \| u_m \|_{L^2}^{2*} \\
\leq \frac{1}{2} t_m^2 \| u_m \|_{H_\mu}^2 - \frac{1}{2} t_m^2 \| u_m \|_{L^2}^{2*}.
$$

From Lemma 3.3, set $\varepsilon = m^{-h}$, $h > 0$, and choose $h$ large enough such that $\| u_m \|_{H_\mu}$ and $\| u_m \|_{L^2}^{2*}$ are bounded; then as $t_m \to \infty$, we have

$$
I_{\lambda, \mu}(t_m u_m) \to -\infty,
$$

which contradicts

$$
I_{\lambda, \mu}(t_m u_m) \geq \frac{d}{N} S_{\mu}^{\frac{N}{d}} > 0.
$$

Thus, $t_m$ is bounded, and hence up to a subsequence, $t_m \to t_0 \geq 0$. But if $t_0 = 0$, then from the definition of $\lambda_1$ and the continuous embedding, we know that $\| u_m \|_{L^2}$ and $\| u_m \|_{L^2}^{2*}$ are bounded provided that $\| u_m \|_{H_\mu}^2$ is bounded. Thus, as in the argument above, choosing $h$ large enough such that $\| u_m \|_{H_\mu}^2$ is bounded, we have

$$
I_{\lambda, \mu}(t_m u_m) = \frac{1}{2} t_m^2 \| u_m \|_{H_\mu}^2 - \frac{1}{2} t_m^2 \lambda \| u_m \|_{L^2}^2 - \frac{1}{2} t_m^2 \| u_m \|_{L^2}^{2*} \to 0
$$

which contradicts $\beta > 0$. 

Theorem 4.2 (Non-Resonant Case). Problem (1.1) has a non-trivial solution, if parameters $N, a, b, c, \lambda, \mu$ satisfy the following conditions

$$
\begin{align*}
N > 2(\sqrt{M} + 1), & \quad M = \max \left\{ b^2, \frac{c^2}{4} \right\} \\
-\infty < a < \frac{N-2}{2}, & \quad a \leq b < a+1, \ c > 0 \\
\left( \frac{N-2(a+1)}{2} \right)^2 - \left( \frac{N-2}{2} \right)^2 \leq \mu < \left( \frac{N-2(a+1)}{2} \right)^2 - M \\
\lambda > 0, \ & \lambda \notin \sigma_\mu.
\end{align*}
$$
Claim B. \( I_{\lambda, \mu}(t_m u_m) \leq \frac{d}{N} S_{\mu}^N + C e^{\frac{4}{2m}} (m^2 \sqrt{\mu - \mu} - m^{(1 + l)(2\sqrt{\mu - \mu} - c)}) \), \( m \to +\infty \).

In fact, it is easy to see that
\[
\max_{t \geq 0} \left( \frac{t^2}{2} A - \frac{t^2}{2} B \right) = \frac{2 - 2}{\lambda - 2} A \left( \frac{A}{B} \right)^{\frac{2}{\lambda - 2}},
\]
and the maximum is attained at \( t = \frac{2}{\lambda - 2} \). From (3.5), (3.6) and (4.2), we have that
\[
\frac{1}{2} t_m^2 \| u_m \|^2_{H_{\mu}} - \frac{1}{2} t_m^2 \| u_m \|^2_{L_{2^*}} \leq \frac{2 - 2}{\lambda - 2} \| u_m \|^2_{H_{\mu}} \left( \frac{\| u_m \|^2_{H_{\mu}}}{\| u_m \|^2_{L_{2^*}}} \right)^{\frac{2}{\lambda - 2}} \leq \frac{d}{N} S_{\mu}^N + C e^{\frac{4}{2m}} m^{(1 + l)(2\sqrt{\mu - \mu} - c)}.
\]
Thus, combining with Claim A and (3.7), we get Claim B.

Since \( \mu < \left( \frac{N - 2(1 + 1)}{2} \right)^2 - M \) implies that \( \sqrt{\mu - \mu} - c > 0 \), choose \( h \) and \( l \) large enough such that \( (1 + l) \sqrt{\mu - \mu} - c > 2\sqrt{\mu - \mu} \); then as \( m \to \infty \), \( I_{\lambda, \mu}(t_m u_m) < \frac{d}{N} S_{\mu}^N \) which contradicts the assumption \( I_{\lambda, \mu}(t_m u_m) \geq \frac{d}{N} S_{\mu}^N \).

2. \( \lambda > \lambda_1 \) case.

In this case, we shall apply the linking argument (cf. Section 2.8 in [21]). Suppose that \( \lambda \in (\lambda_k, \lambda_{k+1}) \), \( k \geq 1 \). Let
\[ Q_m^e := [(B_m \cap H_m^{-}) \oplus [0, R][u_m^e]] \]
\[ \Gamma := \{ \gamma \in C(Q_m^e, H_{\mu}) \mid \gamma(u) = u, \forall u \in \partial Q_m^e \}. \]
In the spirit of Lemma 4 in [13], one can show that \( S_\rho \cap H^+ \) and \( \partial Q_m^e \) link for \( R > \rho \), \( m \) large enough, and \( I_{\lambda, \mu} \) has a (PS)\(_c\) sequence at level
\[ \beta = \inf \max_{\gamma \in \Gamma} I_{\lambda, \mu}(\gamma(t)) > 0. \]

From Lemma 3.3, we only need prove that there exists \( m \) large enough such that
\[ \inf \max_{u \in Q_m^e} I_{\lambda, \mu}(u) \leq \max_{u \in Q_m^e} I_{\lambda, \mu}(u) < \frac{d}{N} S_{\mu}^N. \]

By contradiction, assume that for any \( m > 0 \), there exists \( v_m = w_m + t_m u_m \in Q_m^e \) such that
\[ I_{\lambda, \mu}(v_m) = \max_{u \in Q_m^e} I_{\lambda, \mu}(u) \geq \frac{d}{N} S_{\mu}^N, \]
where \( w_m \in H_m^{-}, t_m \in [0, R] \). Since \( |\text{supp}(w_m) \cap \text{supp}(u_m)| = 0 \), we have
\[ I_{\lambda, \mu}(v_m) = I_{\lambda, \mu}(w_m) + I_{\lambda, \mu}(t_m u_m) \geq \frac{d}{N} S_{\mu}^N. \]

From (iii) in Lemma 2.3, it follows that
\[ I_{\lambda, \mu}(w_m) = \frac{1}{2} \| w_m \|^2_{H_{\mu}} - \frac{1}{2} \lambda \| w_m \|^2_{L_{2^*}} - \frac{1}{2} \| w_m \|^2_{L_{2^*}} \]
\[ \leq \frac{1}{2}(\lambda_k + c) m^{-2} \| w_m \|^2_{L^2} \]
\[ \leq 0, \]
for \( m \) large enough. Thus, for any \( m > 0 \), there exists \( t_m \in [0, R] \) such that \( I_{\lambda, \mu}(t_m u_m) \geq \frac{d}{N} S_{\mu}^N \). The rest is similar to the argument in the first case. \( \blacksquare \)
Theorem 4.3 (Resonant Case). Problem (1.1) has a non-trivial solution if parameters \( N, a, b, c, \lambda, \mu \) satisfy the following conditions

\[
\begin{align*}
N &> 2(\sqrt{M} + 1), \quad M = \max \left\{ b^2, \frac{c^2}{4} \right\} \\
-\infty < a < \frac{N - 2}{2}, \quad &a \leq b < a + 1, \quad c > 0, \quad b^- = \max\{-b, 0\} \\
\left( \frac{N - 2(a + 1)}{2} \right)^2 - \left( \frac{N - 2}{2} \right)^2 &\leq \mu < \left( \frac{N + 2d}{2N} c + \frac{2b^-d}{N} \right)^2
\end{align*}
\]

\( \lambda \in \sigma_{\mu} \).

Proof. We follow the argument in [6]. Suppose that \( \lambda = \lambda_k, \ k \geq 1 \). Like for the second case in the proof of Theorem 4.2, we shall apply the linking argument. It suffices to show that there exists \( m \) large enough such that

\[
\inf_{\gamma \in \Gamma} \max_{u \in Q^m_n} I_{\lambda, \mu}(\gamma(u)) \leq \max_{u \in Q^m_n} I_{\lambda, \mu}(u) < \frac{d}{N} S^N_{\mu}. 
\]

By contradiction, assume that for any \( m > 0 \), there exists \( v_m = w_m + t_m u_m \in Q^m_n \) such that

\[
I_{\lambda, \mu}(v_m) = \max_{u \in Q^m_n} I_{\lambda, \mu}(u) \geq \frac{d}{N} S^N_{\mu},
\]

where \( w_m \in H^-_m \), \( t_m \in [0, R] \). Since \( |\text{supp}(w_m) \cap \text{supp}(u_m)| = 0 \), we have

\[
I_{\lambda, \mu}(v_m) = I_{\lambda, \mu}(w_m) + I_{\lambda, \mu}(t_m u_m) \geq \frac{d}{N} S^N_{\mu}.
\]

From (iii) in Lemma 2.3 and (4.2), it follows that

\[
I_{\lambda, \mu}(w_m) = \frac{1}{2} \| w_m \|_{H^-_m}^2 - \frac{\lambda_k}{2} \| w_m \|_{L^2}^2 - \frac{1}{2*} \| w_m \|_{L^{2*}}^2 
\leq \frac{1}{2} \sum_{n \geq 2} C m^{-2\sqrt{\mu - \mu}} \| w_m \|_{L^2}^2 - \frac{1}{2*} \| w_m \|_{L^{2*}}^2 
\leq \frac{1}{2} \sum_{n \geq 2} C m^{-2\sqrt{\mu - \mu}},
\]

for \( m \) large enough. Taking \( \varepsilon = \left( \frac{1}{m} \right)^h \), \( h = (2* - 1)\sqrt{\mu - \mu} \geq (\sqrt{\mu - \mu} + 1) \left( \frac{2+2b^-}{2} \right) \), we have \( I_{\lambda, \mu}(w_m) \leq \varepsilon \sum_{n \geq 2} m^{-2\sqrt{\mu - \mu}} \). Then the same argument as in Theorem 4.2 yields that Claim A is also true in this case, i.e., up to a subsequence, \( t_m \to t_0 > 0 \). Hence we have

\[
I_{\lambda, \mu}(t_m u_m) = \frac{1}{2} \sum_{n \geq 2} t_m \| u_m \|_{L^2}^2 - \frac{1}{2} \sum_{n \geq 2} t_m \| u_m \|_{L^{2*}}^2 
\leq \frac{1}{2} \sum_{n \geq 2} C e^{\frac{4}{\sum_{n \geq 2} m^{2\sqrt{\mu - \mu}}}} + C e^{\frac{4}{\sum_{n \geq 2} m^{(1+)(2\sqrt{\mu - \mu}-c)}}} 
\leq \frac{1}{2*} \sum_{n \geq 2} C e^{\frac{4}{\sum_{n \geq 2} m^{2\sqrt{\mu - \mu}}}} + C e^{\frac{4}{\sum_{n \geq 2} m^{(1+)(2\sqrt{\mu - \mu}-c)}}}.
\]

In the last inequality, we have supposed that

\[
\begin{align*}
(1+)(2\sqrt{\mu - \mu} - c) > 2\sqrt{\mu - \mu} - 2* b \\
(1+)(2\sqrt{\mu - \mu} - c) > 2\sqrt{\mu - \mu}.
\end{align*}
\]
To get the existence of such an \( l \), we need the assumption \( \mu < \left( \frac{N-2(a+1)}{2} \right)^2 - \left( \frac{N+2d}{N}c + \frac{2b-d}{N} \right)^2 \), \( b^- = \max \{-b, 0\} \).

Thus, we have

\[
I_{\lambda, \mu}(v_m) = I_{\lambda, \mu}(w_m) + I_{\lambda, \mu}(t_m u_m) \\
\leq \frac{d}{N} S_{\frac{N}{2} \mu}^{\frac{N}{2}} - C \varepsilon^{\frac{4}{N-2}} m^{(1+l)(2\sqrt{\mu - c})} \\
< \frac{d}{N} S_{\frac{N}{2} \mu}^{\frac{N}{2}}, \quad m \to \infty,
\]

which contradicts the assumption \( I_{\lambda, \mu}(v_m) \geq \frac{d}{N} S_{\frac{N}{2} \mu}^{\frac{N}{2}} \). \( \blacksquare \)

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References