Stability of the Lagrange Points, L_4 and L_5

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Abstract

A proof of the stability of the non collinear Lagrange Points, L_4 and L_5. We will start by covering the basics of stability, stating a theorem (without proof) with a few corollaries, and then turn to the Lagrange points, proving first the stability of all Lagrange points in the z-direction and then restricting our attention to the the points, L_4 and L_5.

Introduction

The three body problem is one that has been studied for many centuries. It consists of considering 3 bodies, subject to only to mutually attracting forces (determined by the inverse square force, gravitation) and to solve for the motions of the three bodies. This task can be extremely complicated and, following Poincaré's example, most mathematicians and physicists have interested themselves in the finding periodic solutions to the problem, that is, where $x(t) = x(t + T)$ where T is the period [3]. Indeed not only does this offer a useful framework from within which to work, but also describes states that are most applicable and useful to real world situations. Finding these periodic solutions is by no means trivial. It is a problem that has been worked on for centuries and still is in modern times, one of the latest publications dealing with it as recent as March of 2013 [9].

Among the first to interest themselves in and find solutions to the 3 body problem were Leonhard Euler in 1765 and Joseph Lagrange in 1772 [3]. In his publication, *Essai sur le problème des trois corps* (*Essay on the 3-Body Problem*), Lagrange proposed a method that had never been used until then, that of considering only the distances between the three bodies rather than their absolute positions [2]. Through this method, he found that there are exactly 5 different configurations the three bodies can be arranged in so that their movement is both circular and periodic. Given the initial position of two of the masses (usually the largest masses), the 5 different locations that the third body can be in such that the solution is circular and periodic are called the 5 Lagrange or Libration points.

Lagrange himself did not actually believe that instances of the 5 Libration points existed in the "real world", as he states at the beginning of his derivation [2]. Indeed, it was not until 1906 that the first example, an asteroid sharing Jupiter's orbit but ahead of it by close to 60° with respect to the sun, was discovered by professor Max Wolf at Heidelberg [8]. Because the asteroid and those that were subsequently discovered at the L_4 and L_5 points of the Jupiter-sun system were named after characters in the Illiad, asteroids in L_4 and L_5 of any system are generally referred to as Trojan asteroids.
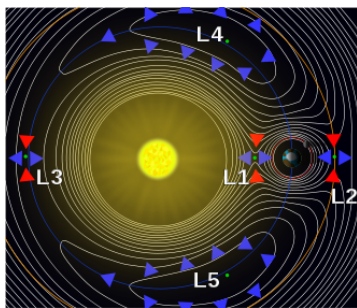
Figure 1: Lagrange Points
The three first points, L_1, L_2, L_3 are all aligned with the earth-sun vector. L_4 and L_5 form an equilateral triangle with the sun and earth.
The arrows indicate potentials and show the potential peaks, wells and saddles.

To find these points, Lagrange needed to make some approximations (indeed, the system is known today to be chaotic otherwise). The results nevertheless are often quite good at accurately describing dynamics of our solar system. In our case we consider the circular restricted 3-body problem as described below.

The circular restricted 3-body problem

The assumptions are as follows:

- Two of the masses, m_1 and m_2, are much heavier than the third one which we thus consider to be negligible. The center of mass is thus on a line between m_1 and m_2.

- The masses follow a circular orbit around the center of mass.

We will describe our system by putting it in the form:

$$\frac{d}{dt}X(t) = AX(t)$$

1 Stability and its consequences

In any dynamical system with one or more equilibrium points, it is important to know whether the equilibrium points are stable, that is, whether a point at equilibrium can be subjected to small perturbations and still stay close to the equilibrium point. We use the following definition:

Definition An equilibrium solution, x_0, to a dynamical system is considered:

- *stable* if for every small $\varepsilon > 0$, there exists a $\delta > 0$ such that every solution, $x(t)$, with initial conditions $||x(t_0) - x_0|| < \delta$ is such that $||x(t) - x_0|| < \varepsilon \;\; \forall t \geq t_0$.

- *asymptotically stable* if, in addition to being *stable*, there exists $\delta_0 > 0$ such that every solution, $x(t)$, with initial conditions $||x(t_0) - x_0|| < \delta_0$ is such that $x(t) \to x_0$ as $t \to \infty$.

In words, an equilibrium point is *stable* if, given a small distance ε, there always exists a distance, δ, such that any solution with initial conditions within δ of the equilibrium point will always stay within ε of the equilibrium for any time t. As we will see, Lagrange Points do not fulfill the much stronger condition of *asymptotically stability* but they are stable in some cases.

Beyond characterizing the dynamical system in question, considering stability is extremely important for any real-world applications. Indeed it is virtually impossible for any object to be precisely at an actual equilibrium point. Thus it is crucial to know what an object will do when close to an equilibrium for these are the actual dynamics that the object will necessarily follow.

Before starting, we need a theorem and a few corollaries.

Theorem 1.1 *Given a system* $X'(t) = AX(t)$ *where* A *has distinct paired complex eigenvalues,* $\alpha_1 + i\beta_1, \alpha_1 - i\beta_1 \ldots, \alpha_k + i\beta_k, \alpha_k - i\beta_k$. *Let* T *be the matrix such that* $T^{-1}AT$ *is in canonical form:*

$$T^{-1}AT = \begin{pmatrix} B_1 & & \\ & \ddots & \\ & & B_k \end{pmatrix}$$

where

$$B_i = \begin{pmatrix} \alpha_i & \beta_i \\ -\beta_i & \alpha_i \end{pmatrix}$$

Then the general solution of $X'(t) = AX(t)$ *is* $TY(t)$ *where*

$$Y(t) = \begin{pmatrix} a_1 e^{\alpha_1 t} \cos\beta_1 t + b_1 e^{\alpha_1 t} \sin\beta_1 t \\ -a_1 e^{\alpha_1 t} \sin\beta_1 t + b_1 e^{\alpha_1 t} \cos\beta_1 t \\ \vdots \\ a_k e^{\alpha_k t} \cos\beta_k t + b_k e^{\alpha_k t} \sin\beta_k t \\ -a_k e^{\alpha_k t} \sin\beta_k t + b_k e^{\alpha_k t} \cos\beta_k t \end{pmatrix} \tag{1.1}$$

The proof of this theorem is beyond the scope of this paper, but can be found in chp. 6 of *Differential Equations, Dynamical Systems, and an Introduction to Chaos* [4].

Corollary 1.2 *A point in a dynamical system whose matrix of equations of motion,* A, *has an eigenvalue with positive real part is unstable.*

Proof Let m be the index s.t. $\alpha_m > 0$. To show the point is stable, it suffices to show that the point is unstable in only one direction. To do so we let $a_i = b_i = 0 \ \forall i \neq m$ and assume that either $a_m \neq 0$ or $b_m \neq 0$ (or both). This is effectively considering small perturbations of the equilibrium point in only one direction. We denote $c_{i,j} = (T)_{i,j}$ (where T is the matrix described in theorem 1.1).

We begin by noting that T is invertible so every column must have at least one non-zero entry and no column is the multiple of another column. This means that, for every $1 \leq j \leq k$, there exists an i such that $c_{i,2j-1} \neq 0$ or $c_{i,2j} \neq 0$ and $c_{i,2j-1} \neq \pm c_{i,2j}$ (if this second condition were not the case then we would have $(T)_{2j-1} = \pm(T)_{2j}$ which is impossible).

Suppose the point is stable. Then we have $||TY(0) - TY(t)|| < \varepsilon$ for some ε and all $t > 0$. For any row, n, we get:

$$\varepsilon > \bigg| a_m(c_{n,2m-1} - c_{n,2m})$$

$$- e^{\alpha_m t}\left[c_{n,2m-1}\left(a_m \cos\beta_m t + b_m \sin\beta_m t\right) + c_{n,2m}\left(-a_m \cos\beta_m t + b_m \sin\beta_m t\right)\right]\bigg| \tag{1.2}$$

There are two cases to consider:

- **Case 1:** $b_m \neq 0$.

  Let n be the index s.t. $c_{n,2m-1} \neq 0$ or $c_{n,2m} \neq 0$ and $c_{n,2m-1} \neq -c_{n,2m}$. For simplicity we consider only the values $t = (k2\pi + \pi/2)/\beta_i$ for $k \in \mathbb{Z}$. Our equation then becomes:

  $$\varepsilon > \left| e^{\alpha_m t} b_m \left[c_{n,2m-1} + c_{n,2m} \right] + a_m (c_{n,2m-1} - c_{n,2m}) \right|$$

  As $t \to \infty$, $e^{\alpha_m t} b_m \left[c_{n,2m-1} + c_{n,2m} \right]$ dominates and the value goes to $\pm\infty$ depending on the sign of $b_m \left[c_{n,2m-1} + c_{n,2m} \right]$. Thus the solution is clearly not stable.

- **Case 2:** $b_m = 0$.

  Similarly to the first case, let n be the index s.t. $c_{n,2m-1} \neq 0$ or $c_{n,2m} \neq 0$ and $c_{n,2m-1} \neq c_{n,2m}$. Again for simplicity, consider only the values $t = (k2\pi)/\beta_i$ for $k \in \mathbb{Z}$. Our equation then becomes:

  $$\varepsilon > \left| e^{\alpha_m t} a_m \left[c_{n,2m-1} - c_{n,2m} \right] + a_m (c_{n,2m-1} - c_{n,2m}) \right|$$

  As $t \to \infty$, $e^{\alpha_m t} a_m \left[c_{n,2m-1} - c_{n,2m} \right]$ dominates and the value goes to $\pm\infty$ depending on the sign of $a_m \left[c_{n,2m-1} - c_{n,2m} \right]$. Thus the solution is clearly not stable.

Thus in either case, the point is not stable. ∎

Corollary 1.3 *A point in a dynamical system whose matrix of equations of motion, A, has purely imaginary (non-zero) eigenvalues is stable.*

Proof Suppose that $\alpha_i = 0$ $\forall i$ in (1.1). As seen in (1.2) the components of the solution becomes:

$$\sum_{i=1}^{k} \left[c_{n,2i-1} \left(a_i \cos \beta_i t + b_i \sin \beta_i t \right) + c_{n,2i} \left(-a_i \cos \beta_i t + b_i \sin \beta_i t \right) \right]$$

Since time, t, only appears in sines and cosines, it is clear that these values are completely bounded. Thus by choosing $a_1, \ldots, a_k$ and $b_1, \ldots, b_k$ appropriately (i.e. small enough), we can make the solutions stay within ε of the initial state for any $\varepsilon > 0$ and we thus conclude that the points are stable. ∎

2 The Lagrange Points

Recall from lecture that the first step we used to finding the Lagrange Points is to consider the bodies in a rotating reference frame such that the two heavier masses m_1 and m_2 do not move. Furthermore, since the movement of the three planets is planar, we can define the plane they move in to be the x-y plane with both heavier masses on the x-axis (with the origin at the center of mass). Let R be the distance between m_1 and m_2, we then have the positions of the two heavier masses as:

$$\mathbf{r_1} = \left(-\frac{m_2 R}{m_1 + m_2}, 0, 0 \right) \qquad \mathbf{r_2} = \left(\frac{m_1 R}{m_1 + m_2}, 0, 0 \right) \tag{2.1}$$

Proposition 2.1 *The angular frequency of the rotating reference frame, Ω, is given by:*

$$\Omega^2 R^3 = G(M_1 + M_2) \tag{2.2}$$

This only holds because we are considering the circular restricted 3-body problem as discussed above and follows directly from *Kepler's third law* (for more information see Landau & Livshits p. 23 [1]).

Solving the results found in lecture, We find our Lagrange points, L_4 and L_5:

$$L_4 = \left(\frac{R}{2} \left(\frac{m_1 - m_2}{m_1 + m_2} \right), \frac{\sqrt{3}}{2} R, 0 \right), \qquad L_5 = \left(\frac{R}{2} \left(\frac{m_1 - m_2}{m_1 + m_2} \right), -\frac{\sqrt{3}}{2} R, 0 \right) \qquad (2.3)$$

To find out the dynamical stability of motion near the equilibrium points, we need to look at the generalized potential about the mass m_3. Because we are in a rotating frame we must add both the *coriolis acceleration* and the *centrifugal acceleration*. Let $\mathbf{r}$ be the position vector of m_3 and $\mathbf{\Omega}$ the angular velocity such that $\mathbf{\Omega} = (0, 0, \Omega)$. Furthermore let d_1 and d_2 be the distances between m_3 and m_1 and m_2 respectively:

$$d_1^2 = \left(x + \frac{m_2 R}{m_1 + m_2} \right)^2 + y^2 + z^2, \qquad d_2^2 = \left(x - \frac{m_1 R}{m_1 + m_2} \right)^2 + y^2 + z^2 \qquad (2.4)$$

Proposition 2.2 *The total acceleration, $\ddot{\mathbf{r}}$, and the generalized potential, U are given by*

$$\ddot{\mathbf{r}} = -\frac{Gm_1(\mathbf{r} - \mathbf{r_1})}{d_1^3} - \frac{Gm_2(\mathbf{r} - \mathbf{r_2})}{d_2^3} - 2\mathbf{\Omega} \times \dot{\mathbf{r}} - \mathbf{\Omega} \times \mathbf{\Omega} \times \mathbf{r} \qquad (2.5)$$

$$U = -\frac{Gm_1}{d_1} - \frac{Gm_2}{d_2} - 2\Omega(x\dot{y} - y\dot{x}) - \frac{\Omega^2}{2}(x^2 + y^2) \qquad (2.6)$$

The second to last and last elements correspond to the coriolis and centrifugal accelerations respectively (for further information on these forces see Landau & Livshits p. 128 [1]).

Note that, in addition to position, the potential is dependent on velocity and, although it has no effect on the position of the Lagrange points, it must be taken into account when looking at their stability. To do so we separate the components dependent on velocity:

$$U' = U + 2\Omega(x\dot{y} - y\dot{x}) = -\frac{Gm_1}{d_1} - \frac{Gm_2}{d_2} - \frac{\Omega^2}{2}(x^2 + y^2) \qquad (2.7)$$

Reducing (2.5) by components we get:

$$\ddot{x} = -\frac{Gm_1 \left(x + \frac{m_2 R}{m_1 + m_2} \right)}{d_1^3} - \frac{Gm_2 \left(x - \frac{m_1 R}{m_1 + m_2} \right)}{d_2^3} + 2\Omega\dot{y} + \Omega^2 x = -\frac{\partial U'}{\partial x} + 2\Omega\dot{y} \qquad (2.8)$$

$$\ddot{y} = -\frac{Gm_1 y}{d_1^3} - \frac{Gm_2 y}{d_2^3} - 2\Omega\dot{x} + \Omega^2 y = -\frac{\partial U'}{\partial y} - 2\Omega\dot{x} \qquad (2.9)$$

$$\ddot{z} = -\frac{Gm_1 z}{d_1^3} - \frac{Gm_2 z}{d_2^3} = -\frac{\partial U'}{\partial z} \qquad (2.10)$$

Finally, it can be useful to rewrite the generalized potential as a sum of partial derivatives. Using the Taylor series expansion of U' around the Lagrange point (x_0, y_0, z_0) we get (to second order):

$$U' = U'_0 + U'_x(x - x_0) + U'_y(y - y_0) + U'_z(z - z_0)$$
$$+ \frac{1}{2} \left[U'_{xx}(x - x_0)^2 + U'_{yy}(y - y_0)^2 + U'_{zz}(z - z_0)^2 \right]$$
$$+ U'_{xy}(x - x_0)(y - y_0) + U'_{xz}(x - x_0)(z - z_0) + U'_{yz}(y - y_0)(z - z_0)$$

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where $U_0' = U'\big|_{(x_0,y_0,z_0)}$ and $U_X' = \frac{\partial U}{\partial X}\big|_{(x_0,y_0,z_0)}$ for any variable X. Note, however, that $U_{xz} = U_{yz} = 0$ and furthermore we have that by definition, $U_x' = U_y' = U_z' = 0$ for Lagrange points. We thus get:

$$U' = U_0' + \frac{1}{2}\left[U_{xx}'(x - x_0)^2 + U_{yy}'(y - y_0)^2 + U_{zz}'(z - z_0)^2\right] + U_{xy}'(x - x_0)(y - y_0) \tag{2.11}$$

3 Linearization and stability in the z-direction

To analyze the stability about the equilibrium points in a system such as the 3-body problem, we linearize the equations of motion and look at the small perturbations. Note that because of how we have set up the reference frame, the Lagrange points are *fixed points*. We thus have the following equations:

$$x = x_0 + \delta x \qquad \dot{x} = \delta\dot{x} \tag{3.1}$$
$$y = y_0 + \delta y \qquad \dot{y} = \delta\dot{y} \tag{3.2}$$
$$z = z_0 + \delta z \qquad \dot{z} = \delta\dot{z} \tag{3.3}$$

Plugging in these values into (2.11) then gives us:

$$U' = U_0' + \frac{1}{2}\left[U_{xx}'(\delta x)^2 + U_{yy}'(\delta y)^2 + U_{zz}'(\delta z)^2\right] + U_{xy}'\delta x\delta y \tag{3.4}$$

Using (2.8)-(2.10) and (3.4) we get our equations:

$$\delta\ddot{x} = -U_{xx}'\delta x - U_{xy}'\delta y + 2\Omega\delta\dot{y} \tag{3.5}$$
$$\delta\ddot{y} = -U_{yy}'\delta y - U_{xy}'\delta x - 2\Omega\delta\dot{x} \tag{3.6}$$
$$\delta\ddot{z} = -U_{zz}'\delta z \tag{3.7}$$

We thus get the following:

$$\frac{d}{dt}\begin{pmatrix} x \\ y \\ z \\ \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \frac{d}{dt}\begin{pmatrix} \delta x \\ \delta y \\ \delta z \\ \delta\dot{x} \\ \delta\dot{y} \\ \delta\dot{z} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -U_{xx}' & -U_{xy}' & 0 & 0 & 2\Omega & 0 \\ -U_{xy}' & -U_{yy}' & 0 & -2\Omega & 0 & 0 \\ 0 & 0 & -U_{zz}' & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \delta x \\ \delta y \\ \delta z \\ \delta\dot{x} \\ \delta\dot{y} \\ \delta\dot{z} \end{pmatrix} \tag{3.8}$$

We start by considering only the z-direction.

Theorem 3.1 *All Lagrange points are stable in the z-direction.*

Proof Let $\delta x = \delta y = 0$. By (3.5)-(3.7) we can see that δz and $\delta\dot{z}$ are independent of δx, $\delta\dot{x}$, δy, and $\delta\dot{y}$ and vice-versa so we can reduce our matrix to

$$\frac{d}{dt}\begin{pmatrix} \delta z \\ \delta\dot{z} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -U_{zz}' & 0 \end{pmatrix}\begin{pmatrix} \delta z \\ \delta\dot{z} \end{pmatrix}$$

By (2.10) we have that

$$U'_{zz} = \frac{Gm_1}{d_1^3} + \frac{Gm_2}{d_2^3} \tag{3.9}$$

Furthermore, since d_1, d_2 are distances we have $d_1, d_2 > 0$ and thus $U'_{zz} > 0$.
The eigenvalues of the above matrix are:

$$\pm i \sqrt{U'_{zz}} \tag{3.10}$$

Since these are always imaginary, we have by corollary 1.3 that the point is stable and thus conclude that all Lagrange points are stable in the z-direction. ∎

In the same way (and for the same reason) that we were able to consider only the z-direction, we can do the opposite and consider only the x and y directions. Our equation becomes:

$$\frac{d}{dt} \begin{pmatrix} \delta x \\ \delta y \\ \delta \dot{x} \\ \delta \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -U'_{xx} & -U'_{xy} & 0 & 2\Omega \\ -U'_{xy} & -U'_{yy} & -2\Omega & 0 \end{pmatrix} \begin{pmatrix} \delta x \\ \delta y \\ \delta \dot{x} \\ \delta \dot{y} \end{pmatrix} \tag{3.11}$$

It is this equation that we consider for the rest of the paper.

4 Stability of L$_4$ and L$_5$

The discovery of asteroids or other astrological bodies at the L$_4$ and L$_5$ points of almost every planet-sun system seems to indicate quite strongly that these points are indeed very stable. However, this is not always the case and requires that the heaviest mass, m_1 be significantly heavier than the second heaviest mass, m_2. We have the following theorem:

Theorem 4.1 *The Lagrange points, L$_4$ and L$_5$ are stable in all directions if and only if*

$$\frac{m_1}{m_2} \geq \frac{25 + 3\sqrt{69}}{2} \approx 24.9599 \tag{4.1}$$

Proof We start by using (2.7) to compute the partial derivatives found in (3.11). Note that for L$_4$ and L$_5$, $d_1 = d_2 = R$. Evaluating our partial double derivatives at L$_4$ and L$_5$ gives:

$$U'_{xx} = \left(\frac{Gm_1}{d_1^3} + \frac{Gm_2}{d_2^3} - \frac{3Gm_1\left(x + \frac{m_2 R}{m_1+m_2}\right)^2}{d_1^5} - \frac{3Gm_2\left(x - \frac{m_1 R}{m_1+m_2}\right)^2}{d_2^5} - \Omega^2 \right)\Bigg|_{x=\frac{R}{2}\left(\frac{m_1-m_2}{m_1+m_2}\right)}$$

$$= \frac{G(m_1+m_2)}{R^3} - \frac{3Gm_1\left(\frac{(m_1+m_2)R}{2(m_1+m_2)}\right)^2 + m_2\left(\frac{-(m_1+m_2)R}{2(m_1+m_2)}\right)^2}{R^5} - \Omega^2$$

$$= \frac{1}{4}\frac{G(m_1+m_2)}{R^3} - \Omega^2$$

$$U'_{yy} = \left(\frac{Gm_1}{d_1^3} + \frac{Gm_2}{d_2^3} - \frac{3Gm_1 y^2}{d_1^5} - \frac{3Gm_2 y^2}{d_2^5} - \Omega^2 \right)\Bigg|_{y=\pm\frac{\sqrt{3}}{2}R}$$

$$= \frac{G(m_1+m_2)}{R^3} - \frac{3Gm_1\frac{3}{4}R^2 + 3Gm_2\frac{3}{4}R^2}{R^5} - \Omega^2$$

$$= -\frac{5}{4}\frac{G(m_1+m_2)}{R^3} - \Omega^2$$

$$U'_{xy} = \left(-\frac{3Gm_1\left(x+\frac{m_2 R}{m_1+m_2}\right)y}{d_1^5} - \frac{3Gm_2\left(x-\frac{m_1 R}{m_1+m_2}\right)y}{d_2^5} \right)\Bigg|_{x=\frac{R}{2}\left(\frac{m_1-m_2}{m_1+m_2}\right),\, y=\pm\frac{\sqrt{3}}{2}R}$$

$$= \mp\frac{3Gm_1\left(\frac{R}{2}\right)\frac{\sqrt{3}}{2}R + 3Gm_2\left(-\frac{R}{2}\right)\frac{\sqrt{3}}{2}R}{R^5}$$

$$= \mp\frac{3\sqrt{3}G(m_1-m_2)}{4R^3} = -\frac{3\sqrt{3}}{4}\kappa_\pm\frac{G(m_1+m_2)}{R^3}$$

where $\kappa_\pm = \pm(m_1-m_2)/(m_1+m_2)$. Using (2.2) we get:

$$U'_{xx} = -\frac{3}{4}\Omega^2 \qquad U'_{yy} = -\frac{9}{4}\Omega^2 \qquad U'_{xy} = -\frac{3\sqrt{3}}{4}\kappa_\pm\Omega^2 \qquad (4.2)$$

Our matrix from (3.11) thus becomes

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{3}{4}\Omega^2 & \frac{3\sqrt{3}}{4}\kappa_\pm\Omega^2 & 0 & 2\Omega \\ \frac{3\sqrt{3}}{4}\kappa_\pm\Omega^2 & \frac{9}{4}\Omega^2 & -2\Omega & 0 \end{pmatrix} \qquad (4.3)$$

The matrix has the four following eigenvalues:

$$\lambda_\pm = \pm i\frac{\Omega}{2}\sqrt{2 - \sqrt{27\kappa_\pm^2 - 23}} \qquad \sigma_\pm = \pm i\frac{\Omega}{2}\sqrt{2 + \sqrt{27\kappa_\pm^2 - 23}} \qquad (4.4)$$

By corollary 1.2 and corollary 1.3, we have that all the eigenvalues must all be imaginary (otherwise there will necessarily be at least one eigenvalue with $Re(\lambda) > 0$, making the point unstable) so $\sqrt{2 - \sqrt{27\kappa_\pm^2 - 23}}$ must be completely real. Note that since $|\kappa_\pm| \leq 1$, $\sqrt{27\kappa_\pm^2 - 23} \leq 2$ so the only remaining condition is:

$$27\kappa_\pm^2 - 23 \geq 0 \qquad (4.5)$$

After a little algebraic manipulation this gives us

$$\frac{m_1}{m_2} \geq \frac{1 + \sqrt{\frac{23}{27}}}{1 - \sqrt{\frac{23}{27}}} = \frac{25 + 3\sqrt{69}}{2} \qquad (4.6)$$

which gives us the expected condition for the stability of the Lagrange points, L_4 and L_5. ∎

5 Conclusion

This result would in fact be completely unexpected and surprising were it not for the discovery of actual instances of astrological bodies at these points in our Solar system. Indeed, by looking at the second partial derivatives in (4.2) we find that $U_{xx} = U'_{xx} = -\frac{3}{4}\Omega^2 < 0$ and $U_{yy} = U'_{yy} = -\frac{9}{4}\Omega^2 < 0$. This would indicate that L_4 and L_5 are at peaks (local maxima) of the potential in the x-y plane and would thus imply that that these points are extremely unstable. What gives L_4 and L_5 their stability is simply the coriolis force discussed previously. Initially, a body at L_4 and L_5 start moving away from the equilibrium point but, as the body pick up speed, the coriolis force takes effect sending the body into an effective orbit around the Lagrange point.

Because of this effect, the areas around L_4 and L_5 that are effectively stable are in fact quite large as is illustrated by Figure 1. Indeed, this is the reason that so many trojan asteroids exist, some more than $5°$ off of $60°$, where the L_4 and L_5 are located. No satellites have been placed at these locations (unlike L_1 and L_2 hosting the SOHO satellite [7] and the WMAP satellite [6] respectively despite their inherent instability), however, they were visited in 2009 by the STEREO satellites. As the only stable Lagrange points, L_4 and L_5 are unique phenomena in the solar system and the areas around them have been and are of great interest (for example as possible places of origin of the moon or locations from which to better observe solar storms [5]) to the astrophysics community.

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