Stereo 3D Simulation of Rigid Body Inertia Ellipsoid for The Purpose of Unmanned Helicopter Autopilot Tuning

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Stereo 3D Simulation of Rigid Body Inertia Ellipsoid for The Purpose of Unmanned Helicopter Autopilot Tuning

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ABSTRACT : *The current paper aims at presenting the capabilities and benefits of an online stereo 3D simulation for the purpose of unmanned helicopter autopilot tuning. The parameters of the helicopter airframe are important for tuning the gains in the autopilot. The airframe is modelled as a rigid body whose inertial properties are fully described by the inertia ellipsoid. The inertia ellipsoid is another form of presenting the moment of inertia tensor of rigid bodies but instead of using a numerical approach the described method implements 3D graphical visualization. The current paper focuses on the benefits from stereoscopic graphical 3D presentation of the inertia ellipsoid and how such a method helps designers and researchers analyse, synthesise and tune unmanned helicopter autopilot algorithms. The simulation, subject to the current material, may be observed at the following web address: http://ialms.net/sim/.*

KEYWORDS: *Inertia ellipsoid in stereo 3D simulation, Rigid body inertia ellipsoid, Unmanned helicopter autopilot tuning.*

I. INTRODUCTION

The current article shows a new method of presenting the inertia ellipsoid of rigid bodies and its properties using a visual online environment. A simulation that has unrestricted access on the Internet demonstrates the inertia tensor of various rigid bodies to researchers in universities and institutes around the world. The major features of the advised simulation are the stereo 3D environment, in which the inertia ellipsoid of various rigid bodies is demonstrated along with translations of the ellipsoid and its moment of inertia tensor to any point in the body reference frame. Along with the inertia ellipsoid, the principal axes of inertia are also displayed. The simulation prints the diagonalized translated moment of inertia tensor and the diagonalizing rotation matrix (Figure 1). The simulation, described in the current paper, may be observed on web address: http://ialms.net/sim/.

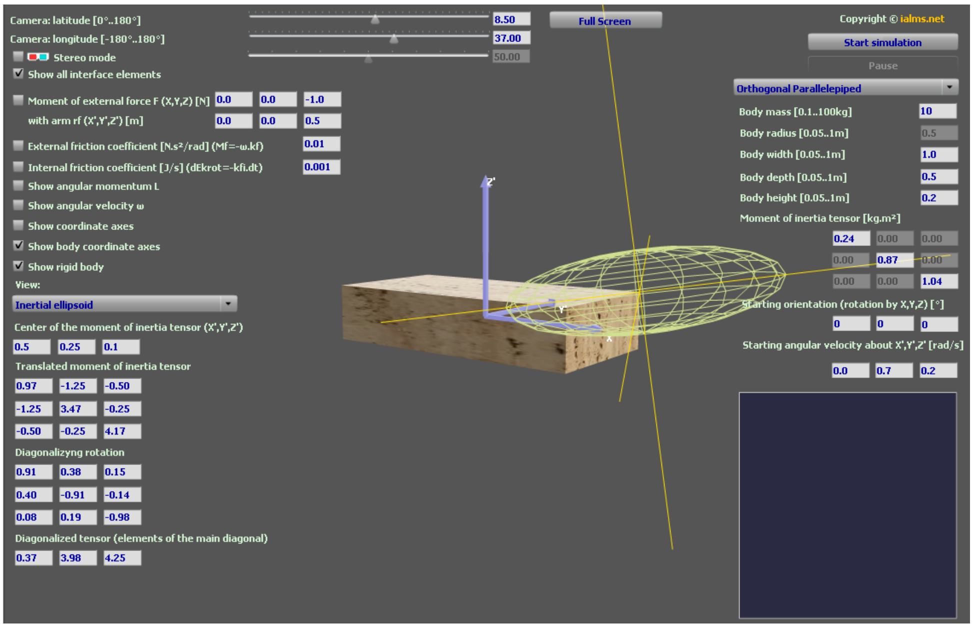

Figure 1. Translated moment of inertia tensor and inertia ellipsoid to the vertex of a rectangular parallelepiped-shaped homogeneous rigid body and 3D graphical visualization

II. MATHEMATICAL FOUNDATIONS OF THE DESCRIBED 3D SIMULATION

To prove the consistency and fidelity of the presented simulation, a concise introduction to the inertial properties of rigid bodies follows. Moment of inertia of a rigid body about a given axis describes quantitatively the body inertia behaviour during a rotation about that axis. If a rigid body has volume V then its moment of inertia of about axis OO' is:

$$I_{OO'} = \int_V r^2 dm = \int_V r^2 \rho dV ,$$

where ρ is the body density at the location of elementary mass $dm = \rho dV$ and r is the perpendicular radius-vector from the axis of rotation OO' to the elementary mass. About any given axis the rigid body has a certain and generally different moment of inertia. Each moment of inertia could be calculated using the equation above. An alternative way of calculating the moment of inertia is by using the relation between the angular momentum and angular velocity:

$$\vec{L} = \vec{\omega} I ,$$

Here vector $\vec{L}$ is the angular momentum, and vector $\vec{\omega}$ is the angular velocity of the rotational motion. Although useful, this equation is applicable only to rotations that are realized about a principal axis of inertia. Only then $\vec{L}$ and $\vec{\omega}$ are parallel to each other and the relation between them is presented using a product with a scalar value, as in the example above. Such a rotation is the rotation of an axially symmetric homogenous rigid body about its axis of symmetry.

Generally, vectors $\vec{L}$ and $\vec{\omega}$ are not parallel, but a relation connecting them still exists and it is a tensor of second rang called moment of inertia tensor:

$$\mathbf{I}_O = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} \end{bmatrix}$$

This tensor is in respect to a point O, having that axis of rotation passes through this point. To define the relation between $\vec{L}$ and $\vec{\omega}$, these two vectors should be presented in a matrix form (one-row-matrixes):

$$\mathbf{L} = \boldsymbol{\omega} \mathbf{I}_O$$

It follows that components of vector $\vec{L}$ are:

$$L_x = \omega_x I_{xx} + \omega_y I_{xy} + \omega_z I_{xz}$$
$$L_y = \omega_x I_{xy} + \omega_y I_{yy} + \omega_z I_{yz}$$
$$L_z = \omega_x I_{xz} + \omega_y I_{yz} + \omega_z I_{zz}$$

In this general case, the momentary axis of rotation coincides with $\vec{\omega}$. The inertia about this axis of rotation creates an angular momentum about this same axis that is the projection of the angular momentum vector $\vec{L}$ along vector $\vec{\omega}$. In matrix form we have:

$$L_{\boldsymbol{\omega}} = \frac{\mathbf{L}\boldsymbol{\omega}^{\sim}}{\omega} \quad (\boldsymbol{\omega}^{\sim} \text{ is the transposed matrix of } \boldsymbol{\omega})$$

Substituting the angular momentum with the product of angular velocity and the moment of inertia tensor yields:

$$L_{\boldsymbol{\omega}} = \frac{\boldsymbol{\omega}\mathbf{I}\boldsymbol{\omega}^{\sim}}{\omega} = \mathbf{n}_{\boldsymbol{\omega}} \mathbf{I} \mathbf{n}_{\boldsymbol{\omega}}^{\sim} \omega = I_{\boldsymbol{\omega}} \omega ,$$

or the projection of the $\vec{L}$ along the axis of rotation is derived from the magnitude of $\vec{\omega}$ and the scalar quantity $I_{\boldsymbol{\omega}}$. It is called a reduced moment of inertia from the moment of inertia tensor about a given axis of rotation defined by its direction unit vector $\mathbf{n}_\omega$ or $\vec{n}_{\vec{\omega}} = \dfrac{\vec{\omega}}{|\vec{\omega}|}$ in vector form. Thus the reduced moment of inertia may be expressed using the components of the unit vector $\vec{n}_{\vec{\omega}}$ and the moment of inertia tensor:

$$I_\omega = \mathbf{n}_\omega \mathbf{I} \mathbf{n}_\omega^{\sim} = n_x^2 I_{xx} + n_y^2 I_{yy} + n_z^2 I_{zz} + 2n_x n_y I_{xy} + 2n_x n_z I_{xz} + 2n_y n_z I_{yz},$$

Because vector $\vec{n}_{\vec{\omega}}$ has unit length, its components could be substituted with the direction cosines defining the axis of rotation:

$$I_\omega = I_{xx}\cos^2\alpha + I_{yy}\cos^2\beta + I_{zz}\cos^2\gamma + 2I_{xy}\cos\alpha\cos\beta + 2I_{xz}\cos\alpha\cos\gamma + 2I_{yz}\cos\beta\cos\gamma$$

At the same time, the reduced moment of inertia may be expressed through $\vec{L}$ and $\vec{\omega}$ as follows:

$$I_\omega = \frac{L_\omega}{\omega} = \frac{\mathbf{L}\boldsymbol{\omega}^{\sim}}{\omega^2}$$

The inertial ellipsoid enables the researcher to observe in stereoscopic 3D graphical scene the moment of inertia tensor properties, which is essential for the following autopilot synthesis labour. The ellipsoid is defined in such a manner that its three axes are the reciprocals of the square roots of the principal moments of inertia (Figure 1, 2, 3 and 4):

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \Leftrightarrow I_{xx}x^2 + I_{yy}y^2 + I_{zz}z^2 = 1$$

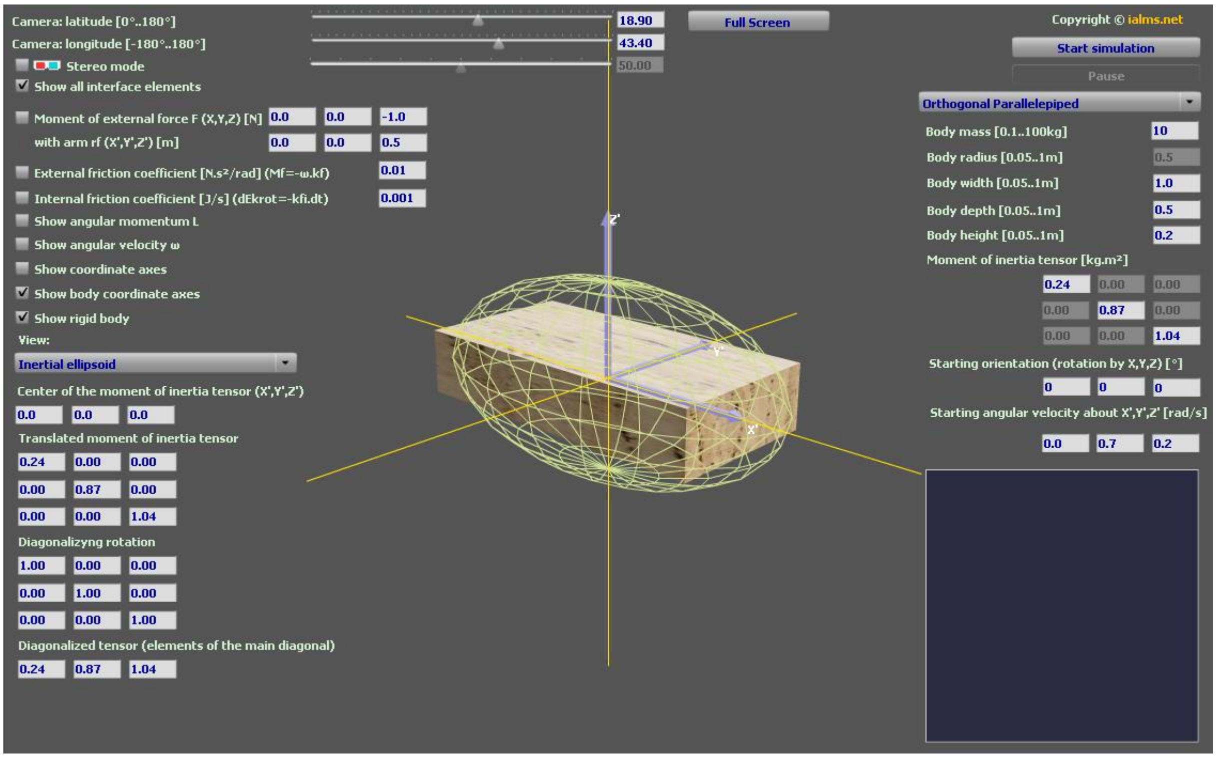

Figure 2. The inertia ellipsoid of the non-displaced moment of inertia tensor

What the simulation offers to the researcher

The simulation was developed in order to facilitate researchers and scientists while studying the mechanical properties of unmanned helicopter airframes and their input in autopilot algorithms. Further the simulation helps engineers in unmanned helicopter autopilot analysis and synthesis. The simulation helps the construction of the inertia ellipsoid of different rigid bodies and the observation of the principal moments of inertia. To make clear the effectiveness of the simulation while solving mechanics problems, several tasks are disclosed as examples. The solutions could be observed in 3D stereo mode. Each step of a solution has its graphical representation, which clarifies the notion of the used mathematical formulae and presents the acquired results and solutions in graphical manner. Hence the simulation is useful in drawing practical understanding among researchers.

Task 1

Let's have a homogenous rigid body. Its shape is a rectangular parallelepiped and its mass is 10 kg (Figure 2). Body reference frame origin coincides with the centre of mass and its axes are parallel to the body edges. The body dimensions are $1.0, 0.5, 0.2$ m along the Ox, Oy, Oz axes respectively. The task is to calculate the moment of inertia tensor for the centre of mass $\mathbf{I}_c$.

Note: Without applying similarity transformation, the moment of inertia tensor is diagonal $\mathbf{I}_c = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}$. I_{xx}, I_{yy} and I_{zz} correspond to the principle moments of inertia of the centre of mass along the body reference frame coordinate axes. This follows from the body homogenous and symmetrical properties.

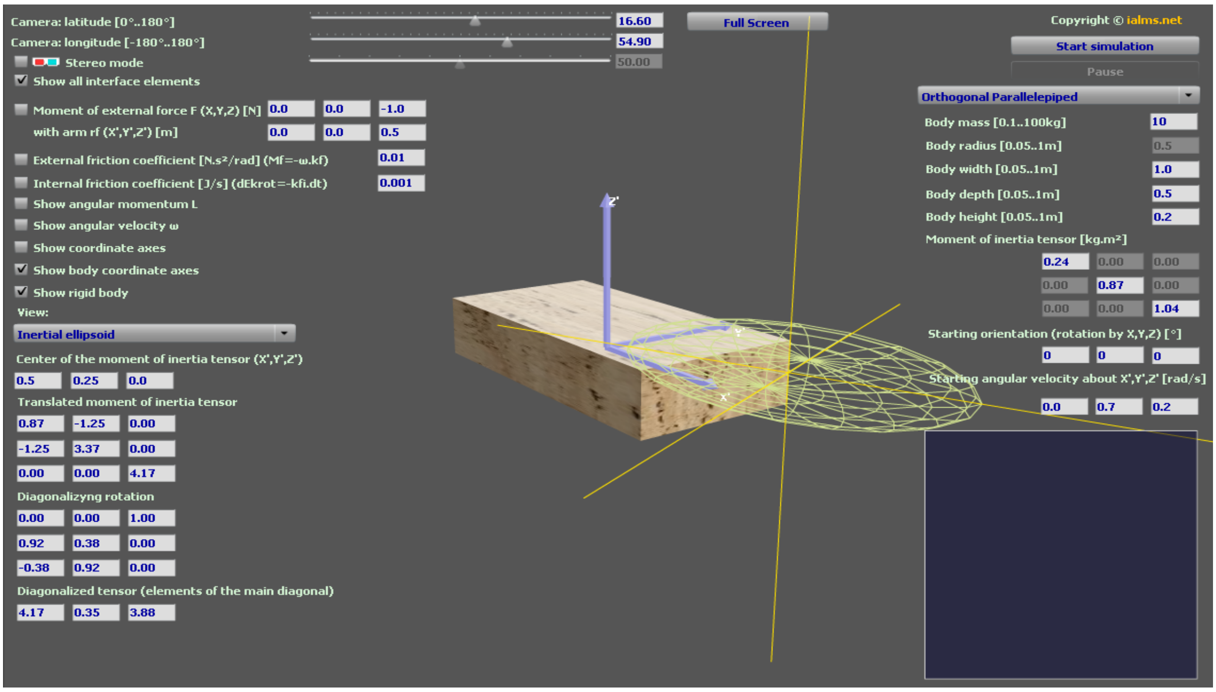

Figure 3. Displacement of the moment of inertia tensor along the Ox and Oy axes.

Solution to task 1:

Homogenous rectangular parallelepiped moment of inertia tensor at its centre of mass in a reference frame oriented along the body edges is derived by the following formula:

$$\mathbf{I}_c = \frac{1}{12} m \begin{bmatrix} b^2 + c^2 & 0 & 0 \\ 0 & a^2 + c^2 & 0 \\ 0 & 0 & a^2 + b^2 \end{bmatrix},$$

Here $m = 10$ kg is the body mass, and $a = 1$ m, $b = 0.5$ m and $c = 0.2$ m are the body dimensions. Hence, the answer follows to be:

$$\mathbf{I}_c = \begin{bmatrix} 0.24 & 0 & 0 \\ 0 & 0.87 & 0 \\ 0 & 0 & 1.04 \end{bmatrix}.$$

Task 2

Calculate the translated moment of inertia tensor from task 1 at the vertex of the body that has only positive coordinates in the body reference frame.

Solution to task 2:

The parallel axis theorem (Huygens-Steiner theorem) solves this problem.

(1) $$\mathbf{I}_t = \mathbf{I}_c + m\left(\mathbf{r}\mathbf{r}^{\sim}\mathbf{1} - \mathbf{r}\otimes\mathbf{r}\right) \Rightarrow$$
$$\mathbf{I}_t = \begin{bmatrix} I_{xx} + m\left(r_y^2 + r_z^2\right) & -mr_x r_y & -mr_x r_z \\ -mr_x r_y & I_{yy} + m\left(r_x^2 + r_z^2\right) & -mr_y r_z \\ -mr_x r_z & -mr_y r_z & I_{zz} + m\left(r_x^2 + r_y^2\right) \end{bmatrix} = \begin{bmatrix} 0.97 & -1.25 & -0.50 \\ -1.25 & 3.47 & -0.25 \\ -0.50 & -0.25 & 4.17 \end{bmatrix}$$

In the above equation matrix $\mathbf{1}$ is the 3×3 identity matrix, vector $\mathbf{r}$ is the translation vector. The body is homogenous and its centre of mass and geometrical centre coincide. It follows that the translation vector pointing to the vertex of the body with only positive coordinates is equal to $[0.5 \quad 0.25 \quad 0.1]$ (see Figure 1). Figure 3 shows a translation of the moment of inertia tensor along both the Ox and Oy axes. Note that in the case of translation along a single coordinate axis, a translated tensor is diagonal. In the case of moment of inertia tensor translation along two axes, four of the translated tensor's non-diagonal components are zeroes.

Task 3

Diagonalize the translated moment of inertia tensor from task 2.

Solution to task 3: Using the theory of symmetric matrixes one could always diagonalize a 3×3 symmetric matrix or symmetric rang 2 tensor. The diagonalized tensor will have only three non-zero values and they will be found in the main diagonal. The other three values (products of inertia) found in the non-dagonalized tensor will be represented by the transformation used to diagonalize the tensor. This transformation is a similarity transformation realized through applying a rotation matrix $\mathbf{R}$. We can prove this assumption, taking into account that under a certain rotation $\mathbf{R}$ the principle axes of the moment of inertia tensor coincide with the axes of the body reference frame and the rotated tensor becomes diagonalized:

(2) $$\mathbf{I}_D = \mathbf{R}^{\sim}\mathbf{I}\mathbf{R} = \begin{bmatrix} I_{Dxx} & 0 & 0 \\ 0 & I_{Dyy} & 0 \\ 0 & 0 & I_{Dzz} \end{bmatrix}$$

From the eigenvectors $\mathbf{n}_{Di} = [x_{Di} \quad y_{Di} \quad z_{Di}]$ and eigenvalues λ_i of the pursued diagonalized tensor we have:

(3) $$\mathbf{n}_{Di}\mathbf{I}_D = \mathbf{n}_{Di}\lambda_i \Rightarrow \mathbf{n}_{Di}\mathbf{I}_D - \mathbf{n}_{Di}\lambda_i = 0$$

This matrix equation shows a homogenous system of three linear equations having a non-trivial solution only if the determinant of its coefficients is zero:

(4) $$\left\|\mathbf{I}_D - \mathbf{1}\lambda_i\right\| = 0 \Rightarrow \left(I_{Dxx} - \lambda_i\right)\left(I_{Dyy} - \lambda_i\right)\left(I_{Dzz} - \lambda_i\right) = 0$$

Formula (4) has three real roots: $\lambda_1 = I_{Dxx}$, $\lambda_2 = I_{Dyy}$ and $\lambda_3 = I_{Dzz}$, hence the diagonalized tensor is:

(5) $$\mathbf{I}_D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

After we apply the similarity transformation (2) to equation (3) we get:

$$\mathbf{n}_{Di}\mathbf{I}_D = \mathbf{n}_{Di}\mathbf{R}^{\sim}\mathbf{I}\mathbf{R} = \mathbf{n}_{Di}\lambda_i \Rightarrow \mathbf{n}_{Di}\mathbf{R}^{\sim}\mathbf{I} = \mathbf{n}_{Di}\mathbf{R}^{\sim}\lambda_i \Rightarrow$$

(6) $$\mathbf{n}_i\mathbf{I} = \mathbf{n}_i\lambda_i \Rightarrow \mathbf{n}_i\mathbf{I} - \mathbf{n}_i\lambda_i = 0$$

It follows that vectors $\mathbf{n}_i = \begin{bmatrix} x_i & y_i & z_i \end{bmatrix}$ are the eigenvectors of $\mathbf{I}$.

(7) $\mathbf{n}_i = \mathbf{n}_{Di}\mathbf{R}^{\sim} \Rightarrow \mathbf{n}_{Di} = \mathbf{n}_i\mathbf{R}$

We see that $\mathbf{I}_D$ and $\mathbf{I}$ have the same eigenvalues. Calculating the eigenvalues of $\mathbf{I}$ and substituting them in (5) gives the wanted diagonalized tensor $\mathbf{I}_D$. Analyse equation (6) helps in finding the eigenvalues of $\mathbf{I}$. Similarly to (3), this matrix equation derives a homogenous system of three linear equations. This system has non-trivial solution only if the determinant of its coefficients is zero:

(8) $$\left\|\mathbf{I} - \mathbf{1}\lambda_i\right\| = 0 \Rightarrow \begin{Vmatrix} I_{xx} - \lambda_i & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} - \lambda_i & I_{yx} \\ I_{xz} & I_{yx} & I_{zz} - \lambda_i \end{Vmatrix} = 0$$

The cubic equation (8) yields three roots λ_i for $i = 1..3$ - $\lambda_1 = 0.37$, $\lambda_2 = 3.98$ and $\lambda_3 = 4.25$. Substituting these eigenvalues in (5) leads to the sought solution:

$$\mathbf{I}_D = \begin{bmatrix} 0.37 & 0 & 0 \\ 0 & 3.98 & 0 \\ 0 & 0 & 4.25 \end{bmatrix}$$ (see Figure 1).

Task 4

Calculate the rotation matrix $\mathbf{R}$ that diagonalizes the tensor from task 3.

Solution to task 4:

Each eigenvalue of $\mathbf{I}$ substituted in (3) yields a corresponding solution for $\mathbf{n}_{Di}$, if $\mathbf{n}_{Di}$ is constraint to a unit eigenvector:

(9) $$\begin{aligned} \lambda_1 &= I_{Dxx} \Rightarrow \mathbf{n}_{D1} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} = \mathbf{i} \\ \lambda_2 &= I_{Dyy} \Rightarrow \mathbf{n}_{D2} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} = \mathbf{j} \\ \lambda_3 &= I_{Dzz} \Rightarrow \mathbf{n}_{D3} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} = \mathbf{k} \end{aligned}$$

From (7) and (9) it follows that the rotation matrix $\mathbf{R}$ transforms vectors $\mathbf{n}_i$ to the orthogonal basis of the body reference frame $\mathbf{i}, \mathbf{j}, \mathbf{k}$. However, the rotation matrix is defined by the direction cosines of one basis rotated to another basis:

(10) $$\mathbf{R} = \begin{bmatrix} \cos\alpha_x & \cos\alpha_y & \cos\alpha_z \\ \cos\beta_x & \cos\beta_y & \cos\beta_z \\ \cos\gamma_x & \cos\gamma_y & \cos\gamma_z \end{bmatrix}$$

Vectors $\mathbf{n}_i$ are unit vectors and their components are equal to their direction cosines, so:

(11) $$\mathbf{R} = \begin{bmatrix} \mathbf{n}_1 \\ \mathbf{n}_2 \\ \mathbf{n}_3 \end{bmatrix}$$

The last step is to find the eigenvectors $\mathbf{n}_i$. We use equation (6) along with the unit length of the eigenvectors:

$$(12) \qquad \left| \begin{array}{l} x_i(I_{xx}-\lambda_i)+y_iI_{xy}+z_iI_{xz}=0 \\ x_iI_{xy}+y_i(I_{yy}-\lambda_i)+z_iI_{yz}=0 \\ x_iI_{xz}+y_iI_{yz}+z_i(I_{zz}-\lambda_i)=0 \\ \qquad x_i^2+y_i^2+z_i^2=1 \end{array} \right.$$

System (12) enables y_i and z_i to be expressed through x_i, using the first three equations, and substituting in the last equation:

$$(13) \qquad \left| \begin{array}{c} y_i = x_i \dfrac{(I_{xx}-\lambda_i)I_{yz}-I_{xy}I_{xz}}{(I_{yy}-\lambda_i)I_{xz}-I_{xy}I_{yz}} \\ z_i = x_i \dfrac{(I_{xx}-\lambda_i)I_{yz}-I_{xy}I_{xz}}{(I_{zz}-\lambda_i)I_{xy}-I_{xz}I_{yz}} \\ x_i^2\left[1+\left(\dfrac{(I_{xx}-\lambda_i)I_{yz}-I_{xy}I_{xz}}{(I_{yy}-\lambda_i)I_{xz}-I_{xy}I_{yz}}\right)^2+\left(\dfrac{(I_{xx}-\lambda_i)I_{yz}-I_{xy}I_{xz}}{(I_{zz}-\lambda_i)I_{xy}-I_{xz}I_{yz}}\right)^2\right]=1 \end{array} \right.$$

From the last equation of system (13) x_i is found as follows:

$$(14) \qquad x_i = \pm\sqrt{\frac{1}{1+\left(\dfrac{(I_{xx}-\lambda_i)I_{yz}-I_{xy}I_{xz}}{(I_{yy}-\lambda_i)I_{xz}-I_{xy}I_{yz}}\right)^2+\left(\dfrac{(I_{xx}-\lambda_i)I_{yz}-I_{xy}I_{xz}}{(I_{zz}-\lambda_i)I_{xy}-I_{xz}I_{yz}}\right)^2}}$$

We find the values of y_i and z_i by substituting x_i in the first two equations of (13). Note that the components x_i, y_i, z_i of eigenvector $\mathbf{n}_i$ do not have defined sign. There are two possibilities yielding an eigenvector $\mathbf{n}_i$ along a defined line, but with two possible opposite directions. The correct direction will be found later. After finding all three eigenvectors $\mathbf{n}_1$, $\mathbf{n}_2$ and $\mathbf{n}_3$ their components are substituted in (11) to generate the sought rotation matrix $\mathbf{R}$ (see Figure 1):

$$\mathbf{R} = \begin{bmatrix} 0.91 & 0.38 & 0.15 \\ 0.40 & -0.91 & -0.14 \\ 0.08 & 0.19 & -0.98 \end{bmatrix}$$

However, if the incorrect directions (signs in (14)) were chosen, matrix $\mathbf{R}$ would yield an inverse (improper) rotation. This fact can be verified by calculating the determinant of $\mathbf{R}$, which should be $+1$:

$$\|\mathbf{R}\| = 1$$

This means that matrix $\mathbf{R}$ is a proper rotation. In the case when the determinant is -1, matrix $\mathbf{R}$ should be corrected to a proper rotation by multiplying it with a negative identity (changing direction of all eigenvectors) or by changing the sign of one of its rows (changing direction of one of the eigenvectors). Visit http://ialms.net/sim/ to simulate other examples.

III. CONCLUSION

The described in this article simulation of rigid body properties in a stereo 3D virtual environment enables researchers and engineers to observe setups that are impossible to be created in laboratory conditions. Such an approach reveals important insights for the scientists working on unmanned helicopter autopilot analysis, synthesis and tuning.

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