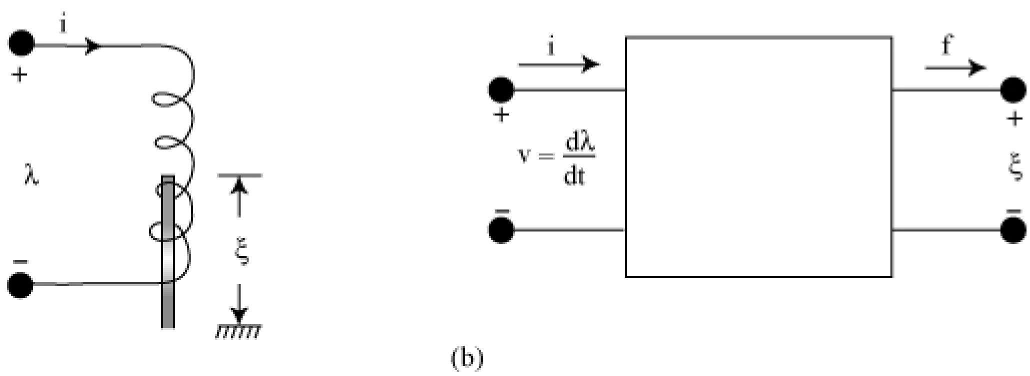
6.641, Electromagnetic Fields, Forces, and Motion
Prof. Markus Zahn
Lecture 13: Magnetoquasistatic Forces

I. MQS Energy Method of Forces



(a) Magnetoquasistatic system having one electrical terminal pair and one mechanical degree of freedom. (b) Schematic representation of MQS subsystem with coupling to external mechanical system represented by a mechanical terminal pair.

A. Circuit Approach

$$v = \frac{d\lambda}{dt} = \frac{d}{dt}\left[L(\xi)i\right] = L(\xi)\frac{di}{dt} + i\frac{dL(\xi)}{dt}$$

$$p = vi = L(\xi)i\frac{di}{dt} + i^2\frac{dL(\xi)}{dt}$$

$$= L(\xi)\frac{d}{dt}\left(\frac{1}{2}i^2\right) + i^2\frac{dL(\xi)}{dt}$$

$$= \frac{d}{dt}\left[\frac{1}{2}L(\xi)i^2\right] + \frac{1}{2}i^2\frac{dL(\xi)}{dt}$$

$$= \frac{d}{dt}\left[\frac{1}{2}L(\xi)i^2\right] + \frac{1}{2}i^2\frac{dL(\xi)}{d\xi}\frac{d\xi}{dt}$$

$$v\,i = \frac{dW_m}{dt} + f_\xi\frac{d\xi}{dt} \Rightarrow W_m = \frac{1}{2}L(\xi)i^2, \quad f_\xi = \frac{1}{2}i^2\frac{dL(\xi)}{d\xi}$$

$$\lambda = L(\xi)i \Rightarrow f_\xi = \frac{1}{2}i^2\frac{dL(\xi)}{d\xi}$$

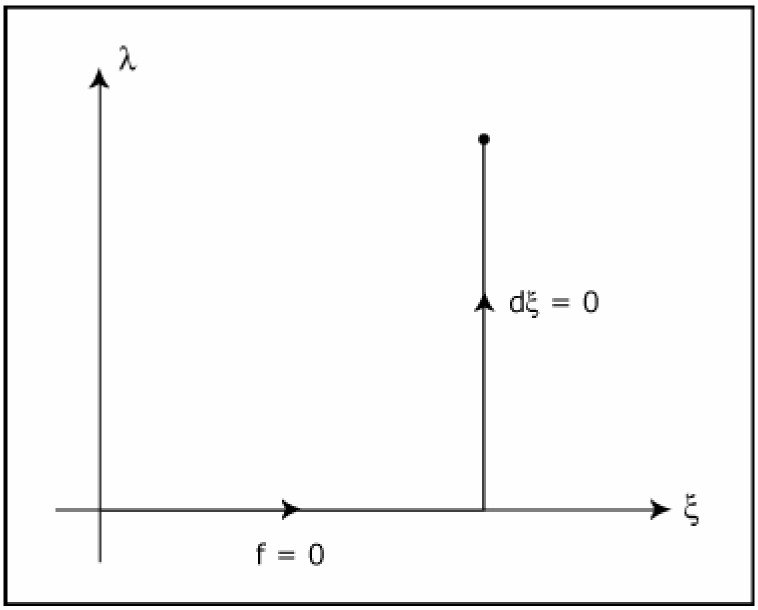
$$= \frac{1}{2}\frac{\lambda^2}{L^2(\xi)}\frac{dL(\xi)}{d\xi}$$

$$= -\frac{1}{2}\lambda^2 \frac{d}{d\xi}\left[1/L(\xi)\right]$$

B. Energy Method

$$vi = i\frac{d\lambda}{dt} = \frac{dW_m}{dt} + f_\xi \frac{d\xi}{dt} \Rightarrow dW_m = i\,d\lambda - f_\xi\, d\xi$$

$$f_\xi = -\frac{\partial W_m}{\partial \xi}\bigg|_{\lambda = \text{constant}} \quad , \quad i = \frac{\partial W_m}{\partial \lambda}\bigg|_{\xi=\text{constant}}$$



$$W_m = \int_{\lambda=0} -\cancelto{0}{f_\xi}\, d\xi + \int_{\xi=\text{constant}} i\, d\lambda$$

$$i = \frac{\lambda}{L(\xi)}$$

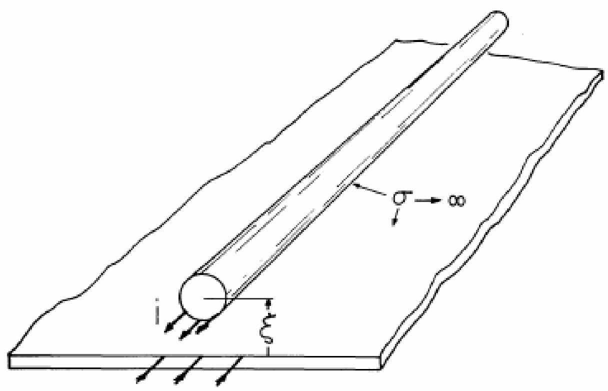
$$W_m = \int_{\xi=\text{constant}} \frac{\lambda}{L(\xi)}\, d\lambda = \frac{\lambda^2}{2L(\xi)}$$

$$f_\xi = \frac{-\partial W_m}{\partial \xi}\bigg|_{\lambda=\text{constant}} = -\frac{1}{2}\lambda^2 \frac{d}{d\xi}\left(1/L(\xi)\right)$$

$$= -\frac{1}{2}\lambda^2\left(-\frac{1}{L^2(\xi)}\right)\frac{dL(\xi)}{d\xi}$$

$$= \frac{1}{2}i^2\frac{dL(\xi)}{d\xi}$$

II. Force on a Wire over a Perfectly Conducting Plane



Depth D

Figure 11.7.3 Cross-section of perfectly conducting current-carrying wire over a perfectly conducting ground plane.

Courtesy of Hermann A. Haus and James R. Melcher. Used with permission.

$$L(\xi) = \frac{\mu_0 D}{2\pi}\ln\left[\frac{\xi}{R} + \sqrt{\left(\frac{\xi}{R}\right)^2 - 1}\right]$$

[See Haus & Melcher p. 343, take ½ of Eq. (12) which is the inductance between 2 cylinders]

A. Energy Method

$$f_\xi = \frac{1}{2}i^2\frac{dL(\xi)}{d\xi} = \frac{\mu_0 i^2 D}{4\pi R}\frac{1}{\sqrt{\left(\xi/R\right)^2 - 1}}$$

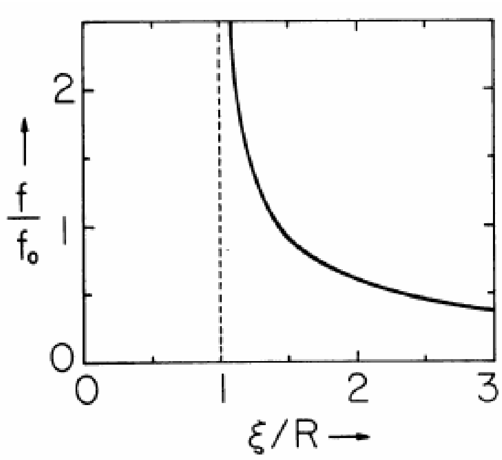
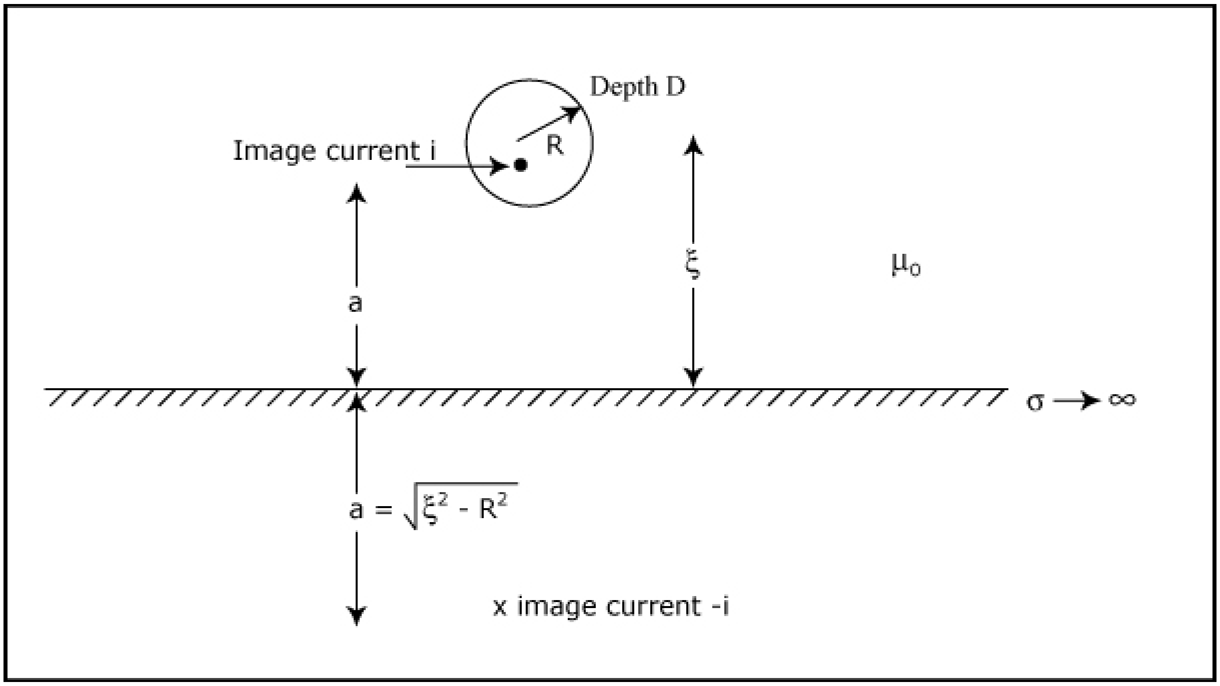


Figure 11.7.4 The force tending to levitate the wire of Figure 11.7.3 as a function of the distance to the ground plane normalized to the radius R of wire.

Courtesy of Hermann A. Haus and James R. Melcher. Used with permission.

B. Method of Images Approach with Lorentz Force



$$f_\xi = iD\left(\frac{\mu_0 i}{2\pi(2a)}\right) = \frac{\mu_0 i^2 D}{4\pi a} = \frac{\mu_0 i^2 D}{4\pi\sqrt{\xi^2 - R^2}}$$

C. Demonstration: Steady State Magnetic Leviation

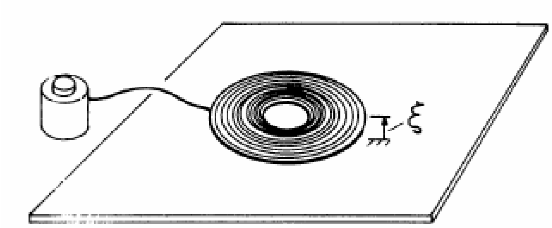


Figure 11.7.5 When the pancake coil is driven by an ac current, it floats above the aluminum plate. In this experiment, the coil consists of 250 turns of No. 10 aluminum wire with an outer radius of 16 cm and an inner one of 2.5 cm. The aluminum sheet has a thickness of 1.3 cm. With a 60 Hz current i of about 20 amp rms, the height above the plate is 2 cm.

III. One Turn Loop

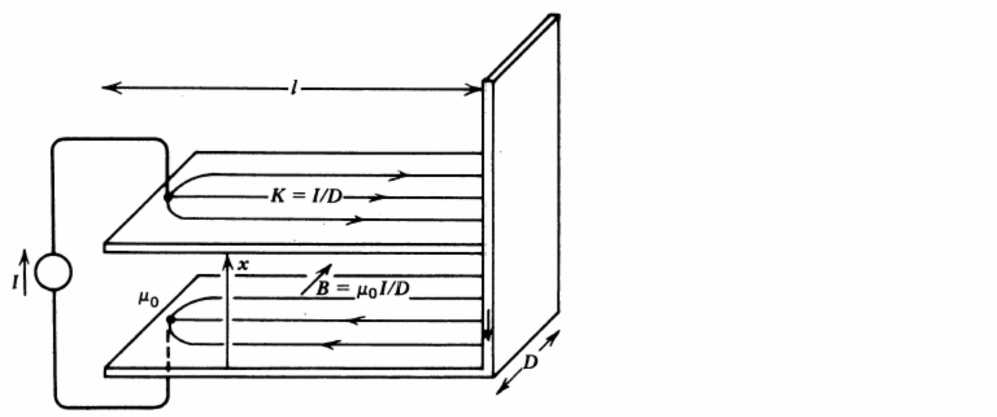


Figure 6-35 The magnetic force on a current-carrying loop tends to expand the loop.

$$H_z = \frac{I}{D}\ , \quad \Phi = \mu_0 H_z x l\ , \quad L(x) = \Phi/I = \frac{\mu_0 x l}{D}$$

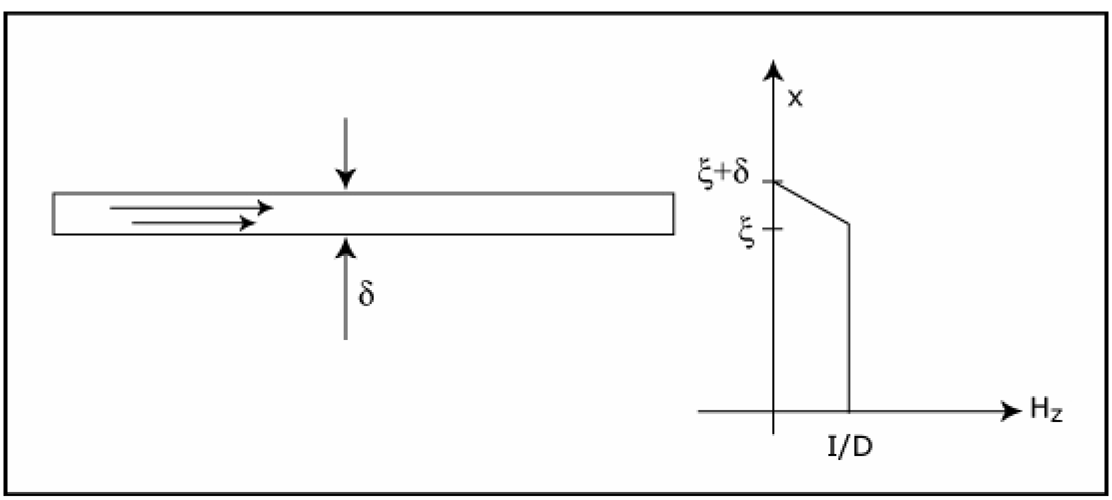
$$= \frac{\mu_0 x l}{D} I$$

A. Energy Method

$$f_x = \frac{1}{2} I^2 \frac{dL(x)}{dx} = \frac{1}{2} I^2 \frac{\mu_0 l}{D}$$

B. Lorentz Force Law

$$\bar{f} = \int_V \bar{J} \times \bar{B}\ dV$$



Model surface current $K_y = \frac{I}{D}$ as volume current of small thickness δ

$$J_y = \frac{I}{D\delta}$$

$$\nabla \times \bar{H} = \bar{J} \Rightarrow \frac{\partial H_z}{\partial x} = -J_y = -\frac{I}{D\delta} \Rightarrow H_z = -\frac{I}{D\delta}\left(x - (\xi + \delta)\right)$$

$$f_x = \int_V J_y \, \mu_0 \, H_z \, dx \, dy \, dz$$

$$= \int_{x=\xi}^{\xi+\delta} \frac{I}{D\delta}\left(\frac{-\mu_0 I}{\not{D}\delta}\right)\left(x - (\xi+\delta)\right) l \, \not{D} \, dx$$

$$= \frac{-\mu_0 I^2 l}{D\delta^2}\left[\frac{x^2}{2} - (\xi+\delta)x\right]\Bigg|_{x=\xi}^{\xi+\delta}$$

$$= \frac{-\mu_0 I^2 l}{D\delta^2}\left[\frac{(\xi+\delta)^2}{2} - \frac{\xi^2}{2} - (\xi+\delta)^2 + \xi(\xi+\delta)\right]$$

$$= \frac{-\mu_0 I^2 l}{D\delta^2}\left[-\frac{1}{2}(\xi+\delta)^2 + \frac{\xi^2}{2} + \xi\delta\right]$$

$$= \frac{-\mu_0 I^2 l}{D\delta^2}\left[-\frac{1}{2}\delta^2\right]$$

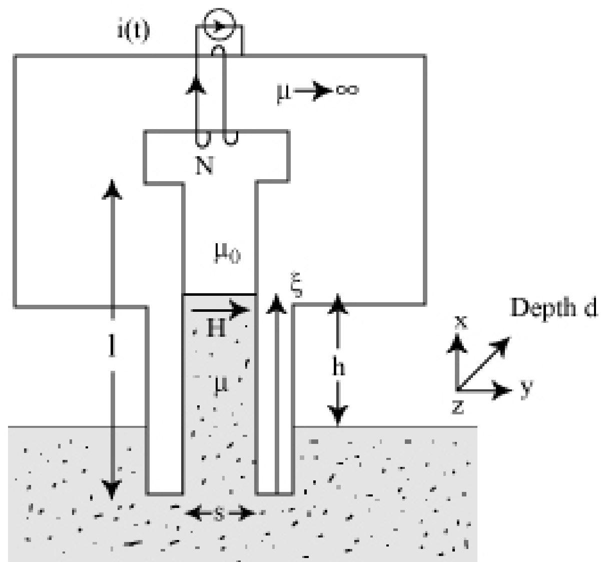
$$= +\frac{1}{2}\frac{\mu_0 I^2 l}{D} \Rightarrow \bar{f} = \int_S \frac{1}{2}\bar{K} \times \bar{B} \, dS$$

½ comes from integrating uniform volume current over small thickness δ

General formula: $\bar{f} = \int_S \bar{K} \times \bar{B}_{av} \, dS$

For our case: $B_{av} = \frac{\cancel{B_{metal}}^{\,0} + B_{air}}{2} = \frac{1}{2} B_{air}$

IV. Lifting of Magnetic Fluid



A magnetizable liquid is pushed upward into the field region between the pole faces by the forces on magnetic dipoles in the fringing region at the bottom.

A. Energy Method Approach

$$H = \frac{Ni}{s}$$

$$\Phi = H\left[\mu\xi + \mu_0\left(l - \xi\right)\right]d$$

$$= \frac{Nd}{s}\left[\mu\xi + \mu_0\left(l - \xi\right)\right]i$$

$$\lambda = N\Phi = \frac{N^2 d}{s}\left[\mu\xi + \mu_0\left(l - \xi\right)\right]i$$

$$L\left(\xi\right) = \frac{\lambda}{i} = \frac{N^2 d}{s}\left[\mu\xi + \mu_0\left(l - \xi\right)\right]$$

$$f_\xi = \frac{1}{2} i^2 \frac{dL}{d\xi} = \frac{1}{2}\frac{N^2 i^2 d}{s}\left(\mu - \mu_0\right)$$

$$f_\xi = \rho_m g h d s = \frac{1}{2}\frac{N^2 i^2 d}{s}\left(\mu - \mu_0\right)$$

$$h = \frac{1}{2}\frac{N^2 i^2 \not{d}}{s^2 \rho_m g \not{d}}\left(\mu - \mu_0\right)$$

B. Magnetization force

$$F_x = \mu_0 \left(\overline{M} \bullet \nabla\right) H_x$$

$$= \mu_0 \left[M_x \frac{\partial H_x}{\partial x} + M_y \frac{\partial H_x}{\partial y} + \underbrace{M_z \cancel{\frac{\partial H_x}{\partial z}}}_{\frac{\partial}{\partial z} = 0} \right]$$

$$\nabla \times \overline{H} = \overline{J} = 0 \Rightarrow \frac{\partial H_x}{\partial y} = \frac{\partial H_y}{\partial x}$$

$$F_x = \mu_0 \left[M_x \frac{\partial H_x}{\partial x} + M_y \frac{\partial H_y}{\partial x} \right]$$

$$\overline{B} = \mu \overline{H} = \mu_0 \left(\overline{H} + \overline{M}\right) \Rightarrow \overline{M} = \left(\frac{\mu}{\mu_0} - 1\right) \overline{H}$$

$$F_x = \mu_0 \left[\left(\frac{\mu}{\mu_0} - 1\right) H_x \frac{\partial H_x}{\partial x} + \left(\frac{\mu}{\mu_0} - 1\right) H_y \frac{\partial H_y}{\partial x} \right]$$

$$= \mu_0 \left(\frac{\mu}{\mu_0} - 1\right) \frac{\partial}{\partial x} \left[\frac{1}{2} \left(H_x^{\ 2} + H_y^{\ 2}\right) \right]$$

$$f_x = \int F_x \, dx \, dy \, dz$$

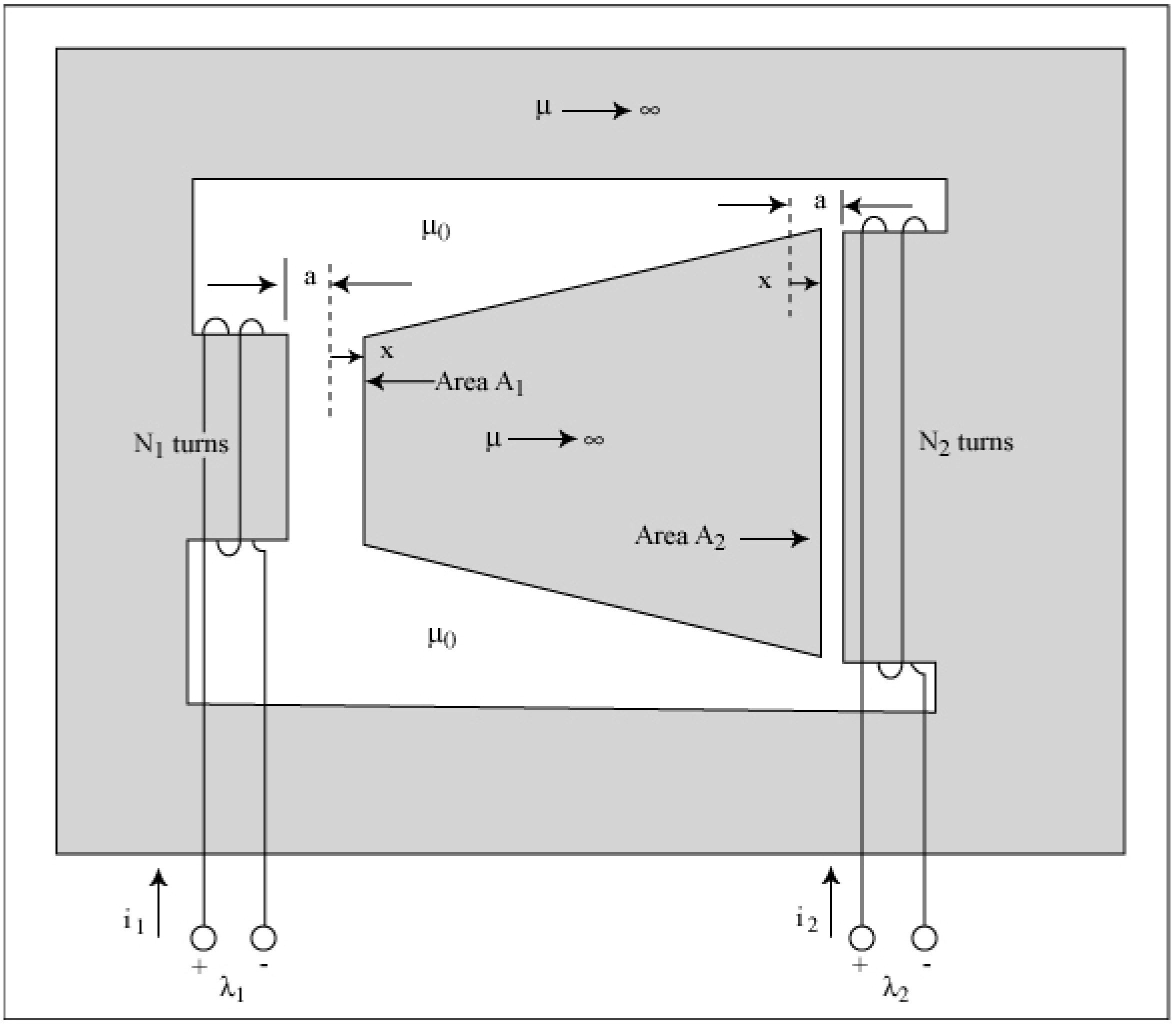
$$= \frac{\left(\mu - \mu_0\right)}{2} \int_{x=-\infty}^{h} \int_{y=0}^{s} \int_{z=0}^{d} \frac{\partial}{\partial x} \left(H_x^{\ 2} + H_y^{\ 2}\right) dx \, dy \, dz$$

$$= \frac{\left(\mu - \mu_0\right)}{2} ds \left(H_x^{\ 2} + H_y^{\ 2}\right) \Big|_{x=-\infty}^{h}$$

$$= \frac{\left(\mu - \mu_0\right)}{2} d\cancel{s} \frac{N^2 i^2}{s^{\cancel{2}}}$$

$$= \frac{1}{2} \left(\mu - \mu_0\right) d \frac{N^2 i^2}{s}$$

V. Magnetic Actuator



$$\oint_C \overline{H} \cdot d\overline{s} = H_1 (x + a) + H_2 (a - x) = N_1 i_1 + N_2 i_2$$

$$\mu_0 H_1 A_1 = \mu_0 H_2 A_2 \Rightarrow H_1 = \frac{H_2 A_2}{A_1}$$

$$H_2 \left[(a - x) + (a + x) \frac{A_2}{A_1} \right] = N_1 i_1 + N_2 i_2$$

$$H_2 = \frac{(N_1 i_1 + N_2 i_2) A_1}{A_1 (a - x) + (a + x) A_2}$$

$$H_1 = \frac{(N_1 i_1 + N_2 i_2) A_2}{A_1 (a - x) + (a + x) A_2}$$

$$\lambda_1 = N_1 \mu_0 H_1 A_1 = \frac{\mu_0 N_1 A_1 A_2 (N_1 i_1 + N_2 i_2)}{A_1 (a - x) + (a + x) A_2}$$

$$\lambda_2 = N_2 \mu_0 H_2 A_2 = \frac{\mu_0 N_2 A_1 A_2 (N_1 i_1 + N_2 i_2)}{A_1 (a - x) + (a + x) A_2}$$

$$\lambda_1 = L_1(x) i_1 + M(x) i_2$$

$$\lambda_2 = M(x) i_1 + L_2(x) i_2$$

$$L_1(x) = \frac{\mu_0 A_1 A_2 N_1^2}{A_1 (a - x) + (a + x) A_2} \; ; \quad L_2(x) = \frac{\mu_0 A_1 A_2 N_2^2}{A_1 (a - x) + (a + x) A_2} \; ; \quad M(x) = \frac{\mu_0 A_1 A_2 N_1 N_2}{A_1 (a - x) + (a + x) A_2}$$

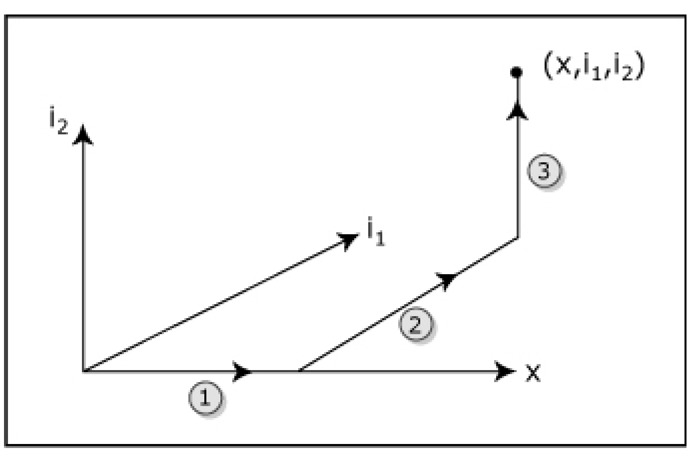
$$= \sqrt{L_1(x) L_2(x)}$$

$$dw = i_1 \, d\lambda_1 + i_2 \, d\lambda_2 - f \, dx$$

$$d\underbrace{(i_1 \lambda_1 + i_2 \lambda_2 - w)}_{w' \text{ (co–energy)}} = \lambda_1 \, di_1 + \lambda_2 \, di_2 + f \, dx$$

$$dw' = \lambda_1 \, di_1 + \lambda_2 \, di_2 + f \, dx$$

$$f = + \left. \frac{\partial w'}{\partial x} \right|_{i_1, i_2 \text{ constant}}$$



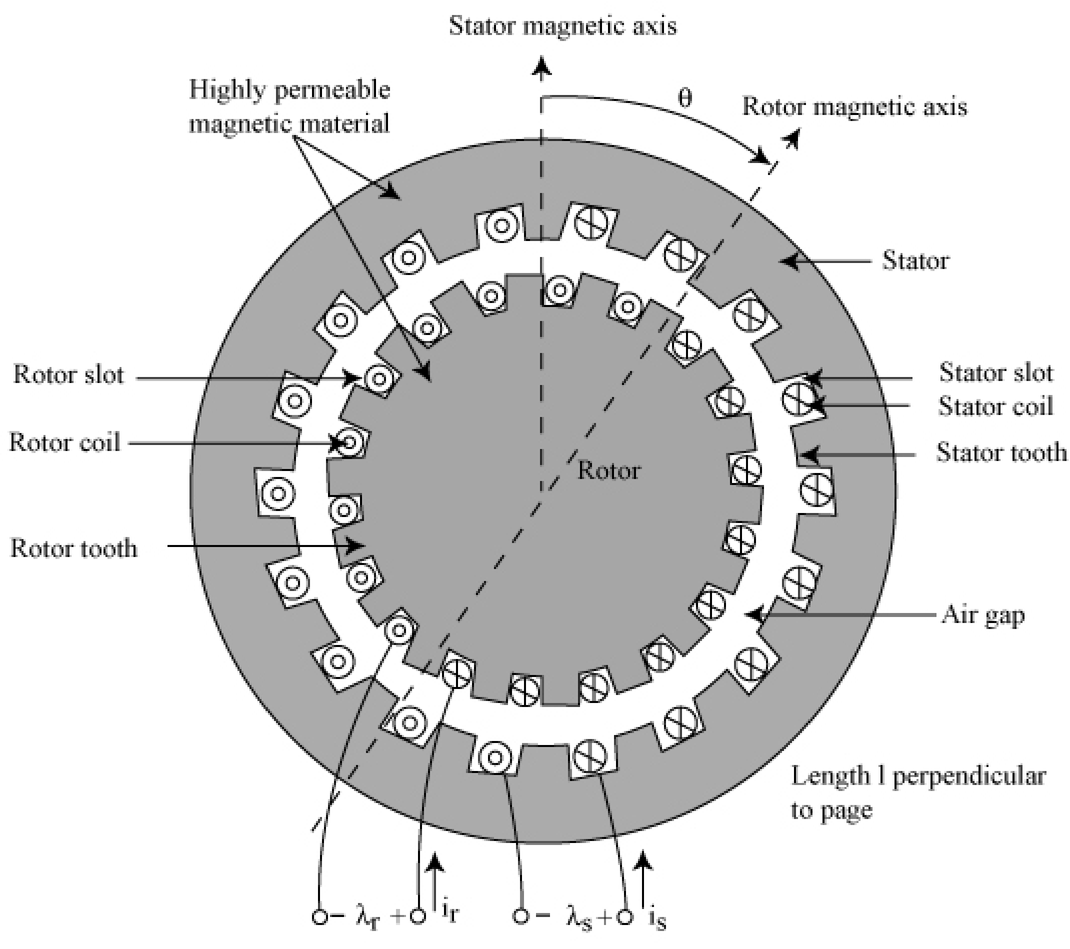
$$dw' = \int_{\substack{1 \\ i_1=i_2=0}} f\,dx + \int_{\substack{2 \\ i_2=0 \\ x=\text{constant}}} \lambda_1\, di_1 + \int_{\substack{3 \\ i_1=\text{constant} \\ x=\text{constant}}} \lambda_2\, di_2$$

$$dw' = \int_{\substack{i_2=0 \\ x=\text{constant}}} L_1(x) i_1\, di_1 + \int_{\substack{i_1=\text{constant} \\ x=\text{constant}}} \left(M(x) i_1 + L_2(x) i_2\right) di_2$$

$$= \frac{1}{2} L_1(x) i_1^2 + M(x) i_1 i_2 + \frac{1}{2} L_2(x) i_2^2$$

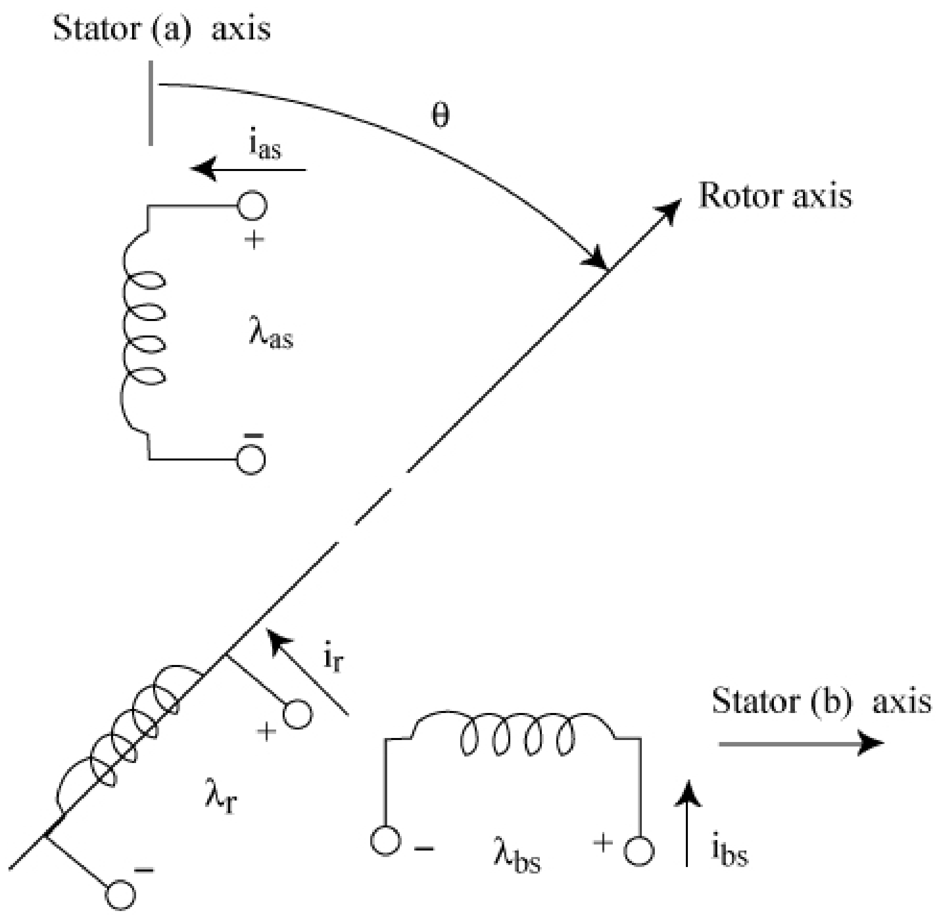
$$f = +\left.\frac{\partial w'}{\partial x}\right|_{i_1, i_2} = \frac{1}{2} i_1^2 \frac{dL_1}{dx} + \frac{1}{2} i_2^2 \frac{dL_2}{dx} + i_1 i_2 \frac{dM}{dx}$$

VI. Synchronous Machine



Geometry of smooth-air-gap rotating machine showing distributed windings on stator and rotor of a single-phase machine.

Smooth- Air-Gap Machines



Schematic representation of smooth-air-gap synchronous machine with field (dc) winding on the rotor and balanced two-phase stator (armature) winding.

$$\lambda_{as} = L_s i_{as} + M i_r \cos\theta$$

$$\lambda_{bs} = L_s i_{bs} + M i_r \sin\theta$$

$$\lambda_r = L_r i_r + M\left(i_{as} \sin\theta + i_{bs} \sin\theta\right)$$

$$dw = i_{as}\, d\lambda_{as} + i_{bs}\, d\lambda_{bs} + i_r\, d\lambda_r - T^e\, d\theta$$

$$d\left(w - i_{as}\lambda_{as} - i_{bs}\lambda_{bs} - i_r\lambda_r\right) = -dw'$$

$$w' = i_{as}\lambda_{as} + i_{bs}\lambda_{bs} + i_r\lambda_r - w \qquad \text{co–energy}$$

$$dw' = \lambda_{as}\, di_{as} + \lambda_{bs}\, di_{bs} + \lambda_r\, di_r + T^e\, d\theta$$

$$w' = \int\limits_{\substack{i_{as}=0\\ i_{bs}=0\\ i_r=0}} T\,d\theta + \int\limits_{\substack{\theta=\text{constant}\\ i_{bs}=0\\ i_r=0}} \lambda_{as}\,di_{as} + \int\limits_{\substack{\theta=\text{constant}\\ i_{as}=\text{constant}\\ i_r=0}} \lambda_{bs}\,di_{bs} + \int\limits_{\substack{\theta=\text{constant}\\ i_{as}=\text{constant}\\ i_{bs}=\text{constant}}} \lambda_r\,di_r$$

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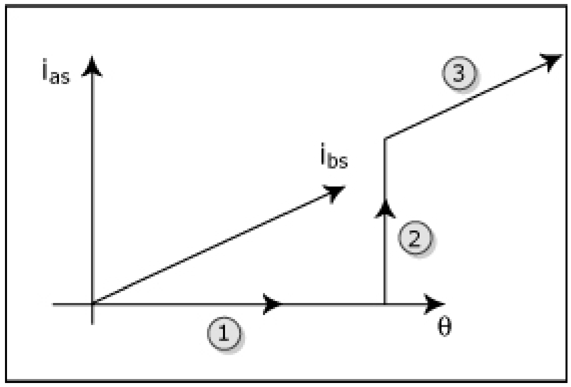

$$w' = \int\limits_{\substack{i_{as}=0\\ i_{bs}=0\\ i_r=0}} \overset{0}{\cancel{T}}\,d\theta + \int L_s\,i_{as}\,di_{as} + \int L_s\,i_{bs}\,di_{bs} + \int\limits_{\substack{\theta=\text{constant}\\ i_{as}=\text{constant}\\ i_{bs}=\text{constant}}} \left[L_r i_r + M\left(i_{as}\cos\theta + i_{bs}\sin\theta\right)\right] di_r$$

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$$w' = \frac{1}{2}L_s\,i_{as}^{\,2} + \frac{1}{2}L_s\,i_{bs}^{\,2} + \frac{1}{2}L_r\,i_r^{\,2} + M i_r\left(i_{as}\cos\theta + i_{bs}\sin\theta\right)$$

$$T^e = +\left.\frac{\partial w'}{\partial\theta}\right|_{i_{as},\,i_{bs},\,i_r} = M i_r\left(-i_{as}\sin\theta + i_{bs}\cos\theta\right)$$

Balanced 2 phase currents

$$i_{as} = I_s\cos\omega t,\ \ i_{bs} = I_s\sin\omega t,\ \ i_r = I_r,\ \ \theta = \omega_m t + \gamma$$

$$T^e = M I_r\,I_s\left(-\cos\omega t\ \sin\theta + \sin\omega t\ \cos\theta\right) = M I_r\,I_s\sin\left(\omega t - \theta\right)$$
$$= M I_r\,I_s\sin\left(\left(\omega - \omega_m\right)t - \gamma\right)$$

$$< T^e > \neq 0 \Rightarrow \omega = \omega_m$$

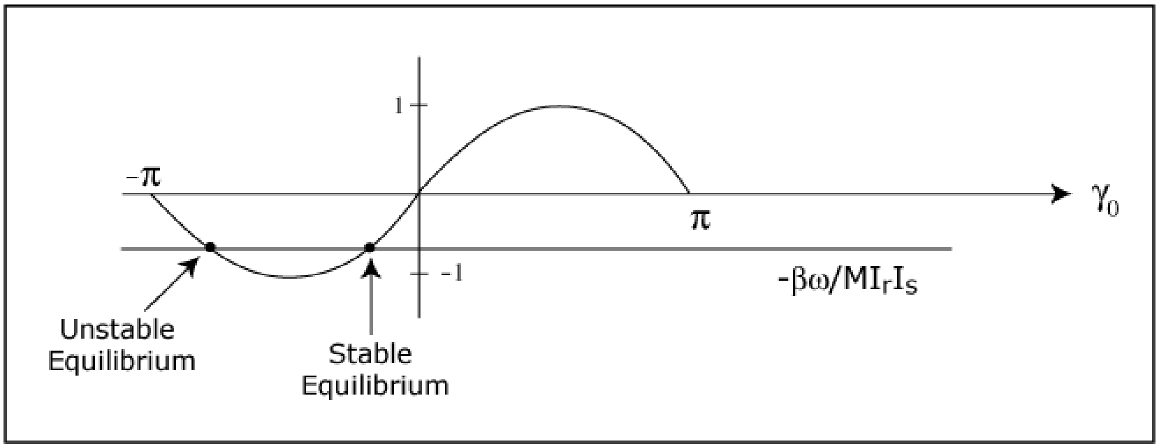
$$T^e = -M I_r\,I_s\sin\gamma$$

$$J\frac{d^2\theta}{dt^2} = T^e - \beta\frac{d\theta}{dt}$$

$$\theta = \omega_m t + \gamma_0 + \gamma'(t) \quad , \quad \gamma'(t) << \gamma_0$$

$$-M I_r I_s \sin\gamma_0 - \beta\omega = 0$$

$$\sin\gamma_0 = -\frac{\beta\omega}{M I_r I_s}$$



Pullout when $\left|\sin\gamma_0\right| = 1 \Rightarrow \beta\omega = M I_r I_s$

Hunting transients: $\sin(\gamma_0 + \gamma') \approx \sin\gamma_0 \cos\gamma' + \cos\gamma_0 \sin\gamma' \approx \sin\gamma_0 + \gamma'\cos\gamma_0$

$$J\frac{d^2\gamma'}{dt^2} = -M I_r I_s \cos\gamma_0 \gamma' - \beta\gamma' = -\left(M I_r I_s \cos\gamma_0 + \beta\right)\gamma'$$

$$\frac{d^2\gamma'}{dt^2} + \omega_0^{\ 2}\gamma' = 0 \quad ; \quad \omega_0^{\ 2} = \left[M I_r I_s \cos\gamma_0 + \beta\right]/J$$

$$\gamma' = A_1 \sin\omega_0 t + A_2 \cos\omega_0 t$$

Stable if $\omega_0^{\ 2} > 0$ (ω_0 real)

Unstable if $\omega_0^{\ 2} < 0$ (ω_0 imaginary)