# Derivation of the Lorentz transformation without the use of Einstein's second postulate 

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#### Abstract

Derivation of the Lorentz transformation without the use of Einstein's Second Postulate is provided along the lines of Ignatowsky, Terletskii, and others. This is a write-up of the lecture first delivered in PHYS 4202 E\&M class during the Spring semester of 2014 at the University of Georgia. The main motivation for pursuing this approach was to develop a better understanding of why the faster-than-light neutrino controversy (OPERA experiment, 2011) was much ado about nothing.


## Special relativity as a theory of space and time

All physical phenomena take place in space and time. The theory of space and time (in the absence of gravity) is called the Special Theory of Relativity.

We do not get bogged down with the philosophical problems related to the concepts of space and time. We simply acknowledge the fact that in physics the notions of space and time are regarded as basic and cannot be reduced to something more elementary or fundamental. We therefore stick to pragmatic operational definitions:

Time is what clocks measure. Space is what measuring rods measure.

In order to study and make conclusions about the properties of space and time we need an observer. A natural choice is an observer who moves freely (the one who is free from any external influences). An observer is not a single person sitting at the origin of a rectangular coordinate grid. Rather, it is a bunch of friends (call it Team $K$ ) equipped with identical clocks distributed throughout the grid who record the events happening at their respective locations.

How do we know that this bunch of friends is free from any external influences? We look around and make sure that nothing is pulling or pushing on any member of the bunch; no strings, no springs, no ropes are attached to them. An even better way is to use a collection of "floating-ball detectors" (Fig. 1) distributed throughout the grid [1]. When detector balls are released, they should remain at rest inside their respective capsules. If any ball touches the touch-sensitive surface of the capsule, the frame is not inertial.

In the reference frame associated with a freely moving observer (our rectangular coordinate grid), Galileo's Law of Inertia is satisfied: a point mass, itself free from any external influences, moves with constant velocity. To be able to say what "constant velocity" really means, and thus to verify the law of inertia, we need to be able to measure distances and time intervals between events happening at different grid locations.


FIG. 1: (Color online.) A floating-ball inertial detector. After [1].

## Definition of clock synchronization

It is pretty clear how to measure distances: the team simply uses its rectangular grid of rods.

It is also clear how to measure time intervals at a particular location: the team member situated at that location simply looks at his respective clock. What's not so clear, however, is how the team measures time intervals between events that are spacially separated.

A confusion about measuring this kind of time intervals was going on for two hundred years or so, until one day Einstein said: "We need the notion of synchronized clocks! Clock synchronization must be operationally defined."

The idea that clock synchronization and, consequently, the notion of simultaneity of spacially separated events, has to be defined (and not assumed apriori) is the single most important idea of Einstein's, the heart of special relativity. Einstein proposed to use light pulses. The procedure then went like this:

In frame $K$, consider two identical clocks equipped with light detectors, sitting some distance apart, at $A$ and $B$. Consider another clock equipped with a light emitter at location $C$ which is half way between $A$ and $B$ (we can verify that $C$ is indeed half way between $A$ and
$B$ with the help of the grid of rods that had already been put in place when we constructed our frame $K$ ). Then, at some instant, emit two pulses from $C$ in opposite directions, and let those pulses arrive at $A$ and $B$. If the clocks at $A$ and $B$ show same time when the pulses arrive then the clocks there are synchronized, by definition.

The light pulses used in the synchronization procedure can be replaced with two identical balls initially sitting at $C$ and connected by a compressed spring. The spring is released (say, the thread holding the spring is cut in the middle), the balls fly off in opposite directions towards $A$ and $B$, respectively.

How do we know that the balls are identical? Because Team $K$ made them in accordance with a specific manufacturing procedure.

How do we know that all clocks at $K$ are identical? Because Team $K$ made all of them in accordance with a specific manufacturing procedure.

How do we know that a tic-toc of any clock sitting in frame $K$ corresponds to 1 second? Because Team $K$ called a tic-toc of a clock made in accordance with the manufacturing procedure "a second".

Similarly, clocks in $K^{\prime}$ are regarded as identical and tick-tocking at 1 second intervals because in that frame all of the clocks were made in accordance with the same manufacturing procedure.

Now, how do we know that the manufacturing procedures in $K$ and $K^{\prime}$ are the same? (Say, how do we know that a Swiss shop in frame $K$ makes watches the same way as its counterpart in frame $K^{\prime} ?$ ) Hmm.... That's an interesting question to ponder about.

When studying spacetime from the point of view of inertial frames of reference discussed above, people discovered the following.

## Properties of space and time:

1. At least one inertial reference frame exists. (Geocentric is OK for crude experiments; geliocentric is better; the frame in which microwave background radiation is uniform is probably closest to ideal).
2. Space is uniform (translations; 3 parameters).
3. Space is isotropic (rotations; 3 parameters).
4. Time is uniform (translation; 1 parameter).
5. Space is continuous (down to $\sim 10^{-18}[\mathrm{~m}]$ ).
6. Time is continuous (down to $\sim 10^{-26}[\mathrm{~s}]$ ).
7. Space is Euclidean (apart from local distortions, which we ignore; cosmological observations put the limit at $\sim 10^{26}[\mathrm{~m}]$, the size of the visible Universe; this property is what makes rectangular grids of rods possible).
8. Relativity Principle (boosts; 3 parameters).

Einstein constructed his theory of relativity on the basis of (1) The Principle of Relativity (laws of nature are the same in all inertial reference frames), and (2) The Postulate of the Constancy of the Speed of Light (the speed of light measured by any inertial observer is independent of the state of motion of the emitting body). [NOTE: This is not the same as saying that the speed of light emitted and measured in $K$ is the same as the speed of light emitted and measured in $K^{\prime}$. This latter type of constancy of the speed of light is already implied by the principle of relativity.]

Here we want to stick to mechanics and push the derivation of the coordinate transformation as far as possible without the use of the highly counterintuitive Einstein's Second Postulate. The method that achieves this will be presented below and was originally due to Vladimir Ignatowsky [2].
[DISCLAIMER: I never read Ignatowsky's original papers, but the idea is well-known within the community, often mentioned and discussed. Anyone with time to burn can reproduce the steps without much difficulty. The derivation below consists of 14 steps. If you can reduce that number, let me know.]

## Step 1: Galileo's Law of Inertia for freely moving particles

....implies the linearity of the coordinate transformation between $K$ and $K^{\prime}$ (see Fig. 2),

$$
\begin{align*}
x^{\prime} & =\alpha_{11}(v) x+\alpha_{12}(v) y+\alpha_{13}(v) z+\alpha_{14}(v) t  \tag{1}\\
y^{\prime} & =\alpha_{21}(v) x+\alpha_{22}(v) y+\alpha_{23}(v) z+\alpha_{24}(v) t  \tag{2}\\
z^{\prime} & =\alpha_{31}(v) x+\alpha_{32}(v) y+\alpha_{33}(v) z+\alpha_{34}(v) t  \tag{3}\\
t^{\prime} & =\alpha_{41}(v) x+\alpha_{42}(v) y+\alpha_{43}(v) z+\alpha_{44}(v) t \tag{4}
\end{align*}
$$

Here we assumed that the origins of the two coordinate systems coincide, that is event $(0,0,0,0)$ in $K$ has coordinates $(0,0,0,0)$ in $K^{\prime}$.


FIG. 2: (Color online.) Two inertial reference frames (orthogonal grids of rods equipped with synchronized clocks) in relative motion along the $x$-axis.

## Step 2: Isotropy and homogeneity of space and homogeneity of time

... imply that (i) $x^{\prime}$ is independent of $y$ and $z,(i i) y^{\prime}$ is independent of $z, x$, and $t$; (iii) $z^{\prime}$ is independent of $x$, $y$, and $t ;(i v) t^{\prime}$ is independent of $y$ and $z$, so

$$
\begin{align*}
x^{\prime} & =\alpha_{11}(v) x+\alpha_{14}(v) t  \tag{5}\\
y^{\prime} & =\alpha_{22}(v) y  \tag{6}\\
z^{\prime} & =\alpha_{33}(v) z  \tag{7}\\
t^{\prime} & =\alpha_{41}(v) x+\alpha_{44}(v) t \tag{8}
\end{align*}
$$

NOTE: The fact that $y^{\prime}$ and $z^{\prime}$ are independent of $x$ and $t$ follows from the requirement that the $x^{\prime}$-axis (the line $y^{\prime}=z^{\prime}=0$ ) always coincides with the $x$-axis (the line $y=z=0$ ); this would not be possible if $y^{\prime}$ and $z^{\prime}$ depended on $x$ and $t$.

IMPORTANT: Eq. (8) indicates that it is possible to have two spacially separated events $A$ and $B$ that are simultaneous in frame $K$ and, yet, non-simultaneous in frame $K^{\prime}$, that is

$$
\begin{equation*}
\Delta t_{A B}=0, \quad \Delta x_{A B} \neq 0: \quad \Delta t_{A B}^{\prime}=\alpha_{41} \Delta x_{A B} \neq 0 \tag{9}
\end{equation*}
$$

This is not as obvious as might seem: for example, before Einstein it was assumed that whenever $\Delta t_{A B}$ is zero, $\Delta t_{A B}^{\prime}$ must also be zero. So keeping $\alpha_{41}(v)$ in (8) is a significant departure from classical Newtonian mechanics.

Once the standard method of clock synchronization is adopted, it is, however, relatively easy to give an example of two events satisfying (9). Try that on your own!

## Step 3: Isotropy of space

$\ldots$ also implies that $y^{\prime}$ and $z^{\prime}$ are physically equivalent, so that $\alpha_{22}(v)=\alpha_{33}(v) \equiv k(v)$, and thus

$$
\begin{align*}
x^{\prime} & =\alpha_{11}(v) x+\alpha_{14}(v) t  \tag{10}\\
y^{\prime} & =k(v) y  \tag{11}\\
z^{\prime} & =k(v) z  \tag{12}\\
t^{\prime} & =\alpha_{41}(v) x+\alpha_{44}(v) t . \tag{13}
\end{align*}
$$

## Step 4: Motion of $O^{\prime}$ (the origin of frame $K^{\prime}$ )

$\ldots$ as seen from $K$ gives $x_{O^{\prime}}=v t_{O^{\prime}}$, or $x_{O^{\prime}}-v t_{O^{\prime}}=0$. On the other hand, as seen from $K^{\prime}, x_{O^{\prime}}^{\prime}=0$. For this to be possible, we must have $x^{\prime} \propto(x-v t)$, and thus

$$
\begin{align*}
x^{\prime} & =\alpha(v)(x-v t)  \tag{14}\\
y^{\prime} & =k(v) y  \tag{15}\\
z^{\prime} & =k(v) z  \tag{16}\\
t^{\prime} & =\delta(v) x+\gamma(v) t \tag{17}
\end{align*}
$$

where we have re-labeled $\alpha_{41} \equiv \delta$ and $\alpha_{44} \equiv \gamma$.
NOTE: The $\gamma$ just introduced will soon become the celebrated gamma factor.

Step 5: Inversion $\begin{gathered}\tilde{x}=-x, \tilde{y}=-y \text {, and } \tilde{x}^{\prime}=-x^{\prime}, \\ \tilde{y}^{\prime}=-y^{\prime}\end{gathered}$


FIG. 3: (Color online.) "Inverted" frames in relative motion.
... which is just a relabeling of coordinate marks, preserves the right-handedness of the coordinate systems and is physically equivalent to (inverted) frame $\tilde{K}^{\prime}$ moving with velocity $\tilde{v}=-v$ relative to (inverted) frame $\tilde{K}$ (see Fig. 3), so that

$$
\begin{align*}
\tilde{x}^{\prime} & =\alpha(-v)(\tilde{x}-v t)  \tag{18}\\
\tilde{y}^{\prime} & =k(-v) \tilde{y}  \tag{19}\\
z^{\prime} & =k(-v) z  \tag{20}\\
t^{\prime} & =\delta(-v) \tilde{x}+\gamma(-v) t \tag{21}
\end{align*}
$$

or,

$$
\begin{align*}
-x^{\prime} & =\alpha(-v)(-x-v t)  \tag{22}\\
-y^{\prime} & =-k(-v) y  \tag{23}\\
z^{\prime} & =k(-v) z  \tag{24}\\
t^{\prime} & =-\delta(-v) x+\gamma(-v) t \tag{25}
\end{align*}
$$

which gives

$$
\begin{align*}
\alpha(-v) & =\alpha(v)  \tag{26}\\
k(-v) & =k(v)  \tag{27}\\
\delta(-v) & =-\delta(v)  \tag{28}\\
\gamma(-v) & =\gamma(v) \tag{29}
\end{align*}
$$

## Step 6: Relativity principle and isotropy of space

... tell us that the velocity of $K$ relative to $K^{\prime}$, as measured by $K^{\prime}$ using primed coordinates $\left(x^{\prime}, t^{\prime}\right)$, is equal to $-v$. REMINDER: the velocity of $K^{\prime}$ relative to $K$, as measured by $K$ using unprimed coordinates $(x, t)$, is $v$.

I justify this by considering two local observers comoving with $O$ and $O^{\prime}$, respectively, and firing identical
spring guns in opposite directions at the moment when they pass each other (for a more formal approach, see [3]). If the ball shot in the $+x$ direction by $O$ stays next to $O^{\prime}$ then, by the relativity principle and isotropy of space, the ball shot in the $-x^{\prime}$ direction by $O^{\prime}$ should stay next to $O$. This means that the velocity of $O$ relative to $O^{\prime}$ as measured by $K^{\prime}$ is negative of the velocity of $O^{\prime}$ relative to $O$ as measured by $K$. Thus,

$$
\begin{align*}
x & =\alpha(-v)\left(x^{\prime}+v t^{\prime}\right)  \tag{30}\\
y & =k(-v) y^{\prime}  \tag{31}\\
z & =k(-v) z^{\prime}  \tag{32}\\
t & =\delta(-v) x^{\prime}+\gamma(-v) t^{\prime} \tag{33}
\end{align*}
$$

and since

$$
\begin{equation*}
y=k(-v) y^{\prime}=k(-v) k(v) y=k^{2}(v) y \tag{34}
\end{equation*}
$$

we get

$$
\begin{equation*}
k(v)= \pm 1 \tag{35}
\end{equation*}
$$

Choosing $k(v)=+1$, which corresponds to parallel relative orientation of $y$ and $y^{\prime}$ (as well as of $z$ and $z^{\prime}$ ), gives, for the direct transformation,

$$
\begin{align*}
x^{\prime} & =\alpha(v)(x-v t)  \tag{36}\\
y^{\prime} & =y  \tag{37}\\
z^{\prime} & =z  \tag{38}\\
t^{\prime} & =\delta(v) x+\gamma(v) t \tag{39}
\end{align*}
$$

and, for the inverse transformation,

$$
\begin{align*}
& x=\alpha(v)\left(x^{\prime}+v t^{\prime}\right)  \tag{40}\\
& y=y^{\prime}  \tag{41}\\
& z=z^{\prime}  \tag{42}\\
& t=-\delta(v) x^{\prime}+\gamma(v) t^{\prime} \tag{43}
\end{align*}
$$

## Step 7: Motion of $O$ (the origin of frame $K$ )

$\ldots$ as seen from $K$ gives $x_{O}=0$; also, as seen from $K^{\prime}$, it gives $x_{O}^{\prime}=\alpha(v)\left(x_{O}-v t_{O}\right)=-v \alpha(v) t_{O}$ and $t_{O}^{\prime}=$ $\delta(v) x_{O}+\gamma(v) t_{O}=\gamma(v) t_{O}$. From this, $\frac{x_{O}^{\prime}}{t_{O}^{\prime}}=-v \frac{\alpha(v)}{\gamma(v)}$. But $\frac{x_{O}^{\prime}}{t_{O}^{\prime}}=-v$, which gives

$$
\begin{equation*}
\alpha(v)=\gamma(v) \tag{44}
\end{equation*}
$$

As a result,

$$
\begin{align*}
x^{\prime} & =\gamma(v)(x-v t)  \tag{45}\\
y^{\prime} & =y  \tag{46}\\
z^{\prime} & =z  \tag{47}\\
t^{\prime} & =\delta(v) x+\gamma(v) t \tag{48}
\end{align*}
$$

and

$$
\begin{align*}
x & =\gamma(v)\left(x^{\prime}+v t^{\prime}\right)  \tag{49}\\
y & =y^{\prime}  \tag{50}\\
z & =z^{\prime}  \tag{51}\\
t & =-\delta(v) x^{\prime}+\gamma(v) t^{\prime} \tag{52}
\end{align*}
$$

or, in matrix notation,

$$
\left[\begin{array}{l}
x^{\prime}  \tag{53}\\
t^{\prime}
\end{array}\right]=\left[\begin{array}{cc}
\gamma(v) & -v \gamma(v) \\
\delta(v) & \gamma(v)
\end{array}\right]\left[\begin{array}{l}
x \\
t
\end{array}\right]
$$

and

$$
\left[\begin{array}{l}
x  \tag{54}\\
t
\end{array}\right]=\left[\begin{array}{cc}
\gamma(v) & +v \gamma(v) \\
-\delta(v) & \gamma(v)
\end{array}\right]\left[\begin{array}{l}
x^{\prime} \\
t^{\prime}
\end{array}\right] .
$$

## Step 8: The odd function $\delta(v)$

...can be written as

$$
\begin{equation*}
\delta(v)=-v f\left(v^{2}\right) \gamma(v) \tag{55}
\end{equation*}
$$

since $\gamma(v)$ is even. [NOTE: The newly introduced function $f$ of $v^{2}$ will turn out to be a constant! Actually, one of the goals of the remaining steps of this derivation is to show that $f$ is a constant. It will later be identifies with $1 / c^{2}$.] Therefore,

$$
\left[\begin{array}{c}
x^{\prime}  \tag{56}\\
t^{\prime}
\end{array}\right]=\gamma\left[\begin{array}{cc}
1 & -v \\
-v f & 1
\end{array}\right]\left[\begin{array}{l}
x \\
t
\end{array}\right]
$$

and

$$
\left[\begin{array}{l}
x  \tag{57}\\
t
\end{array}\right]=\gamma\left[\begin{array}{cc}
1 & v \\
v f & 1
\end{array}\right]\left[\begin{array}{l}
x^{\prime} \\
t^{\prime}
\end{array}\right]
$$

## Step 9: Lorentz transformation followed by its inverse must give the identity transformation

This seems physically reasonable. We have,

$$
\begin{align*}
{\left[\begin{array}{l}
x^{\prime} \\
t^{\prime}
\end{array}\right] } & =\gamma\left[\begin{array}{cc}
1 & -v \\
-v f & 1
\end{array}\right]\left[\begin{array}{l}
x \\
t
\end{array}\right] \\
& =\gamma^{2}\left[\begin{array}{cc}
1 & -v \\
-v f & 1
\end{array}\right]\left[\begin{array}{cc}
1 & v \\
v f & 1
\end{array}\right]\left[\begin{array}{l}
x^{\prime} \\
t^{\prime}
\end{array}\right] \\
& =\gamma^{2}\left[\begin{array}{cc}
1-v^{2} f & 0 \\
0 & 1-v^{2} f
\end{array}\right]\left[\begin{array}{l}
x^{\prime} \\
t^{\prime}
\end{array}\right], \tag{58}
\end{align*}
$$

and thus

$$
\begin{equation*}
\gamma^{2}\left(1-v^{2} f\right)=1 \tag{59}
\end{equation*}
$$

from where

$$
\begin{equation*}
\gamma= \pm \frac{1}{\sqrt{1-v^{2} f}} \tag{60}
\end{equation*}
$$

To preserve the parallel orientation of the $x$ and $x^{\prime}$ axes we have to choose the plus sign (as can be seen by taking the $v \rightarrow 0$ limit), so that

$$
\begin{equation*}
\gamma=\frac{1}{\sqrt{1-v^{2} f}} \tag{61}
\end{equation*}
$$

Thus,

$$
\left[\begin{array}{l}
x^{\prime}  \tag{62}\\
t^{\prime}
\end{array}\right]=\frac{1}{\sqrt{1-v^{2} f}}\left[\begin{array}{cc}
1 & -v \\
-v f & 1
\end{array}\right]\left[\begin{array}{l}
x \\
t
\end{array}\right]
$$

and

$$
\left[\begin{array}{l}
x  \tag{63}\\
t
\end{array}\right]=\frac{1}{\sqrt{1-v^{2} f}}\left[\begin{array}{cc}
1 & v \\
v f & 1
\end{array}\right]\left[\begin{array}{l}
x^{\prime} \\
t^{\prime}
\end{array}\right]
$$

where, we recall, $f=f\left(v^{2}\right)$.

Step 10: Two Lorentz transformations performed in succession is a Lorentz transformation

This step is crucial for everything that we've been doing so far, for it shows that $f$ is a constant, which will be identified with $1 / c^{2}$, where $c$ is Nature's limiting speed.

We have a sequence of two transformations: from $(x, t)$ to $\left(x^{\prime}, t^{\prime}\right)$, and then from $\left(x^{\prime}, t^{\prime}\right)$ to $\left(x^{\prime \prime}, t^{\prime \prime}\right)$,

$$
\begin{align*}
{\left[\begin{array}{l}
x^{\prime \prime} \\
t^{\prime \prime}
\end{array}\right] } & =\gamma^{\prime}\left[\begin{array}{cc}
1 & -v^{\prime} \\
-v^{\prime} f^{\prime} & 1
\end{array}\right]\left[\begin{array}{l}
x^{\prime} \\
t^{\prime}
\end{array}\right] \\
& =\gamma^{\prime}\left[\begin{array}{cc}
1 & -v^{\prime} \\
-v^{\prime} f^{\prime} & 1
\end{array}\right] \gamma\left[\begin{array}{cc}
1 & -v \\
-v f & 1
\end{array}\right]\left[\begin{array}{l}
x \\
t
\end{array}\right] \\
& =\gamma^{\prime} \gamma\left[\begin{array}{cc}
1+v v^{\prime} f & -\left(v+v^{\prime}\right) \\
-\left(v f+v^{\prime} f^{\prime}\right) & 1+v v^{\prime} f^{\prime}
\end{array}\right]\left[\begin{array}{l}
x \\
t
\end{array}\right], \tag{64}
\end{align*}
$$

where $v$ is the velocity of $K^{\prime}$ relative to $K$ (as measured in $K$ using the ( $x, t$ ) coordinates), and $v^{\prime}$ is the velocity of $K^{\prime \prime}$ relative to $K^{\prime}$ (as measured in $K^{\prime}$ using the ( $x^{\prime}, t^{\prime}$ ) coordinates). But this could also be written as a single transformation from $(x, t)$ to $\left(x^{\prime \prime}, t^{\prime \prime}\right)$,

$$
\left[\begin{array}{l}
x^{\prime \prime}  \tag{65}\\
t^{\prime \prime}
\end{array}\right]=\gamma^{\prime \prime}\left[\begin{array}{cc}
1 & -v^{\prime \prime} \\
-v^{\prime \prime} f^{\prime \prime} & 1
\end{array}\right]\left[\begin{array}{l}
x \\
t
\end{array}\right]
$$

with $v^{\prime \prime}$ being the velocity of $K^{\prime \prime}$ relative to $K$ (as measured in $K$ using the ( $x, t$ ) coordinates). This shows that the $(1,1)$ and $(2,2)$ elements of the transformation matrix (64) must be equal to each other and, thus,

$$
\begin{equation*}
f=f^{\prime} \tag{66}
\end{equation*}
$$

which means that $f$ is a constant that has units of inverse speed squared, $\left[\mathrm{s}^{2} / \mathrm{m}^{2}\right]$. Wow!!

## Step 11: Velocity addition formula (for reference frames)

To derive the velocity addition formula (along the $x$ axis) we use Eqs. (64) and (65) to get

$$
\begin{equation*}
\gamma^{\prime} \gamma\left(v+v^{\prime}\right)=\gamma^{\prime \prime} v^{\prime \prime} \tag{67}
\end{equation*}
$$

Squaring and rearranging give

$$
\begin{equation*}
\left(v^{\prime \prime}\right)^{2}=\left(\frac{v+v^{\prime}}{1+v v^{\prime} f}\right)^{2} \tag{68}
\end{equation*}
$$

or,

$$
\begin{equation*}
v^{\prime \prime}= \pm \frac{v+v^{\prime}}{1+v v^{\prime} f} \tag{69}
\end{equation*}
$$

Choosing the plus sign (to make sure that $v^{\prime \prime}=v$ when $v^{\prime}=0$ ), we get

$$
\begin{equation*}
v^{\prime \prime}=\frac{v+v^{\prime}}{1+v v^{\prime} f} \tag{70}
\end{equation*}
$$

## Step 12: The universal constant $f$ cannot be negative ...

... because in that case the conclusions of relativistic dynamics would violate experimental observations! For example, the force law,

$$
\begin{equation*}
\frac{d}{d t} \frac{m \mathbf{v}_{p}}{\sqrt{1-v_{p}^{2} f}}=\mathbf{F} \tag{71}
\end{equation*}
$$

where $\mathbf{v}_{p}$ is the velocity of a particle, would get messed up. In particular, such law would violate the observed fact that it requires an infinite amount of work (and, thus, energy) to accelerate a material particle from rest to speeds approaching $3 \times 10^{8}[\mathrm{~m} / \mathrm{s}]$ (this argument is due to Terletskii [4]). In fact, it would become "easier" to accelerate the particle, the faster it is moving. Incidentally, this experimental fact is what "replaces" Einstein's Second Postulate in the present derivation! Thus, Eq. (70) is the limit to which our (actually, Ignatowski's) derivation can be pushed.

REMARK: Relativistic dynamics has to be discussed separately. We won't do that here, but maybe you can suggest a different reason for $f$ not to be negative? See [3] and [5] for possible approaches.

## Step 13: Existence of the limiting speed

Denoting

$$
\begin{equation*}
f \equiv \frac{1}{c^{2}} \tag{72}
\end{equation*}
$$

we get the velocity addition formula,

$$
\begin{equation*}
v^{\prime \prime}=\frac{v+v^{\prime}}{1+\frac{v v^{\prime}}{c^{2}}} \tag{73}
\end{equation*}
$$

If we begin with $v^{\prime}<c$ and attempt to take the limit $v^{\prime} \rightarrow c$, we'll get

$$
\begin{equation*}
v^{\prime \prime} \rightarrow \frac{v+c}{1+\frac{v c}{c^{2}}}=c \tag{74}
\end{equation*}
$$

which tells us that $c$ is the limiting speed that a material object can attain. (Notice that "material" here means "the one with which an inertial frame can be associated". The photons do not fall into this category, as will be discussed shortly!)

The possibilities therefore are:

1. $c=+\infty$ (Newtonian mechanics; contradicts (71));
2. $c>0$ and finite (Special Relativity);
3. $c=0$ (Contradics observations.)

So we stick with option 2.
What if an object were created to have $v^{\prime}>c$ from the start (a so-called tachyon), like in the recent superluminal neutrino controversy? We'd get some strange results.

For example, if we take $v=c / 2$ and $v^{\prime}=2 c$, we get

$$
\begin{equation*}
v^{\prime \prime}=\frac{(c / 2)+(2 c)}{1+\frac{(c / 2)(2 c)}{c^{2}}}=\frac{5}{4} c \tag{75}
\end{equation*}
$$

so in $K$ the object would move to the right at a slower speed than relative to $K^{\prime}$, while $K^{\prime}$ itself is moving to the right relative to $K$. Bizarre, but OK, the two speeds are measured by different observers, so maybe it's not a big deal....

## Step 14: Lorentz transformation in standard form

However, if we consider the resulting Lorentz transformation,

$$
\left[\begin{array}{c}
x^{\prime}  \tag{76}\\
t^{\prime}
\end{array}\right]=\frac{1}{\sqrt{1-\frac{v^{2}}{c^{2}}}}\left[\begin{array}{cc}
1 & -v \\
-\frac{v}{c^{2}} & 1
\end{array}\right]\left[\begin{array}{l}
x \\
t
\end{array}\right]
$$

or,

$$
\begin{align*}
t^{\prime} & =\frac{t-\frac{v}{c^{2}} x}{\sqrt{1-\frac{v^{2}}{c^{2}}}}  \tag{77}\\
x^{\prime} & =\frac{x-v t}{\sqrt{1-\frac{v^{2}}{c^{2}}}}  \tag{78}\\
y^{\prime} & =y  \tag{79}\\
z^{\prime} & =z \tag{80}
\end{align*}
$$

we notice that in a reference frame $K^{\prime}$ associated with hypothetical tachyons moving with $v>c$ relative to $K$ (imagine a whole fleet of them, forming a grid which makes up $K^{\prime}$ ), the spacetime coordinates of any event would be imaginary! In order for the spacetime measurements to give real values for $\left(t^{\prime}, x^{\prime}, y^{\prime}, z^{\prime}\right)$, the reference frame $K^{\prime}$ made of tachyons must be rejected.

What about a reference frame made of photons? In that case, coordinates would be infinite and should also be rejected. So a fleet of photons cannot form a "legitimate" reference frame. Nevertheless, we know that
photons exist. Similarly, tachyons may also exist and, like photons, (a) should be created instantaneously (that is, can't be created at rest, and then accelerated), and (b) should not be allowed to form a "legitimate" inertial reference frame.

What about violation of causality?


FIG. 4: (Color online.) Violation of causality by a hypothetical tachyon.

Indeed, the Lorentz transformation shows that tachyons violate causality. If we consider two events, $A$ (tachyon creation) and $B$ (tachyon annihilation) with $t_{B}>t_{A}$ such that tachyon's speed, $v_{p}=\frac{x_{B}-x_{A}}{t_{B}-t_{A}}>c$, is greater than $c$ as measured in $K$ (see Fig. 4), then in frame $K^{\prime}$ moving with velocity $v<c$ relative to $K$ we'll have from (77) and (78),

$$
\begin{align*}
t_{B}^{\prime}-t_{A}^{\prime} & =\gamma\left[\left(t_{B}-t_{A}\right)-\frac{v}{c^{2}}\left(x_{B}-x_{A}\right)\right] \\
& =\gamma\left[1-\frac{v}{c^{2}} \frac{x_{B}-x_{A}}{t_{B}-t_{A}}\right]\left(t_{B}-t_{A}\right) \\
& =\gamma\left(1-\frac{v v_{p}}{c^{2}}\right)\left(t_{B}-t_{A}\right)  \tag{81}\\
x_{B}^{\prime}-x_{A}^{\prime} & =\gamma\left[\left(x_{B}-x_{A}\right)-v\left(t_{B}-t_{A}\right)\right] \\
& =\gamma\left[1-v \frac{t_{B}-t_{A}}{x_{B}-x_{A}}\right]\left(x_{B}-x_{A}\right) \\
& =\gamma\left(1-\frac{v}{v_{p}}\right)\left(x_{B}-x_{A}\right) \tag{82}
\end{align*}
$$

where $\gamma=1 / \sqrt{1-v^{2} / c^{2}}$, which shows that it is possible to find $v<c$ such that $t_{B}^{\prime}-t_{A}^{\prime}<0$; that is, in $K^{\prime}$ event $B$ happens before event $A$. This seems to indicate that tachyons are impossible. However, causality is a consequence of the Second Law of Thermodynamics, which is a statistical law, applicable to macroscopic systems; it does not apply to processes involving individual elementary particles. As a result, the existence of tachyons cannot be so easily ruled out.

## Step 15: Speed of light is the limiting speed for material objects

Finally, returning to Eq. (74), we see that if something moves with $c$ relative to $K^{\prime}$, it also moves with $c$ relative to any other frame $K^{\prime \prime}$. That is: the limiting speed is the same in all inertial reference frames. And there is no mentioning of any emitter. Also, as follows from (73), $c$ is the only speed that has this property (of being the same in all inertial frames). We know that light has this property (ala Michelson-Morley experiment), so the speed of light is the limiting speed for material objects. Since neutrinos have mass, they cannot move faster than light, and thus superluminal neutrinos are not possible.


FIG. 5: (Color online.) Length contraction and relativity of simultaneity.

Here we have a rod of (proper) length $\ell_{0} \equiv x_{B}^{\prime}-x_{A}^{\prime}>0$ sitting at rest in frame $K^{\prime}$, see Fig. 5. Its speed relative to frame $K$ is $v$. The two events, $A$ and $B$, represent the meetings of the two clocks at the ends of the rod with the corresponding clocks in the $K$ frame at $t_{A}=t_{B}$. We have from (77) and (78),

$$
\begin{align*}
t_{B}^{\prime}-t_{A}^{\prime} & =\gamma\left(-\frac{v}{c^{2}}\right)\left(x_{B}-x_{A}\right)  \tag{83}\\
x_{B}^{\prime}-x_{A}^{\prime} & =\gamma\left(x_{B}-x_{A}\right) \tag{84}
\end{align*}
$$

or

$$
\begin{align*}
t_{B}^{\prime}-t_{A}^{\prime} & =\left(-\frac{v}{c^{2}}\right)\left(x_{B}^{\prime}-x_{A}^{\prime}\right)  \tag{85}\\
x_{B}-x_{A} & =\frac{x_{B}^{\prime}-x_{A}^{\prime}}{\gamma} \tag{86}
\end{align*}
$$

Eq. (85) says that $t_{B}^{\prime}-t_{A}^{\prime}<0$, that is, the meeting events are not simultaneous in $K^{\prime}$ (relativity of simultaneity). Eq. (86) says that the length of the rod, $\ell \equiv x_{B}-x_{A}$, as measured in $K$ is smaller than its proper length by the gamma factor,

$$
\begin{equation*}
\ell=\ell_{0} / \gamma \tag{87}
\end{equation*}
$$

the phenomenon of length contraction.

## B. Time dilation



FIG. 6: (Color online.) Time dilation.

This time a single clock belonging to $K^{\prime}$, Fig. 6, passes by two different clocks in $K$. The corresponding two events, $A$ and $B$, have $x_{A}^{\prime}=x_{B}^{\prime}$, and are related to each other by

$$
\begin{align*}
t_{B}^{\prime}-t_{A}^{\prime} & =\gamma\left[\left(t_{B}-t_{A}\right)-\frac{v}{c^{2}}\left(x_{B}-x_{A}\right)\right] \\
& =\gamma\left[1-\frac{v}{c^{2}} \frac{x_{B}-x_{A}}{t_{B}-t_{A}}\right]\left(t_{B}-t_{A}\right) \\
& =\gamma\left(1-\frac{v^{2}}{c^{2}}\right)\left(t_{B}-t_{A}\right) \\
& =\frac{t_{B}-t_{A}}{\gamma} \tag{88}
\end{align*}
$$

This means that upon arrival at $B$ the moving clock will read less time than the $K$-clock sitting at that location. This phenomenon is called time dilation (moving clocks run slower).

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