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Analytical mechanics

Runge-Lenz-Symmetries

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Abstract

In this report we study the symmetries that correspond to the conservation of the Runge-Lenz vector in the Kepler problem. In section 2 we use Noether's theorem to define a Runge-Lenz vector as a consequence of an invariance of the action integral. It's shown that such a vector exists for all central potentials. In section 3 we describe the Kepler problem in space-time. By choosing a nice parametrization we show that the equations of motion and the conservation of energy describe a harmonic oscillator with an extra derivative in four dimensions and a four dimensional sphere, respectively. From this we define a conserved tensor. The components of this tensor correspond to the Runge-Lenz vector and angular momentum.

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1 Introduction

One of the most famous problems in classical mechanics is the Kepler problem. This is the problem of a point mass in a central force field of the form

$$\mathbf{F}(r) = \frac{-k}{r^2}\mathbf{e}_r. \tag{1}$$

A special thing about this problem is that there exists an extra conserved quantity besides the total energy and the angular momentum. This quantity is a vector called the *Runge-Lenz vector*. The Runge-Lenz vector $\mathbf{A}$ for a particle of mass m moving in a central force field $\mathbf{F} = -\frac{k}{r^2}\mathbf{e}_r$ is defined as

$$\mathbf{A} := \mathbf{p} \times \mathbf{L} - \frac{mk}{|\mathbf{r}|}\mathbf{r}. \tag{2}$$

Here $\mathbf{p}$ is the momentum of the particle, $\mathbf{L}$ is the angular momentum, m the mass and $\mathbf{r}$ the position vector of the particle. In the Kepler-problem the angular momentum and energy are conserved. One might then think that there exist seven conserved quantities. This is not the case, because the variables are not independent of each other. From Figure 1 one sees that the Runge-Lenz-vector lies in the plane of motion and thus $\mathbf{A}\cdot\mathbf{L} = 0$. Further on by taking the dot product $\mathbf{A} \cdot \mathbf{A}$ one obtains $A^2 = m^2k^2 + 2mEL^2$. From this one can see that there are only five independent constants of motion in the Kepler-problem [1]. The orbits in the Kepler-problem are conic-sections. A nice way to realize this is by using the Runge-Lenz-vector. By denoting θ as the angle between the position vector and the Runge-Lenz-vector one has

$$\mathbf{A} \cdot \mathbf{r} = Arcos\theta = \mathbf{r} \cdot \mathbf{p} \times \mathbf{L} - mkr = \mathbf{L} \cdot \mathbf{r} \times \mathbf{p} - mkr = L^2 - mkr. \tag{3}$$

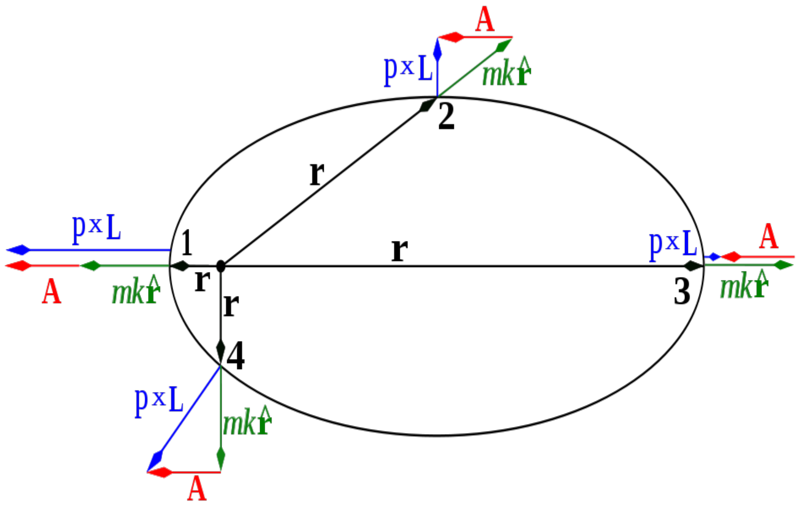

Figure 1: Illustration of how the Runge-Lenz-vector is oriented in the Kepler orbits [3].

Thus we can solve for r as

$$\frac{1}{r} = \frac{mk}{L^2}\left[1 + \frac{A}{mk} cos\theta\right], \tag{4}$$

which is the equation of a conic section with eccentricity $e = A/mk$ provided A is constant [3]. Thus we see that the conservation of the Runge-Lenz-vector actually is the reason that the orbits of the Kepler-problem are closed. The Runge-Lenz-vector can in principle be generalized to any central potential as we shall see in Section 2.2. However these generalized Runge-Lenz-vectors are often complicated functions and usually not expressible in closed form[3]. Since the conservation of the Runge-Lenz-vector implies closed orbits for the Kepler-problem one might expect that there exists some analogue of this derivation for the isotropic harmonic oscillator. This is indeed the case but we shall leave this question here [1].

2 Runge-Lenz-vector from a symmetry of the action integral

2.1 General aspects of conserved quantities

Noether's theorem relates conservation laws with symmetries. A somewhat simplified way to state this theorem is to say that if the action integral is invariant under some transformation then there is some conserved quantity. Hamilton's principle says that the physical path of a system is such that the action is stationary. This simply means that the action is invariant under an infinitesimal variation of the path as $q(t) \mapsto q_i'(t) = q_i(t) + \delta\eta_i$. Besides the Euler-Lagrange-equations, this also implies conservations laws. A trivial example is a Lagrangian with some cyclic coordinate, then the canonical momentum conjugate to this coordinate is conserved. This follows directly from the Euler-Lagrange-equations and thus this conservation law follows from Hamilton's principle. Noether's theorem can somewhat simplified be stated as, an invariance of the Lagrangian corresponds to a conservation law. To see how this works we vary the path as

$$q_i \mapsto q_i' = q_i + \delta q_i, \tag{5}$$

where the variation δq_i is such that it is zero at the endpoints. The velocity then becomes

$$\dot{q}_i \mapsto \dot{q}_i' = \dot{q}_i + \delta\dot{q}_i. \tag{6}$$

The Lagrangian of the new coordinates can be expanded in a power series as

$$L(q_i', \dot{q}_i') = L(q_i, \dot{q}_i) + \sum_i \frac{\partial L}{\partial q_i}\delta q_i + \sum_i \frac{\partial L}{\partial \dot{q}_i}\delta\dot{q}_i + \dots \tag{7}$$

Taking only the linear terms we can write the variation of the Lagrangian as

$$\delta L = \sum_i \frac{\partial L}{\partial q_i}\delta q_i + \sum_i \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i. \tag{8}$$

We now assume that the variation of the action can be written as the integral of the variation of the Lagrangian

$$0 = \delta S = \int_{t_1}^{t_2} dt\delta L = \int_{t_1}^{t_2} dt \sum_i \left[\frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i\right] = \int_{t_1}^{t_2} dt \sum_i \left[\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\right]\delta q_i. \tag{9}$$

In the last equality we used integration by parts and that the variation is zero at the endpoints of integration. By the fundamental lemma of the calculus of variations this integral is zero only if the integrand is zero. Also we assume that the coordinates are independent, which implies that the coefficient in front of each δq_i is zero, and thus we obtain the Euler-Lagrange-equations

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0. \tag{10}$$

These are one equation for each coordinate q_i. By the Euler-Lagrange-equations we can write

$$\sum_i \frac{\partial L}{\partial q_i} = \sum_i \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}. \tag{11}$$

By inserting this expression in the expression for the variation of the Lagrangian we can write

$$\delta L = \sum_i \left[\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\delta q_i\right] + \sum_i \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i = \sum_i \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}}\delta q_i\right]. \tag{12}$$

This expressions is a good way of defining constants of motion. As an example we consider the case of a central potential. The Lagrangian can then be written as

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(r). \tag{13}$$

This Lagrangian of this system is obviously invariant under rotations, e.g rotation about the x-axis by a small angle $\delta\theta$ as

$$\mathbf{r} \mapsto \mathbf{r}' = \mathbf{r} + \delta\theta \mathbf{r} \times \mathbf{e}_x. \tag{14}$$

From Equation (12) we can now define a constant of motion as

$$\frac{1}{\delta\theta} \sum_i \left[\frac{\partial L}{\partial \dot{q}_i} \delta q_i \right] = \sum_i p_i (\mathbf{r} \times \mathbf{e}_x)_i = z p_y - p_z y = L_x, \tag{15}$$

which is the x-component of angular momentum. Although this seems as a useful method it is not possible to define the Runge-Lenz-vector as a consequence of an invariance of the Lagrangian, but for the whole action integral. By this method it is only possible to define constants of motions such as linear and angular momentum which are conserved due to some cyclic coordinate.

2.2 Runge-Lenz-vector

This section is based on the work done in [2]. We will now see how we can define the Runge-Lenz-vector by a similar variational calculation. We study a system with n generalized coordinates $q_1, \ldots q_n$, described by a time independent Lagrangian $L(q_k, \dot{q}_k)$. We now try a slightly more complicated variation of both the path and time and then demand that the action integral is invariant. The path and time is varied as

$$q_i \mapsto q_i' = q_i + \delta\alpha_i(q_k, t) \qquad \text{and} \tag{16}$$

$$t \mapsto t' = t + \delta\beta, \tag{17}$$

with some small parameter δ. Here the variation of the coordinates is a function of the original variables and time. The variation added to time $\delta\beta$ is taken to be constant. Since the variation of time is a constant and the Lagrangian is assumed to be time-independent, the transformation of time will not change the action integral. But the variation of time is still introduced for later calculations. Also here the variations are such that they vanish at the endpoints of integration. The velocities then become

$$\dot{q}_i' = \dot{q}_i + \delta\dot{\alpha}_i. \tag{18}$$

We can as before expand the Lagrangian in a power series and write the variation of the Lagrangian as

$$\delta L = \sum_i \frac{\partial L}{\partial q_i}\delta\alpha + \sum_i \frac{\partial L}{\partial \dot{q}_{i_i}}\delta\dot{\alpha}_i. \tag{19}$$

We now demand that the action integral is invariant under this transformation i.e

$$\int_{t_1}^{t_2} L(q_k, \dot{q}_k)dt = \int_{t_1}^{t_2} dt\big[L(q_k', \dot{q}_k') + \frac{dg(q,t)}{dt}\big], \tag{20}$$

where we added a total time derivative which we can do since the integrand is determined only up to a total time derivative. The variation of the action integral then becomes

$$0 = \delta S = \int_{t_1}^{t_2} dt\left[\delta L + \frac{df}{dt}\right], \tag{21}$$

where f is defined to be the variation of g due to the coordinate transformation. We now simply focus on the integrand since it is the most important part. We can rewrite the integrand in the variation of the transformed action integral as

$$\begin{aligned}
\delta L + \frac{df}{dt} &= \delta L + \frac{df}{dt} + \frac{dL}{dt}\delta\beta - \frac{dL}{dt}\delta\beta \\
&= \frac{d}{dt}\left[\sum_k \frac{\partial L}{\partial \dot{q}_k}\big(\delta\alpha - \dot{q}_k\delta\beta\big) + L\delta\beta + f\right] - \sum_k \left[\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k}\right]\big(\delta\alpha_k - \dot{q}_k\delta\beta\big) \\
&= \frac{d}{dt}\left[\sum_k \frac{\partial L}{\partial \dot{q}_k}\big(\delta\alpha - \dot{q}_k\delta\beta\big) + L\delta\beta + f\right]. \qquad (22)
\end{aligned}$$

In the last step we simply used the Euler-Lagrange equations. We can see that an invariance of the action integral under this more general transformation implies both the Euler-Lagrange equations and another equation which is a conservation law. As before the variation of the action integral is zero only if the integrand is zero and thus we obtain the expression

$$\frac{d}{dt}\left[\sum_k \frac{\partial L}{\partial \dot{q}_k}\big(\delta\alpha_k - \dot{q}_k\delta t\big) + L\delta\beta + f\right] = 0. \qquad (23)$$

This is an expression involving the conserved quantities of the system. The terms in the time derivative can be identified as a linear combination of the conserved quantities of the system. By rewriting this we get

$$0 = \frac{d}{dt}\left[\sum_k \frac{\partial L}{\partial \dot{q}_k}\delta\alpha_k - H\delta\beta + f\right], \qquad (24)$$

where H is the Hamiltonian of the system. This expression has three terms: the Hamiltonian which comes from the variation of time, the conjugate momentum which comes from the variation of coordinates and the function f which we will see is important for understanding how the Runge-Lenz vector can be conserved. To see more conserved quantities we need to determine the functions $f(q,t)$ and $\delta\alpha(q,t)$ such that this expression holds. To proceed further we restrict ourself to the case of a central potential. The Lagrangian is then given by

$$L(r, \dot{q}_k) = \frac{m}{2} \sum_k \dot{q}_k^2 - V(r), \tag{25}$$

where $r := \left(\sum_i q_i^2\right)^{1/2}$ is the distance between the interacting bodies in the two-body problem. We also assume that the equations that determine the generalized coordinates q_k do not involve time explicitly. The Hamiltonian is then equal to the total energy:

$$H = \frac{m}{2} \sum_k \dot{q}_k^2 + V(r). \tag{26}$$

We can write the time derivatives of the terms in Equation (24) by the chain rule as

$$\frac{d}{dt} H\delta\beta = \delta\beta \sum_k \frac{\partial V}{\partial q_k} \dot{q}_k + \delta\beta \sum_k m\dot{q}_k \ddot{q}_k, \tag{27}$$

$$\frac{d}{dt}\delta\alpha_k = \sum_i \frac{\partial \delta\alpha_k}{\partial q_i} \dot{q}_i + \frac{\partial \delta\alpha_k}{\partial t} \qquad \text{and} \tag{28}$$

$$\frac{df}{dt} = \sum_i \frac{\partial f}{\partial q_i} \dot{q}_i + \frac{\partial f}{\partial t}. \tag{29}$$

If we plug in this information into Equation (24) one obtains

$$\frac{d}{dt}\left[\sum_k \frac{\partial L}{\partial \dot{q}_k} \delta\alpha_k - H\delta\beta + f\right] = \sum_k m\ddot{q}_k \delta\alpha_k + \sum_{k,i} m\dot{q}_k \frac{\partial \delta\alpha_k}{\partial q_i} \dot{q}_i +$$
$$+ \sum_k \dot{q}_k \left(\frac{\partial \delta\alpha_k}{\partial t} + \frac{\partial f}{\partial q_k}\right) - \sum_k \frac{\partial V}{\partial q_k} \dot{q}_k \delta\beta - \sum_k m\ddot{q}_k \dot{q}_k \delta\beta + \frac{\partial f}{\partial t}, \tag{30}$$

and by using that

$$\delta\alpha_k = \delta\beta \dot{q}_k \tag{31}$$

Equation (30) becomes

$$\frac{d}{dt}\left[\sum_k \frac{\partial L}{\partial \dot{q}_k}\delta\alpha_k - H\delta\beta + f\right] = \sum_{k,i}\left[m\dot{q}_i\dot{q}_k\frac{\partial\delta\alpha_k}{\partial q_i}\right] + \sum_k \dot{q}_k\left[m\frac{\partial\delta\alpha_k}{\partial t} + \frac{\partial f}{\partial q_k}\right]$$
$$-\left[\sum_k \frac{\partial V}{\partial q_k}\delta\alpha_k - \frac{\partial f}{\partial t}\right]. \tag{32}$$

Each of the three terms to in square brackets will now be set to zero individually. Since the term with the double sum over k and i is symmetric in its indices we can write

$$\frac{\partial\delta\alpha_k}{\partial q_i} = \xi_{ik}, \tag{33}$$

where $\xi_{ik} = -\xi_{ki}$ is some antisymmetric coefficient matrix. We can now integrate this and obtain an expression for $\delta\alpha_k$. This gives the expression

$$\delta\alpha_k = \sum_i \xi_{ki}q_i + a_k(t), \tag{34}$$

where $a_k(t)$ is some integration constant. Setting the second term in square brackets of the right hand side in Equation (32) equal to zero gives

$$\sum_k m\dot{q}_k\frac{\partial\delta\alpha_k}{\partial t} = \sum_k m\dot{q}_k\dot{a}_k = -\sum_k \frac{\partial f}{\partial q_k}\dot{q}_k. \tag{35}$$

Integrating this equation gives an expression for the function f as

$$f = -\sum_k m\dot{a}_k q_k. \tag{36}$$

We now got an expression for all of the terms in the linear combination of conserved quantities. Denoting this linear combination by B we can write

$$B = \frac{1}{2}\sum_{k,i} m\xi_{ki}\big[\dot{q}_k q_i - \dot{q}_i q_k\big] + \sum_k m\big[\dot{q}_k a_k - \dot{a}_k q_k\big] - H\delta\beta. \tag{37}$$

The first term can be identified as components of the angular momentum and the last term is the energy (Hamiltonian) of the system. The term in the middle is the conserved quantity that is components of the Runge-Lenz-vector. By setting the two last terms of Equation (32) to zero and use that the a_k are independent integration constants gives a differential equation for determining a_k. One obtains

$$\ddot{a}_k + \frac{1}{mr}\frac{dV}{dr}a_k = 0. \tag{38}$$

Since we only assumed that the potential is a central potential, this shows that there exist a conserved vector like the Runge-Lenz-vector for all central potentials. Further on we can see that since a_k is independent integration constants, the time derivative of each term in the second sum of Equation (37) must be zero. This means that the conserved Runge-Lenz like vector has components

$$m\big[\dot{q}_k a_k - \dot{a}_k q_k\big], \tag{39}$$

where a_k is determined by Equation (38). For the case of the Kepler problem $V = \frac{-k}{r}$ a possible solution of the differential equation for a_k is

$$a_k = \sum_i \dot{q}_i q_i. \tag{40}$$

This can be verified by inserting the solution into Equation (38). The components of the conserved vector are in this case

$$m\big[\dot{q}_k a_k - \dot{a}_k q_k\big] = m\dot{q}_k \sum_i q_i \dot{q}_i + q_k \left[\frac{k}{r} - m \sum_k \dot{q}_k^2\right], \tag{41}$$

which is proportional to components of the Runge-Lenz vector as defined in Equation (2).

As mentioned before one would suspect that this differential equation also has a nice solution for the case of the harmonic oscillator. For any central potential $V(r)$ the equations of motion can be written

$$\ddot{q}_i = -\frac{1}{m}\frac{\partial V}{\partial q_i} = -\frac{1}{m}\frac{dV}{dr}\frac{q_i}{r}, \tag{42}$$

which is of the same form as Equation (38). Since the solution to the equations of motion is $q_i(t)$, we would expect that the solution to Equation (38) can be written as

$$a_k = \sum_i b_i q_i, \tag{43}$$

for some coefficients b_i [5]. It is shown in [3] that in the case of a harmonic oscillator the generalized Runge-Lenz vector can be written in this form. But the coefficients b_i are more complicated than in the Kepler problem.

3 Extra symmetries in the Kepler problem

This section is based on the work done by Göransson in [4].

The components of the angular momentum satisfy the following Poisson bracket identities:

$$\{L_i, L_j\} = \sum_k \epsilon_{ijk} L_k. \tag{44}$$

The generating function for an infinitesimal rotation about an axis **n** is the component of angular momentum in the direction of that axis. We thus say that the symmetry group for a system with angular momentum conserved is SO(3). This is the group of all orthogonal matrices of determinant one, i.e all matrices that describe rotations in three dimensions. We now introduce the following scaled version of the Runge-Lenz-vector:

$$\vec{D} := \frac{\vec{A}}{\sqrt{2m|E|}}. \tag{45}$$

This new vector then satisfies the following identities [1]:

$$\{D_i, L_j\} = \sum_k \epsilon_{ijk} D_k, \tag{46}$$

$$\{D_i, D_j\} = \begin{cases} \sum_k \epsilon_{ijk} L_k & \text{if } E < 0, \\ -\sum_k \epsilon_{ijk} L_k & \text{if } E > 0. \end{cases} \tag{47}$$

It is known that the conservation of both angular momentum and the Runge-Lenz vector corresponds to a larger symmetry group. For negative energy (bounded motion) this group is SO(4), i.e the group of rotation matrices of rotations in four dimensions [1]. This can be realized by looking at the Poisson bracket relations above. As discussed in [1](p.413-416) the generator matrices of rotations in four dimensions satisfies the same commutation relations as the Possion bracket relations (44), (46) and (47).

We shall now describe the Kepler problem in space time. By choosing a suitable parametrization we shall see how the resulting four dimensional symmetry can be identified. Henceforth we will only deal with the situation of bounded motion, i.e $E < 0$. The equations of motions in the Kepler problem are simply by using Newton's 2nd law

$$m\ddot{\mathbf{r}} = -k\frac{\mathbf{r}}{r^3}. \tag{48}$$

The conservation of energy follows directly since the force is conservative,

$$\frac{m}{2}|\dot{\mathbf{r}}|^2 - \frac{k}{r} = E. \tag{49}$$

To simplify some expressions we define the following constants:

$$\begin{aligned} \alpha &:= \sqrt{-2E/m}, \\ \beta &:= -k/(2E) \qquad \text{and} \\ \gamma := \beta/\alpha &= -k\sqrt{-m/(2E)^3}. \end{aligned}$$

We will describe this problem in space time. In conventional mechanics we describe the trajectories parametrized with time as parameter. In space time we use time as a coordinate, not as a parameter. Because of this we will describe the trajectories parametrized by some other parameter, which we will denote by τ. Differentiating with respect to τ will be denoted by prime. We demand that this new parameter is such that the following equalities hold:

$$\frac{dt}{d\tau} = t' = \frac{r}{\alpha} \qquad \text{and} \tag{50}$$

$$\frac{d\tau}{dt} = \frac{\alpha}{r}. \tag{51}$$

Using the chain rule we can now write

$$\frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{d\tau}\frac{d\tau}{dt} = \mathbf{r}'\frac{\alpha}{r}. \tag{52}$$

By inserting this into the equations of motion we can rewrite the equations of motion in this new parametrization. Doing this one obtains

$$\mathbf{r}'' = -\frac{\beta \mathbf{r}}{r} + \frac{r'\mathbf{r}'}{r}. \tag{53}$$

The energy conservation in this new parametrization becomes

$$\alpha^2(t' - \gamma)^2 + |\mathbf{r}'|^2 = \beta^2. \tag{54}$$

We can think of this equation as a definition of a metric on space-time, which means that the length of any four-vector $\mathbf{\Lambda} = t\mathbf{e}_t + \mathbf{r}$ in this metric is given by

$$||\mathbf{\Lambda}|| = \sqrt{\alpha^2 t^2 + |\mathbf{r}|^2}. \tag{55}$$

We can thus interpret the energy equation as that the four dimensional "velocity-vector" with time component t' and space components $\mathbf{r}'$ moves on a four dimensional sphere centred at $(\gamma, 0, 0, 0)$. To proceed further we shall look at the radial component of $\mathbf{r}''$, which is $r'' = (\sqrt{\mathbf{r} \cdot \mathbf{r}})''$. Calculating this one gets

$$r'' = (\sqrt{\mathbf{r} \cdot \mathbf{r}})'' = \left(\frac{\mathbf{r}' \cdot \mathbf{r}}{r}\right)' = \frac{\mathbf{r}'' \cdot \mathbf{r} + \mathbf{r}' \cdot \mathbf{r}'}{r} - \frac{r'\mathbf{r}' \cdot \mathbf{r}}{r^2}. \tag{56}$$

Inserting the expression for $\mathbf{r}''$ from Equation (53) one obtains

$$r'' = \frac{|\mathbf{r}'|^2}{r} - \beta. \tag{57}$$

We can rewrite equation (54) as

$$|\mathbf{r}'|^2 = \beta^2 - \alpha^2(t' - \gamma)^2 = \beta^2 - \alpha^2(\frac{r}{\alpha} - \gamma). \tag{58}$$

Inserting this into Equation (57) we obtain the nice formula

$$r'' = \beta - r, \tag{59}$$

and by using the parametrization constraint this can also be written as

$$t''' = \gamma - t'. \tag{60}$$

Differentiating Equation (53) with respect to τ and using the expression for the radial part(Equation (59)) one gets

$$\mathbf{r}''' = -\mathbf{r}'. \tag{61}$$

This parametrization gives the equations of motion a nice form which is closely related to a harmonic oscillator in four dimensions. If we define a four- dimensional vector $\mathbf{v}$ with time component $(t' - \gamma)$ and space components $\mathbf{r}'$ the equations of motion and the energy conservation can be written as

$$||\mathbf{v}||^2 = \beta^2 \qquad \text{and} \qquad \mathbf{v}'' = -\mathbf{v}. \tag{62}$$

These equations are invariant under rotations, thus the SO(4) symmetry becomes apparent. From Equation (62) we observe that we can construct a conserved tensor Γ as

$$\Gamma = \mathbf{v} \wedge \mathbf{v}' = \mathbf{v} \otimes \mathbf{v}' - \mathbf{v}' \otimes \mathbf{v}. \tag{63}$$

In index notation we can write this as

$$\Gamma_{ij} = v_i v_j' - v_i' v_j. \tag{64}$$

Calculating its derivative with respect to τ gives

$$\Big(\Gamma_{ij}\Big)' = v_i'v_j' + v_iv_j'' - v_i'v_j' - v_i''v_j = v_iv_j'' - v_i''v_j = -v_iv_j + v_iv_j = 0, \qquad (65)$$

and by the chain rule we have

$$\frac{d\Gamma_{ij}}{dt} = \Big(\Gamma_{ij}\Big)'\frac{d\tau}{dt} = \Big(\Gamma_{ij}\Big)'\frac{\alpha}{r} = 0, \qquad (66)$$

for all i, j and thus each component of Γ is conserved in time. Since $\mathbf{v}$ is a four-vector, the tensor Γ can be represented as a 4×4 matrix. However by the definition of the wedge product Γ is antisymmetric and all components on the diagonal are zero. Because of this we can only identify six conserved quantities from this tensor. Using Cartesian coordinates we can write

$$\mathbf{v}' = \begin{pmatrix} t'' \\ x'' \\ y'' \\ 0 \end{pmatrix} \qquad \text{and} \qquad \mathbf{v} = \begin{pmatrix} t' - \gamma \\ x' \\ y' \\ 0 \end{pmatrix}, \qquad (67)$$

where we choose coordinates such that the motion is in the x-y-plane. The components of Γ can now be calculated as for example

$$\Gamma_{1,2} = v_1v_2' - v_2v_1' = (t' - \gamma)x'' - x't'' \qquad \text{and} \qquad \Gamma_{1,3} = (t' - \gamma)y'' - y't''. \quad (68)$$

These components can be expressed as a conserved vector $\mathbf{\Pi}$ in space as

$$\mathbf{\Pi} = (t' - \gamma)\mathbf{r}'' - t''\mathbf{r}'. \qquad (69)$$

This can be rewritten using Equation (53) and the parametrization constraint as

$$\begin{aligned}\mathbf{\Pi} = (t' - \gamma)\mathbf{r}'' - t''\mathbf{r}' = (\frac{r}{\alpha} - \frac{\beta}{\alpha})\frac{-\beta}{r}\mathbf{r} + \left[(\frac{r}{\alpha} - \frac{\beta}{\alpha})\frac{r'}{r} - \frac{r'}{\alpha}\right]\mathbf{r}' \\ = \frac{\beta}{r\alpha}\left[(r-\beta)\mathbf{r} + r'\mathbf{r}'\right]. \end{aligned} \tag{70}$$

Rewriting this in terms of time derivatives instead one obtains

$$\mathbf{\Pi} = \frac{\beta}{\alpha r}\left[(r-\beta)\mathbf{r} + \frac{r}{\alpha^2}(\dot{\mathbf{r}} \cdot \mathbf{r}) \quad \dot{\mathbf{r}}\right]. \tag{71}$$

This vector can be identified to be proportional to the Runge-Lenz vector as defined in Equation (2). The other important conserved component of the tensor Γ is $\Gamma_{2,3}$. It can be calculated as

$$\Gamma_{2,3} = v_2 v_3' - v_3 v_2' = x'y'' - x''y'. \tag{72}$$

This can rewritten as

$$\Gamma_{2,3} = x'(\frac{-x}{r} + \frac{\dot{r}}{\alpha}x') + y'(\frac{-y}{r} + \frac{\dot{r}}{\alpha}y'). \tag{73}$$

This component of Γ can be identified to be proportional to the z-component of angular momentum.

To summarize, we saw that this parametrization in space time gives the equations of motions and conservation of energy a nice form. From these equations the symmetry group SO(4) becomes apparent. These equations also provide a way of defining the constants of motion angular momentum and the Runge-Lenz vector. This is also done in [4] for the case of $E > 0$ and $E = 0$.

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