Chapter 24 Physical Pendulum

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.... I had along with me….the Descriptions, with some Drawings of the principal Parts of the Pendulum-Clock which I had made, and as also of them of my then intended Timekeeper for the Longitude at Sea.[1]

John Harrison

24.1 Introduction

We have already used Newton's Second Law or Conservation of Energy to analyze systems like the spring-object system that oscillate. We shall now use torque and the rotational equation of motion to study oscillating systems like pendulums and torsional springs.

24.1.1 Simple Pendulum: Torque Approach

Recall the simple pendulum from Chapter 23.3.1.The coordinate system and force diagram for the simple pendulum is shown in Figure 24.1.
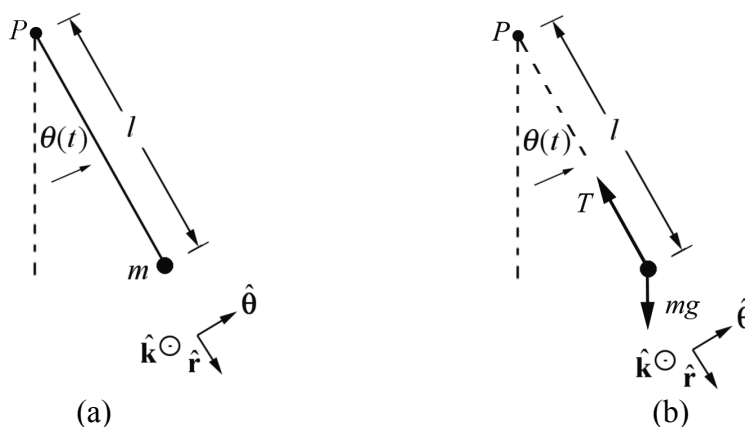

(a)   (b)

Figure 24.1 (a) Coordinate system and (b) torque diagram for simple pendulum

The torque about the pivot point P is given by

$$\vec{\tau}_P = \vec{r}_{P,m} \times m\vec{g} = l\,\hat{r} \times mg(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}) = -l\,mg\sin\theta\,\hat{k} \qquad (24.1.1)$$

The z-component of the torque about point P

$$(\tau_P)_z = -mgl\sin\theta. \qquad (24.1.2)$$

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[1] J. Harrison, *A Description Concerning Such Mechanisms as will Afford a Nice, or True Mensuration of Time;…*(London, 1775), p.19.

When $\theta > 0$, $(\tau_P)_z < 0$ and the torque about P is directed in the negative $\hat{\mathbf{k}}$-direction (into the plane of Figure 24.1b) when $\theta < 0$, $(\tau_P)_z > 0$ and the torque about P is directed in the positive $\hat{\mathbf{k}}$-direction (out of the plane of Figure 24.1b). The moment of inertia of a point mass about the pivot point P is $I_P = ml^2$. The rotational equation of motion is then

$$(\tau_P)_z = I_P \alpha_z \equiv I_P \frac{d^2\theta}{dt^2}$$

$$-mgl\sin\theta = ml^2 \frac{d^2\theta}{dt^2}.$$

(24.1.3)

Thus we have

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta ,$$

(24.1.4)

agreeing with Eq. 23. 3.14. When the angle of oscillation is small, we may use the small angle approximation

$$\sin\theta \cong \theta ,$$

(24.1.5)

and Eq. (24.1.4) reduces to the simple harmonic oscillator equation

$$\frac{d^2\theta}{dt^2} \cong -\frac{g}{l}\theta .$$

(24.1.6)

We have already studied the solutions to this equation in Chapter 23.3. A procedure for determining the period when the small angle approximation does not hold is given in Appendix 24A.

24.2 Physical Pendulum

A *physical pendulum* consists of a rigid body that undergoes fixed axis rotation about a fixed point S (Figure 24.2).
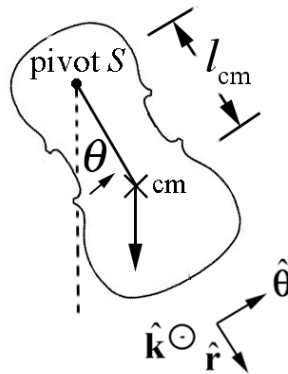

Figure 24.2 Physical pendulum

The gravitational force acts at the center of mass of the physical pendulum. Denote the distance of the center of mass to the pivot point S by l_{cm}. The torque analysis is nearly identical to the simple pendulum. The torque about the pivot point S is given by

$$\vec{\tau}_S = \vec{r}_{S,cm} \times m\vec{g} = l_{cm}\hat{r} \times mg(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}) = -l_{cm}mg\sin\theta\,\hat{k}. \qquad (24.2.1)$$

Following the same steps that led from Equation (24.1.1) to Equation (24.1.4), the rotational equation for the physical pendulum is

$$-mgl_{cm}\sin\theta = I_S \frac{d^2\theta}{dt^2}, \qquad (24.2.2)$$

where I_S the moment of inertia about the pivot point S. As with the simple pendulum, for small angles $\sin\theta \approx \theta$, Equation (24.2.2) reduces to the simple harmonic oscillator equation

$$\frac{d^2\theta}{dt^2} \simeq -\frac{mgl_{cm}}{I_S}\theta. \qquad (24.2.3)$$

The equation for the angle $\theta(t)$ is given by

$$\theta(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t), \qquad (24.2.4)$$

where the angular frequency is given by

$$\omega_0 \simeq \sqrt{\frac{mg\,l_{cm}}{I_S}} \qquad \text{(physical pendulum)}, \qquad (24.2.5)$$

and the period is

$$T = \frac{2\pi}{\omega_0} \simeq 2\pi\sqrt{\frac{I_S}{mg\,l_{cm}}} \qquad \text{(physical pendulum)}. \qquad (24.2.6)$$

Substitute the parallel axis theorem, $I_S = ml_{cm}^2 + I_{cm}$, into Eq. (24.2.6) with the result that

$$T \simeq 2\pi\sqrt{\frac{l_{cm}}{g} + \frac{I_{cm}}{mg\,l_{cm}}} \qquad \text{(physical pendulum)}. \qquad (24.2.7)$$

Thus, if the object is "small" in the sense that $I_{cm} \ll ml_{cm}^2$, the expressions for the physical pendulum reduce to those for the simple pendulum. The z-component of the angular velocity is given by

$$\omega_z(t) = \frac{d\theta}{dt}(t) = -\omega_0 A \sin(\omega_0 t) + \omega_0 B \cos(\omega_0 t). \tag{24.2.8}$$

The coefficients A and B can be determined form the initial conditions by setting $t = 0$ in Eqs. (24.2.4) and (24.2.8) resulting in the conditions that

$$A = \theta(t = 0) \equiv \theta_0$$
$$B = \frac{\omega_z(t = 0)}{\omega_0} \equiv \frac{\omega_{z,0}}{\omega_0}. \tag{24.2.9}$$

Therefore the equations for the angle $\theta(t)$ and $\omega_z(t) = \frac{d\theta}{dt}(t)$ are given by

$$\theta(t) = \theta_0 \cos(\omega_0 t) + \frac{\omega_{z,0}}{\omega_0} \sin(\omega_0 t), \tag{24.2.10}$$

$$\omega_z(t) = \frac{d\theta}{dt}(t) = -\omega_0 \theta_0 \sin(\omega_0 t) + \omega_{z,0} \cos(\omega_0 t). \tag{24.2.11}$$

24.3 Worked Examples

Example 24.1 Oscillating Rod

A physical pendulum consists of a uniform rod of length d and mass m pivoted at one end. The pendulum is initially displaced to one side by a small angle θ_0 and released from rest with $\theta_0 \ll 1$. Find the period of the pendulum. Determine the period of the pendulum using (a) the torque method and (b) the energy method.
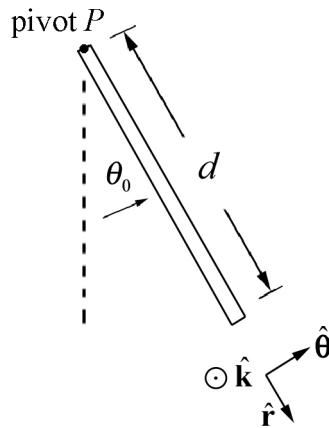

Figure 24.3 Oscillating rod

(a) Torque Method: with our choice of rotational coordinate system the angular acceleration is given by

$$\vec{\alpha} = \frac{d^2\theta}{dt^2}\hat{\mathbf{k}} \,. \tag{24.3.1}$$

The force diagram on the pendulum is shown in Figure 24.4. In particular, there is an unknown pivot force and the gravitational force acts at the center of mass of the rod.
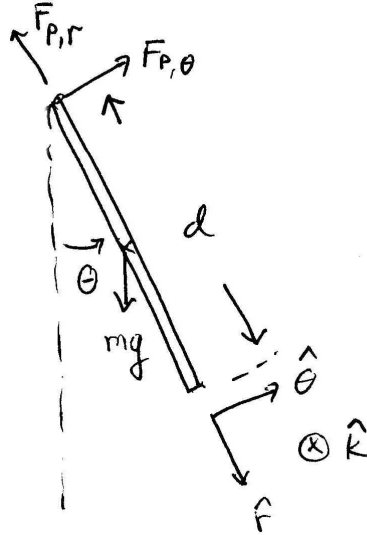

Figure 24.4 Free-body force diagram on rod

The torque about the pivot point P is given by

$$\vec{\tau}_P = \vec{\mathbf{r}}_{P,\text{cm}} \times m\vec{\mathbf{g}} \,. \tag{24.3.2}$$

The rod is uniform, therefore the center of mass is a distance $d/2$ from the pivot point. The gravitational force acts at the center of mass, so the torque about the pivot point P is given by

$$\vec{\tau}_P = (d/2)\hat{\mathbf{r}} \times mg(-\sin\theta\,\hat{\boldsymbol{\theta}} + \cos\hat{\mathbf{r}}) = -(d/2)mg\sin\theta\,\hat{\mathbf{k}} \,. \tag{24.3.3}$$

The rotational equation of motion about P is then

$$\vec{\tau}_P = I_P\vec{\alpha} \,. \tag{24.3.4}$$

Substituting Eqs. (24.3.3) and (24.3.1) into Eq. (24.3.4) yields

$$-(d/2)mg\sin\theta\,\hat{\mathbf{k}} = I_P\frac{d^2\theta}{dt^2}\hat{\mathbf{k}} \,. \tag{24.3.5}$$

When the angle of oscillation is small, we may use the small angle approximation $\sin\theta \cong \theta$, then Eq. (24.3.5) becomes

$$\frac{d^2\theta}{dt^2}+\frac{(d/2)mg}{I_P}\theta \simeq 0, \tag{24.3.6}$$

which is a simple harmonic oscillator equation. The angular frequency of small oscillations for the pendulum is

$$\omega_0 \simeq \sqrt{\frac{(d/2)mg}{I_P}}. \tag{24.3.7}$$

The moment of inertia of a rod about the end point P is $I_P = (1/3)md^2$ therefore the angular frequency is

$$\omega_0 \simeq \sqrt{\frac{(d/2)mg}{(1/3)md^2}} = \sqrt{\frac{(3/2)g}{d}} \tag{24.3.8}$$

with period

$$T = \frac{2\pi}{\omega_0} \simeq 2\pi\sqrt{\frac{2}{3}\frac{d}{g}}. \tag{24.3.9}$$

(b) Energy Method: Take the zero point of gravitational potential energy to be the point where the center of mass of the pendulum is at its lowest point (Figure 24.5), that is, $\theta = 0$.
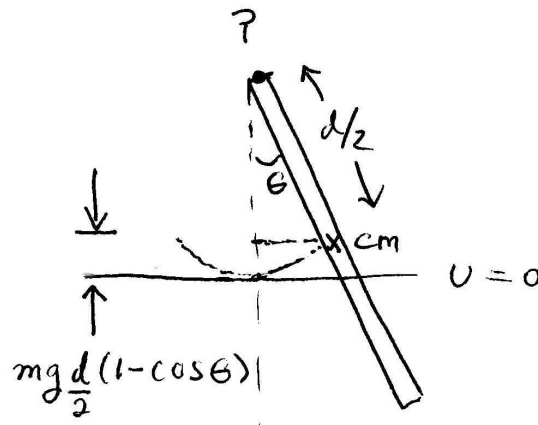

Figure 24.5 Energy diagram for rod

When the pendulum is at an angle θ the potential energy is

$$U = mg\frac{d}{2}(1-\cos\theta). \tag{24.3.10}$$

The kinetic energy of rotation about the pivot point is

$$K^{\text{rot}} = \frac{1}{2} I_p \omega_z^{\,2}. \tag{24.3.11}$$

The mechanical energy is then

$$E = U + K^{\text{rot}} = mg\frac{d}{2}\left(1 - \cos\theta\right) + \frac{1}{2}I_p\omega_z^{\,2}, \tag{24.3.12}$$

with $I_P = (1/3)md^2$. There are no non-conservative forces acting (by assumption), so the mechanical energy is constant, and therefore the time derivative of energy is zero,

$$0 = \frac{dE}{dt} = mg\frac{d}{2}\sin\theta\frac{d\theta}{dt} + I_p\omega_z\frac{d\omega_z}{dt}. \tag{24.3.13}$$

Recall that $\omega_z = d\theta / dt$ and $\alpha_z = d\omega_z / dt = d^2\theta / dt^2$, so Eq. (24.3.13) becomes

$$0 = \omega_z\left(mg\frac{d}{2}\sin\theta + I_p\frac{d^2\theta}{dt^2}\right). \tag{24.3.14}$$

There are two solutions, $\omega_z = 0$, in which case the rod remains at the bottom of the swing,

$$0 = mg\frac{d}{2}\sin\theta + I_p\frac{d^2\theta}{dt^2}. \tag{24.3.15}$$

Using the small angle approximation, we obtain the simple harmonic oscillator equation (Eq. (24.3.6))

$$\frac{d^2\theta}{dt^2} + \frac{mg(d/2)}{I_p}\theta \simeq 0. \tag{24.3.16}$$

Example 24.3 Torsional Oscillator

A disk with moment of inertia about the center of mass I_{cm} rotates in a horizontal plane. It is suspended by a thin, massless rod. If the disk is rotated away from its equilibrium position by an angle θ, the rod exerts a restoring torque about the center of the disk with magnitude given by $\tau_{\text{cm}} = b\theta$ (Figure 24.6), where b is a positive constant. At $t = 0$, the disk is released from rest at an angular displacement of θ_0. Find the subsequent time dependence of the angular displacement $\theta(t)$.
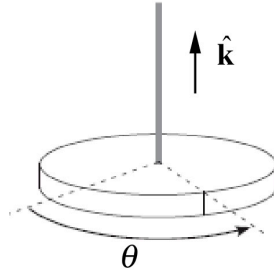
Figure 24.6 Example 24.3 with exaggerated angle θ

Solution: Choose a coordinate system such that $\hat{\mathbf{k}}$ is pointing upwards (Figure 24.6), then the angular acceleration is given by

$$\vec{\alpha} = \frac{d^2\theta}{dt^2}\hat{\mathbf{k}} .$$
(24.3.17)

The torque about the center of mass is given in the statement of the problem as a restoring torque, therefore

$$\vec{\tau}_{cm} = -b\,\theta\,\hat{\mathbf{k}} .$$
(24.3.18)

The z-component of the rotational equation of motion is

$$-b\,\theta = I_{cm}\frac{d^2\theta}{dt^2} .$$
(24.3.19)

This is a simple harmonic oscillator equation with solution

$$\theta(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$$
(24.3.20)

where the angular frequency of oscillation is given by

$$\omega_0 = \sqrt{b/I_{cm}} .$$
(24.3.21)

The z-component of the angular velocity is given by

$$\omega_z(t) = \frac{d\theta}{dt}(t) = -\omega_0 A\sin(\omega_0 t) + \omega_0 B\cos(\omega_0 t) .$$
(24.3.22)

The initial conditions at $t = 0$, are $\theta(t=0) = A = \theta_0$, and $(d\theta/dt)(t=0) = \omega_0 B = 0$. Therefore

$$\theta(t) = \theta_0 \cos(\sqrt{b/I_{cm}}\ t) .$$
(24.3.23)

Example 24.4 Compound Physical Pendulum

A compound physical pendulum consists of a disk of radius R and mass m_d fixed at the end of a rod of mass m_r and length l (Figure 24.7a). (a) Find the period of the pendulum. (b) How does the period change if the disk is mounted to the rod by a frictionless bearing so that it is perfectly free to spin?
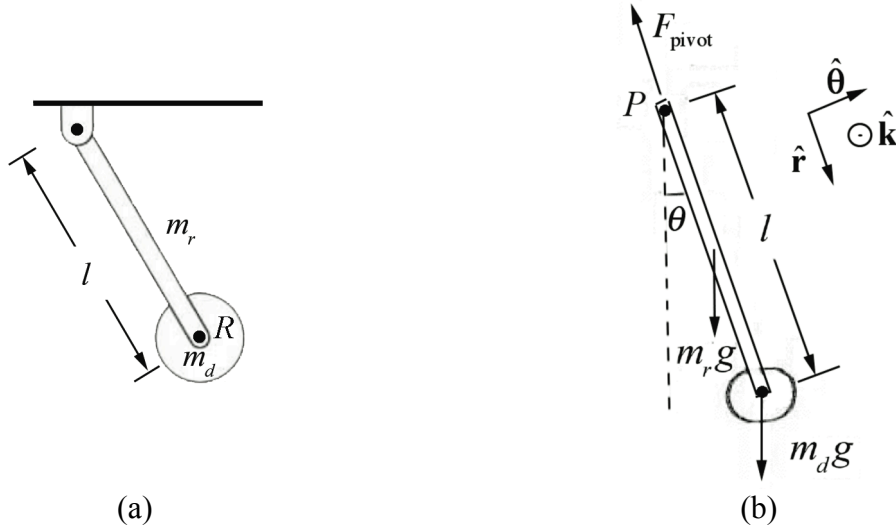

(a)	(b)

Figure 24.7 (a) Example 24.4   (b) Free-body force diagram

Solution: We begin by choosing coordinates. Let $\hat{\mathbf{k}}$ be normal to the plane of the motion of the pendulum pointing out of the plane of the Figure 24.7b. Choose an angle variable θ such that counterclockwise rotation corresponds to a positive z-component of the angular velocity. Thus a torque that points into the page has a negative z-component and a torque that points out of the page has a positive z-component. The free-body force diagram on the pendulum is also shown in Figure 24.7b. In particular, there is an unknown pivot force, the gravitational force acting at the center of mass of the rod, and the gravitational force acting at the center of mass of the disk. The torque about the pivot point is given by

$$\vec{\tau}_P = \vec{r}_{P,\text{cm}} \times m_r \vec{g} + \vec{r}_{P,\text{disk}} \times m_d \vec{g} . \tag{24.3.24}$$

Recall that the vector $\vec{r}_{P,\text{cm}}$ points from the pivot point to the center of mass of the rod a distance $l/2$ from the pivot. The vector $\vec{r}_{P,\text{disk}}$ points from the pivot point to the center of mass of the disk a distance l from the pivot. Torque diagrams for the gravitational force on the rod and the disk are shown in Figure 24.8. Both torques about the pivot are in the negative $\hat{\mathbf{k}}$-direction (into the plane of Figure 24.8) and hence have negative z-components,

$$\vec{\tau}_P = -(m_r(l/2) + m_d l)g \sin\theta \, \hat{\mathbf{k}} . \tag{24.3.25}$$
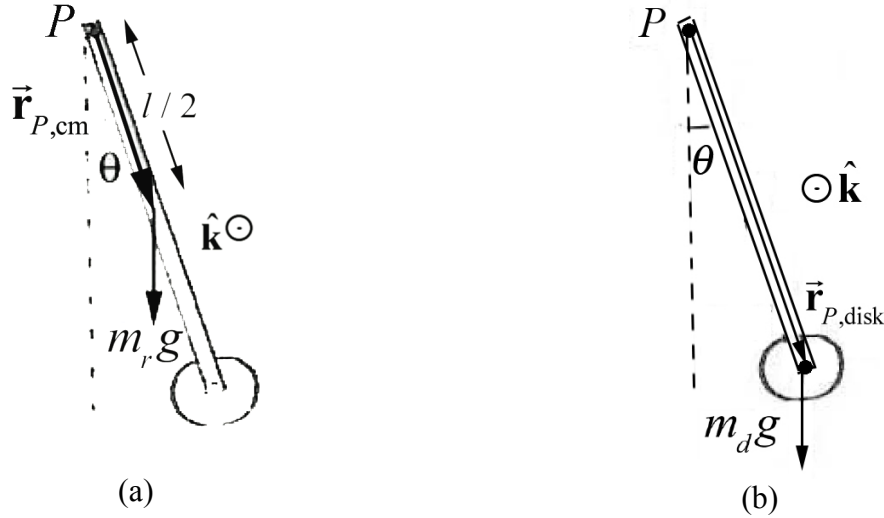
Figure 24.8 Torque diagram for (a) center of mass, (b) disk

In order to determine the moment of inertia of the rigid compound pendulum we will treat each piece separately, the uniform rod of length d and the disk attached at the end of the rod. The moment of inertia about the pivot point P is the sum of the moments of inertia of the two pieces,

$$I_P = I_{P,\text{rod}} + I_{P,\text{disc}}.$$ (24.3.26)

We calculated the moment of inertia of a rod about the end point P (Chapter 16.3.3), with the result that

$$I_{P,\text{rod}} = \frac{1}{3} m_r l^2.$$ (24.3.27)

We can use the parallel axis theorem to calculate the moment of inertia of the disk about the pivot point P,

$$I_{P,\text{disc}} = I_{\text{cm,disc}} + m_d l^2.$$ (24.3.28)

We calculated the moment of inertia of a disk about the center of mass (Example 16.3) and determined that

$$I_{\text{cm,disc}} = \frac{1}{2} m_d R^2.$$ (24.3.29)

The moment of inertia of the compound system is then

$$I_P = \frac{1}{3} m_r l^2 + m_d l^2 + \frac{1}{2} m_d R^2.$$ (24.3.30)

Therefore the rotational equation of motion becomes

$$-((1/2)m_r + m_d)gl\sin\theta\,\hat{\mathbf{k}} = \left(((1/3)m_r + m_d)l^2 + (1/2)m_d R^2\right)\frac{d^2\theta}{dt^2}\hat{\mathbf{k}}\,. \quad (24.3.31)$$

When the angle of oscillation is small, we can use the small angle approximation $\sin\theta \simeq \theta$. Then Eq. (24.3.31) becomes a simple harmonic oscillator equation,

$$\frac{d^2\theta}{dt^2} \simeq -\frac{((1/2)m_r + m_d)gl}{((1/3)m_r + m_d)l^2 + (1/2)m_d R^2}\theta\,. \quad (24.3.32)$$

Eq. (24.3.32) describes simple harmonic motion with an angular frequency of oscillation when the disk is fixed in place given by

$$\omega_{\text{fixed}} = \sqrt{\frac{((1/2)m_r + m_d)gl}{((1/3)m_r + m_d)l^2 + (1/2)m_d R^2}}\,. \quad (24.3.33)$$

The period is

$$T_{\text{fixed}} = \frac{2\pi}{\omega_{\text{fixed}}} \simeq 2\pi\sqrt{\frac{((1/3)m_r + m_d)l^2 + (1/2)m_d R^2}{((1/2)m_r + m_d)gl}}\,. \quad (24.3.34)$$

(b) If the disk is not fixed to the rod, then it will not rotate about its center of mass as the pendulum oscillates. Therefore the moment of inertia of the disk about its center of mass does not contribute to the moment of inertia of the physical pendulum about the pivot point. Notice that the pendulum is no longer a rigid body. The total moment of inertia is only due to the rod and the disk treated as a point like object,

$$I_P = \frac{1}{3}m_r l^2 + m_d l^2\,. \quad (24.3.35)$$

Therefore the period of oscillation is given by

$$T_{\text{free}} = \frac{2\pi}{\omega_{\text{free}}} \simeq 2\pi\sqrt{\frac{((1/3)m_r + m_d)l^2}{((1/2)m_r + m_d)gl}}\,. \quad (24.3.36)$$

Comparing Eq. (24.3.36) to Eq. (24.3.34), we see that the period is smaller when the disk is free and not fixed. From an energy perspective we can argue that when the disk is free, it is not rotating about the center of mass. Therefore more of the gravitational potential energy goes into the center of mass translational kinetic energy than when the disk is free. Hence the center of mass is moving faster when the disk is free so it completes one period is a shorter time.

Appendix 24A Higher-Order Corrections to the Period for Larger Amplitudes of a Simple Pendulum

In Section 24.1.1, we found the period for a simple pendulum that undergoes small oscillations is given by

$$T = \frac{2\pi}{\omega_0} \cong 2\pi\sqrt{\frac{l}{g}} \qquad \text{(simple pendulum)}.$$

How good is this approximation? If the pendulum is pulled out to an initial angle θ_0 that is not small (such that our first approximation $\sin\theta \cong \theta$ no longer holds) then our expression for the period is no longer valid. We shall calculate the first-order (or higher-order) correction to the period of the pendulum.

Let's first consider the mechanical energy, a conserved quantity in this system. Choose an initial state when the pendulum is released from rest at an angle θ_i; this need not be at time $t = 0$, and in fact later in this derivation we'll see that it's inconvenient to choose this position to be at $t = 0$. Choose for the final state the position and velocity of the bob at an arbitrary time t. Choose the zero point for the potential energy to be at the bottom of the bob's swing (Figure 24A.1).
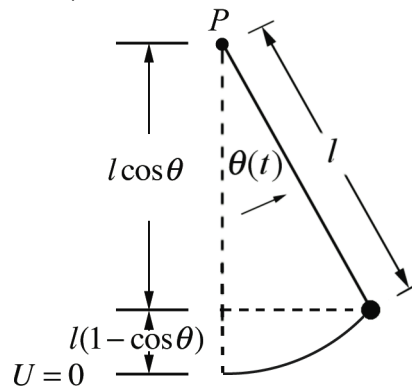

Figure 24A.1 Energy diagram for simple pendulum

The mechanical energy when the bob is released from rest at an angle θ_i is

$$E_i = K_i + U_i = mgl(1 - \cos\theta_i). \qquad (24.C.37)$$

The tangential component of the velocity of the bob at an arbitrary time t is given by

$$v_\theta = l\frac{d\theta}{dt}, \qquad (24.C.38)$$

and the kinetic energy at that time is

$$K_f = \frac{1}{2}mv_\theta{}^2 = \frac{1}{2}m\left(l\frac{d\theta}{dt}\right)^2 . \tag{24.C.39}$$

The mechanical energy at time t is then

$$E_f = K_f + U_f = \frac{1}{2}m\left(l\frac{d\theta}{dt}\right)^2 + mgl(1-\cos\theta) . \tag{24.C.40}$$

Because the tension in the string is always perpendicular to the displacement of the bob, the tension does no work, we neglect any frictional forces, and hence mechanical energy is constant, $E_f = E_i$. Thus

$$\frac{1}{2}m\left(l\frac{d\theta}{dt}\right)^2 + mgl(1-\cos\theta) = mgl(1-\cos\theta_i)$$

$$\left(l\frac{d\theta}{dt}\right)^2 = 2\frac{g}{l}(\cos\theta - \cos\theta_i). \tag{24.C.41}$$

We can solve Equation (24.C.41) for the angular velocity as a function of θ,

$$\frac{d\theta}{dt} = \sqrt{\frac{2g}{l}}\sqrt{\cos\theta - \cos\theta_i} . \tag{24.C.42}$$

Note that we have taken the positive square root, implying that $d\theta/dt \geq 0$. This clearly cannot always be the case, and we should change the sign of the square root every time the pendulum's direction of motion changes. For our purposes, this is not an issue. If we wished to find an explicit form for either $\theta(t)$ or $t(\theta)$, we would have to consider the signs in Equation (24.C.42) more carefully.

Before proceeding, it's worth considering the difference between Equation (24.C.42) and the equation for the simple pendulum in the simple harmonic oscillator limit, where $\cos\theta \simeq 1-(1/2)\theta^2$. Then Eq. (24.C.42) reduces to

$$\frac{d\theta}{dt} = \sqrt{\frac{2g}{l}}\sqrt{\frac{\theta_i^2}{2} - \frac{\theta^2}{2}} . \tag{24.C.43}$$

In both Equations (24.C.42) and (24.C.43) the last term in the square root is proportional to the difference between the initial potential energy and the final potential energy. The final potential energy for the two cases is plotted in Figures 24A.2 for $-\pi < \theta < \pi$ on the left and $-\pi/2 < \theta < \pi/2$ on the right (the vertical scale is in units of mgl).
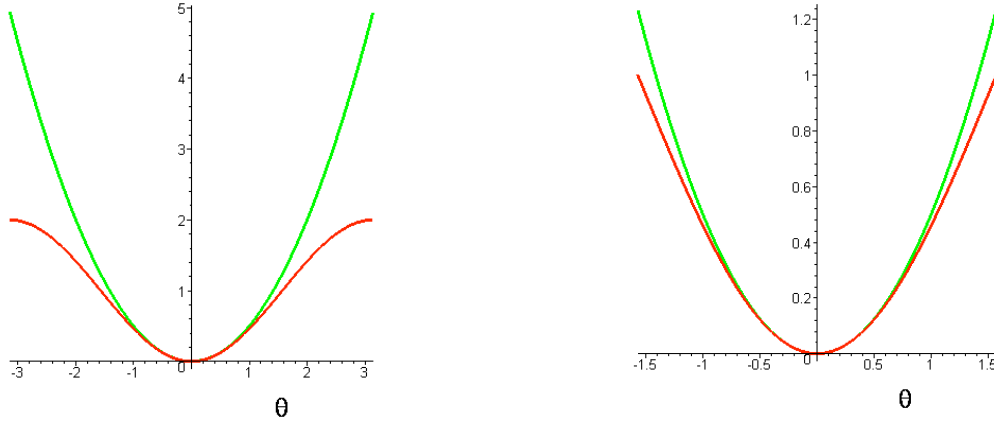
Figures 24A.2 Potential energies as a function of displacement angle

It would seem to be to our advantage to express the potential energy for an arbitrary displacement of the pendulum as the difference between two squares. We do this by first recalling the trigonometric identity

$$1 - \cos\theta = 2\sin^2(\theta/2) \tag{24.C.44}$$

with the result that Equation (24.C.42) may be re-expressed as

$$\frac{d\theta}{dt} = \sqrt{\frac{2g}{l}}\sqrt{2(\sin^2(\theta_i/2) - \sin^2(\theta/2))} \ . \tag{24.C.45}$$

Equation (24.C.45) is separable,

$$\frac{d\theta}{\sqrt{\sin^2(\theta_i/2) - \sin^2(\theta/2)}} = 2\sqrt{\frac{g}{l}}\,dt \tag{24.C.46}$$

Rewrite Equation (24.C.46) as

$$\frac{d\theta}{\sin(\theta_i/2)\sqrt{1 - \dfrac{\sin^2(\theta/2)}{\sin^2(\theta_i/2)}}} = 2\sqrt{\frac{g}{l}}\,dt \ . \tag{24.C.47}$$

The ratio $\sin(\theta/2)/\sin(\theta_i/2)$ in the square root in the denominator will oscillate (but *not* with simple harmonic motion) between -1 and $+1$, and so we will make the identification

$$\sin\phi = \frac{\sin(\theta/2)}{\sin(\theta_i/2)} \ . \tag{24.C.48}$$

Let $b = \sin(\theta_i / 2)$, so that

$$\sin\frac{\theta}{2} = b\sin\phi$$

$$\cos\frac{\theta}{2} = \left(1 - \sin^2\frac{\theta}{2}\right)^{1/2} = (1 - b^2\sin^2\phi)^{1/2}.$$

(24.C.49)

Eq. (24.C.47) can then be rewritten in integral form as

$$\int\frac{d\theta}{b\sqrt{1 - \sin^2\phi}} = 2\int\sqrt{\frac{g}{l}}\,dt\,.$$

(24.C.50)

From differentiating the first expression in Equation (24.C.49), we have that

$$\frac{1}{2}\cos\frac{\theta}{2}\,d\theta = b\cos\phi\,d\phi$$

$$d\theta = 2b\frac{\cos\phi}{\cos(\theta/2)}\,d\phi = 2b\frac{\sqrt{1 - \sin^2\phi}}{\sqrt{1 - \sin^2(\theta/2)}}\,d\phi$$

(24.C.51)

$$= 2b\frac{\sqrt{1 - \sin^2\phi}}{\sqrt{1 - b^2\sin^2\phi}}\,d\phi.$$

Substituting the last equation in (24.C.51) into the left-hand side of the integral in (24.C.50) yields

$$\int\frac{2b}{b\sqrt{1 - \sin^2\phi}}\frac{\sqrt{1 - \sin^2\phi}}{\sqrt{1 - b^2\sin^2\phi}}\,d\phi = 2\int\frac{d\phi}{\sqrt{1 - b^2\sin^2\phi}}\,.$$

(24.C.52)

Thus the integral in Equation (24.C.50) becomes

$$\int\frac{d\phi}{\sqrt{1 - b^2\sin^2\phi}} = \int\sqrt{\frac{g}{l}}\,dt\,.$$

(24.C.53)

This integral is one of a class of integrals known as ***elliptic integrals***. We find a power series solution to this integral by expanding the function

$$(1 - b^2\sin^2\phi)^{-1/2} = 1 + \frac{1}{2}b^2\sin^2\phi + \frac{3}{8}b^4\sin^4\phi + \cdots.$$

(24.C.54)

The integral in Equation (24.C.53) then becomes

$$\int \left(1+\frac{1}{2}b^2\sin^2\phi+\frac{3}{8}b^4\sin^4\phi+\cdots\right)d\phi = \int\sqrt{\frac{g}{l}}\,dt. \qquad (24.C.55)$$

Now let's integrate over one period. Set $t=0$ when $\theta=0$, the lowest point of the swing, so that $\sin\phi=0$ and $\phi=0$. One period T has elapsed the second time the bob returns to the lowest point, or when $\phi=2\pi$. Putting in the limits of the ϕ-integral, we can integrate term by term, noting that

$$\int_0^{2\pi}\frac{1}{2}b^2\sin^2\phi\,d\phi = \int_0^{2\pi}\frac{1}{2}b^2\frac{1}{2}(1-\cos(2\phi))\,d\phi$$

$$=\frac{1}{2}b^2\frac{1}{2}\left(\phi-\frac{\sin(2\phi)}{2}\right)\Bigg|_0^{2\pi} \qquad (24.C.56)$$

$$=\frac{1}{2}\pi b^2 = \frac{1}{2}\pi\sin^2\frac{\theta_i}{2}.$$

Thus, from Equation (24.C.55) we have that

$$\int_0^{2\pi}\left(1+\frac{1}{2}b^2\sin^2\phi+\frac{3}{8}b^4\sin^4\phi+\cdots\right)d\phi = \int_0^T\sqrt{\frac{g}{l}}\,dt$$

$$2\pi+\frac{1}{2}\pi\sin^2\frac{\theta_i}{2}+\cdots = \sqrt{\frac{g}{l}}\,T \qquad (24.C.57)$$

We can now solve for the period,

$$T = 2\pi\sqrt{\frac{l}{g}}\left(1+\frac{1}{4}\sin^2\frac{\theta_i}{2}+\cdots\right). \qquad (24.C.58)$$

If the initial angle $\theta_i \ll 1$ (measured in radians), then $\sin^2(\theta_i/2)\simeq\theta_i^2/4$ and the period is approximately

$$T \cong 2\pi\sqrt{\frac{l}{g}}\left(1+\frac{1}{16}\theta_i^2\right) = T_0\left(1+\frac{1}{16}\theta_i^2\right), \qquad (24.C.59)$$

where

$$T_0 = 2\pi\sqrt{\frac{l}{g}} \qquad (24.C.60)$$

is the period of the simple pendulum with the standard small angle approximation. The first order correction to the period of the pendulum is then

$$\Delta T_1 = \frac{1}{16}\theta_i^2 T_0.$$ (24.C.61)

Figure 24A.3 below shows the three functions given in Equation (24.C.60) (the horizontal, or red plot if seen in color), Equation (24.C.59) (the middle, parabolic or green plot) and the numerically-integrated function obtained by integrating the expression in Equation (24.C.53) (the upper, or blue plot) between $\phi = 0$ and $\phi = 2\pi$. The plots demonstrate that Equation (24.C.60) is a valid approximation for small values of θ_i, and that Equation (24.C.59) is a very good approximation for all but the largest amplitudes of oscillation. The vertical axis is in units of $\sqrt{l/g}$. Note the displacement of the horizontal axis.
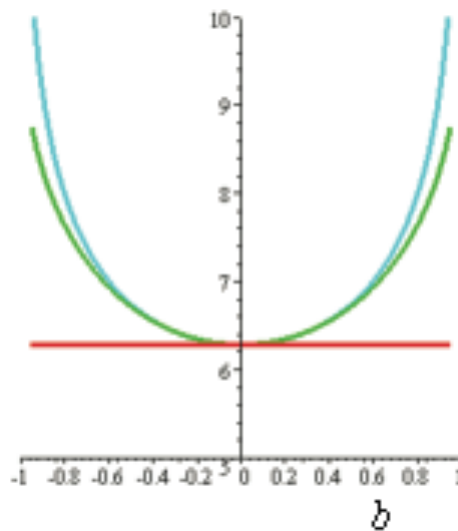

Figure 24A.3 Pendulum Period Approximations as Functions of Amplitude