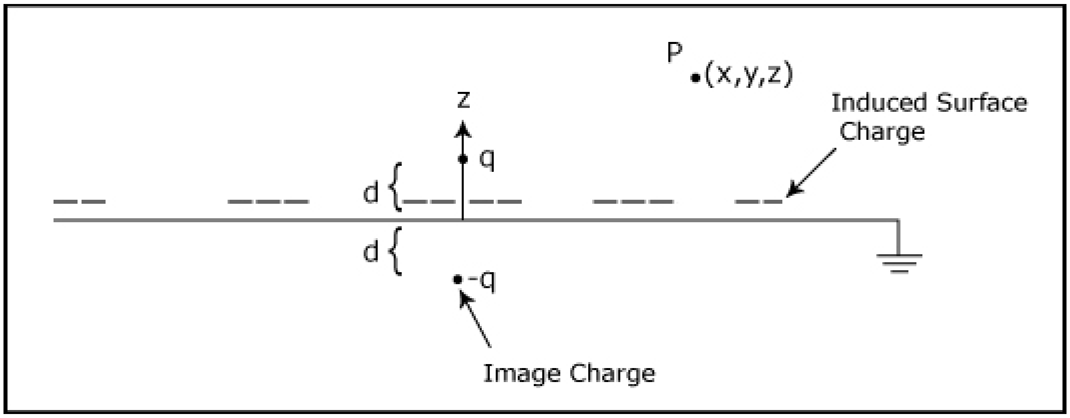
6.641, Electromagnetic Fields, Forces, and Motion
Prof. Markus Zahn
Lecture 5: Method of Images

I. Point Charge Above Ground Plane

1. Potential and Electric Field



$$\Phi_p = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{1}{\sqrt{x^2+y^2+(z+d)^2}}\right]$$

$$\bar{E}_p = -\nabla\Phi_p = -\left[\frac{\partial\Phi_p}{\partial x}\bar{i}_x + \frac{\partial\Phi_p}{\partial y}\bar{i}_y + \frac{\partial\Phi_p}{\partial z}\bar{i}_z\right]$$

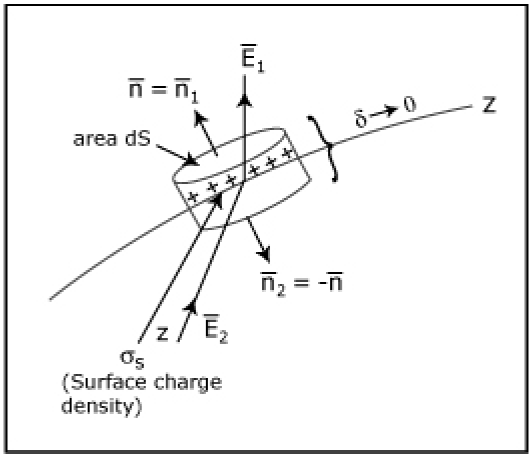
$$= \frac{q}{4\pi\varepsilon_0}\left[\frac{\cancel{2}\left(x\,\bar{i}_x + y\,\bar{i}_y + (z-d)\,\bar{i}_z\right)}{\cancel{2}\left[x^2+y^2+(z-d)^2\right]^{3/2}} - \frac{\cancel{2}\left(x\,\bar{i}_x + y\,\bar{i}_y + (z+d)\,\bar{i}_z\right)}{\cancel{2}\left[x^2+y^2+(z+d)^2\right]^{3/2}}\right]$$

$$\bar{E}_p(z=0) = \frac{q}{2\pi\varepsilon_0}\frac{(-d)}{\left[x^2+y^2+d^2\right]^{3/2}}\bar{i}_z$$

(perpendicular to equipotential ground plane)

2. Gauss's Law Boundary Condition

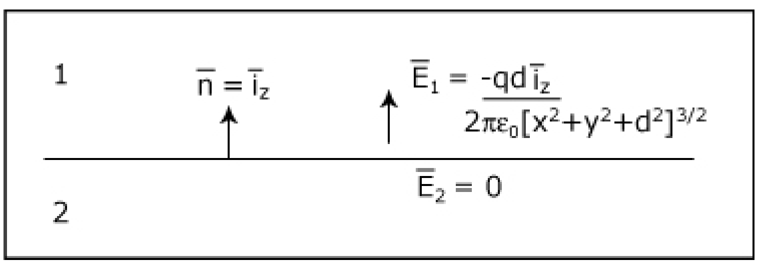
$$\oint_S \varepsilon_0 \bar{E} \cdot \overline{da} = \int_V \rho dV$$



$$\oint_S \varepsilon_0 \bar{E} \cdot \overline{da} = \left(\varepsilon_0 \bar{E}_1 \cdot \bar{n}_1 + \varepsilon_0 \bar{E}_2 \cdot \bar{n}_2\right) dS = \sigma_s dS \quad \text{(total charge inside pillbox)}$$

$$\sigma_s = \varepsilon_0 \bar{n} \cdot \left[\bar{E}_1 - \bar{E}_2\right]$$

3. Back to Point Charge Above Ground Plane



At z=0:

$$\sigma_s = \varepsilon_0 \bar{n} \cdot \left[\bar{E}_1 - \bar{E}_2\right] = \varepsilon_0\, \bar{i}_z \cdot \bar{E}_1 = \varepsilon_0 E_z = \frac{-qd}{2\pi\left[x^2 + y^2 + d^2\right]^{3/2}} = \frac{-qd}{2\pi\left[r^2 + d^2\right]^{3/2}}$$

$$r^2 = x^2 + y^2$$

$$q_T(z=0) = \int_{y=-\infty}^{+\infty} \int_{x=-\infty}^{+\infty} \sigma_s dx dy = \int_{r=0}^{\infty} \int_{\phi=0}^{2\pi} \sigma_s r dr d\phi = \frac{-qd}{\cancel{2}\cancel{\pi}}\left(\cancel{2}\cancel{\pi}\right) \int_{r=0}^{\infty} \frac{r dr}{\left[r^2 + d^2\right]^{3/2}}$$

$$u = r^2 + d^2 \Rightarrow du = 2rdr$$

$$\int \frac{rdr}{\left[r^2 + d^2\right]^{3/2}} = \int \frac{du}{2u^{3/2}} = -u^{-1/2} = -\frac{1}{\sqrt{r^2 + d^2}}$$

$$q_T\left(z = 0\right) = \left.\frac{+qd}{\sqrt{r^2 + d^2}}\right|_0^\infty = -q$$

$$\bar{f}_q = \frac{-q^2}{4\pi\varepsilon_0\left(2d\right)^2}\,\bar{i}_z = \frac{-q^2}{16\pi\varepsilon_0 d^2}$$

II. Point Charge and Sphere

1. Grounded Sphere

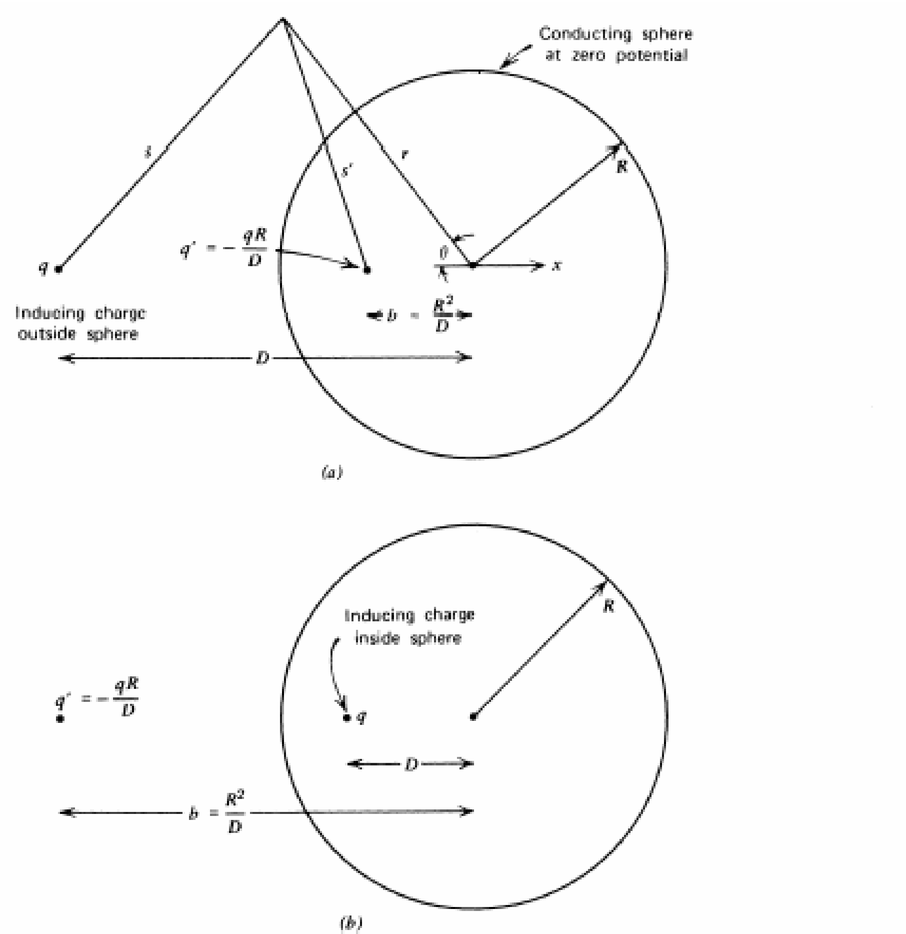


Figure 2-27 (*a*) The field due to a point charge q, a distance D outside a conducting sphere of radius R, can be found by placing a single image charge $-qR/D$ at a distance $b = R^2/D$ from the center of the sphere. (*b*) The same relations hold true if the charge q is inside the sphere but now the image charge is outside the sphere, since $D < R$.

$$\Phi = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{s} + \frac{q'}{s'}\right)$$

$$s = \left[r^2 + D^2 - 2rD\cos\theta\right]^{1/2},\ s' = \left[b^2 + r^2 - 2rb\cos\theta\right]^{1/2}$$

$$\Phi(r = R) = 0 \Rightarrow \frac{q}{s} = \frac{-q'}{s'} \Rightarrow \left(\frac{q}{s}\right)^2 = \left(\frac{q'}{s'}\right)^2$$

$$q^2 s'^2 = q'^2 s^2 \Rightarrow q'^2\left[R^2 + D^2 - 2RD\cos\theta\right] = q^2\left[b^2 + R^2 - 2Rb\cos\theta\right]$$

$$q'^2\left(R^2 + D^2\right) = q^2\left(b^2 + R^2\right)$$

$$+q'^2\,\cancel{2}\,\cancel{R}D\,\cancel{\cos\theta} = +q^2\,\cancel{2}\,\cancel{R}b\,\cancel{\cos\theta} \Rightarrow \frac{q'^2}{q^2} = \frac{b}{D}$$

$$\frac{b}{D}\left(R^2 + D^2\right) = b^2 + R^2 \Rightarrow b^2 - b\left(\frac{R^2}{D} + D\right) + R^2 = 0$$

$$(b - D)\left(b - \frac{R^2}{D}\right) = 0$$

$$b = \frac{R^2}{D}$$

$$q'^2 = q^2\frac{b}{D} = q^2\frac{R^2}{D^2} \Rightarrow q' = -qR/D$$

force on sphere

$$f_x = \frac{qq'}{4\pi\varepsilon_0(D - b)^2} = \frac{-q^2R/D}{4\pi\varepsilon_0\left(D - \frac{R^2}{D}\right)^2} = \frac{-q^2RD}{4\pi\varepsilon_0\left(D^2 - R^2\right)^2}$$

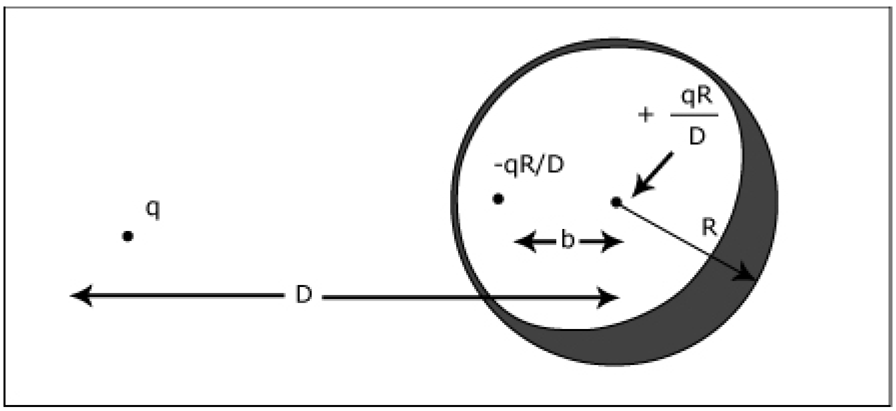
2. Isolated Sphere [Put additional Image Charge $+q' = +qR/D$ at center] (zero charge)

$$\Phi(r = R) = \frac{q'}{4\pi\varepsilon_0 R} = \frac{q}{4\pi\varepsilon_0 D}$$

force on sphere

$$f_x = \frac{q}{4\pi\varepsilon_0}\left[\frac{q'}{(D-b)^2} - \frac{q'}{D^2}\right] = \frac{qq'\left[D^2 - (D-b)^2\right]}{4\pi\varepsilon_0 D^2 (D-b)^2} = \frac{-q^2 R\left[2bD - b^2\right]}{4\pi\varepsilon_0 D^3\left(D - \frac{R^2}{D}\right)^2}$$

$$f_x = \frac{-q^2 R D^2}{4\pi\varepsilon_0 D^3\left(D^2 - R^2\right)^2}\frac{R^2}{D}\left[2D - \frac{R^2}{D}\right] = \frac{-q^2 R^3}{4\pi\varepsilon_0 D^3\left(D^2 - R^2\right)^2}\left[2D^2 - R^2\right]$$



III. Demonstration 4.7.1 – Charge Induced in Ground Plane by Overhead Conductor

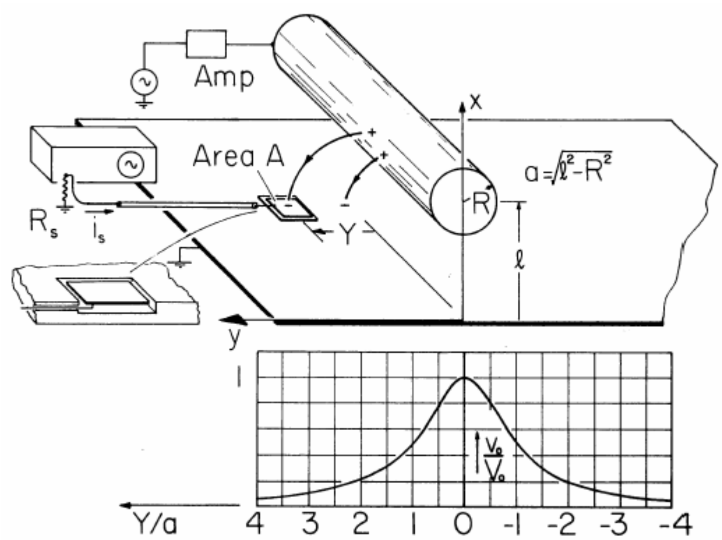
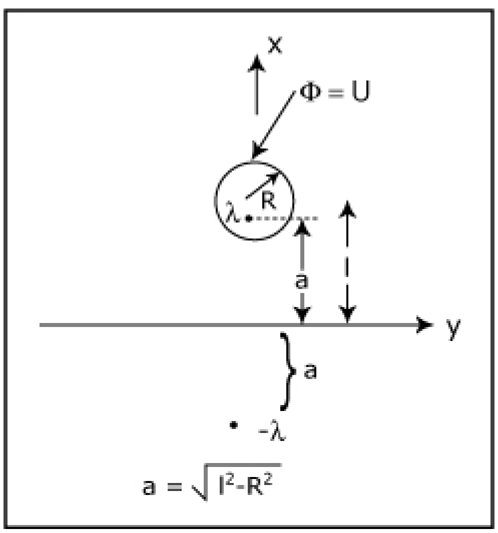


Figure 4.7.2 Charge induced on ground plane by overhead conductor is measured by probe. Distribution shown is predicted by (4.7.7).



$$\Phi = \frac{-\lambda}{2\pi\varepsilon_0} \ln \frac{\left[(a-x)^2 + y^2\right]^{1/2}}{\left[(a+x)^2 + y^2\right]^{1/2}} = \frac{-\lambda}{4\pi\varepsilon_0} \ln\left[\frac{(a-x)^2 + y^2}{(a+x)^2 + y^2}\right]$$

$$C' = \frac{\lambda}{\Phi(x = l - R, y = 0)} = \frac{\lambda}{\frac{-\lambda}{2\pi\varepsilon_0} \ln \frac{a - l + R}{a + l - R}} = \frac{2\pi\varepsilon_0}{\ln\left[\frac{\sqrt{l^2 - R^2} + l}{R}\right]}, \quad \Phi(x = l - R, y = 0) = U$$

$$\sigma_s = \varepsilon_0 E_x (x=0) = -\varepsilon_0 \left.\frac{\partial \Phi}{\partial x}\right|_{x=0}$$

$$= \frac{+\cancel{\varepsilon_0}\lambda}{4\pi \cancel{\varepsilon_0}} \frac{d}{dx}\left[\ln\left[(a-x)^2 + y^2\right] - \ln\left[(a+x)^2 + y^2\right]\right]$$

$$= \frac{\lambda}{4\pi}\left.\left[\frac{-2(a-x)}{(a-x)^2 + y^2} - \frac{2(a+x)}{(a+x)^2 + y^2}\right]\right|_{x=0}$$

$$= \frac{-\lambda a}{\pi\left(a^2 + y^2\right)}$$

Total Charge per unit length on ground plane is:

$$\lambda_T (x=0) = \int_{y=-\infty}^{\infty} \sigma_s dy = \int_{-\infty}^{\infty} \frac{-\lambda a}{\pi\left(a^2+y^2\right)} dy = \frac{-\lambda \cancel{a}}{\pi} \frac{1}{\cancel{a}} \left.\underbrace{\tan^{-1}\frac{y}{a}}_{\pi}\right|_{-\infty}^{\infty}$$

$$= -\lambda$$

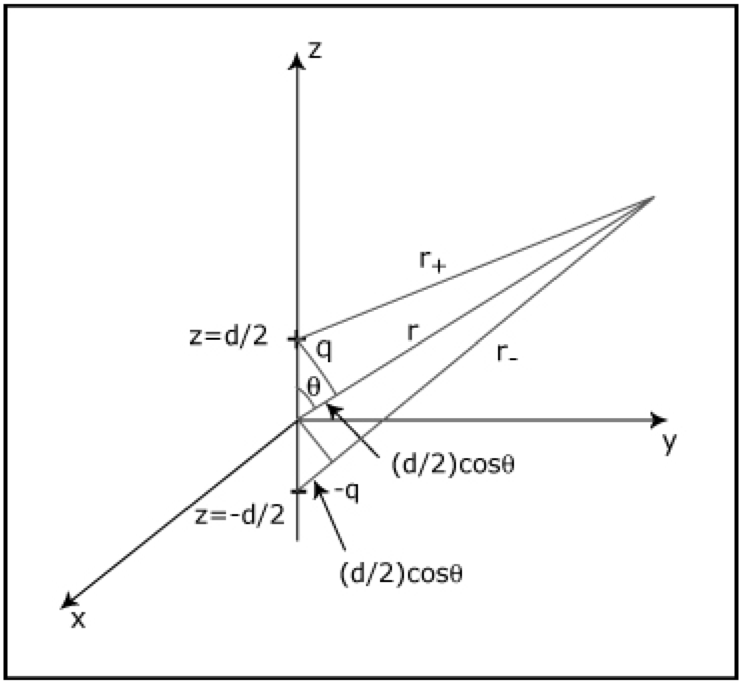
$$i_s = \frac{dq}{dt} \approx A \frac{d\sigma_s}{dt} = \frac{-aA}{\pi\left(a^2+y^2\right)} \frac{d\lambda}{dt} = \frac{-aAC'}{\pi\left(a^2+y^2\right)} \frac{dU}{dt}$$

take $U = U_0 \cos \omega t$

$$v_0 = -i_s R_s = -\frac{C'Aa}{\pi\left(a^2+y^2\right)} U_0 \omega \sin \omega t$$

IV. Point Electric Dipole

1. Potential



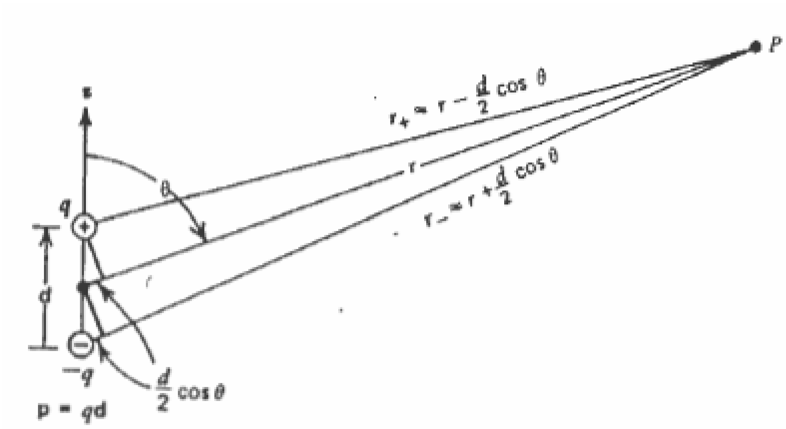
$$\Phi = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{r_+} - \frac{1}{r_-}\right]$$

$$r_+ = \sqrt{x^2 + y^2 + \left(z - \frac{d}{2}\right)^2}$$

$$r_- = \sqrt{x^2 + y^2 + \left(z + \frac{d}{2}\right)^2}$$

Note: $\Phi(z = 0) = 0$

2. Point Electric Dipole (r>>d)



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$$r_+ \approx r - \frac{d}{2}\cos\theta \approx r\left[1 - \frac{d}{2r}\cos\theta\right]$$

$$r_- \approx r + \frac{d}{2}\cos\theta \approx r\left[1 + \frac{d}{2r}\cos\theta\right]$$

$$\Phi \approx \frac{q}{4\pi\varepsilon_0 r}\left[\frac{1}{1 - \frac{d}{2r}\cos\theta} - \frac{1}{1 + \frac{d}{2r}\cos\theta}\right] \approx \frac{q}{4\pi\varepsilon_0 r}\left[1 + \frac{d}{2r}\cos\theta - \left(1 - \frac{d}{2r}\cos\theta\right)\right]$$

$$\approx \frac{qd\cos\theta}{4\pi\varepsilon_0 r^2}$$

$$\lim_{\substack{d\to 0\\ q\to\infty}} p = qd \ \text{(dipole moment)} \Rightarrow \Phi \approx \frac{p\cos\theta}{4\pi\varepsilon_0 r^2}$$

$$\bar{E} = -\nabla\Phi = -\left[\frac{\partial\Phi}{\partial r}\bar{i}_r + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\bar{i}_\theta + \frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}\bar{i}_\phi\right]$$

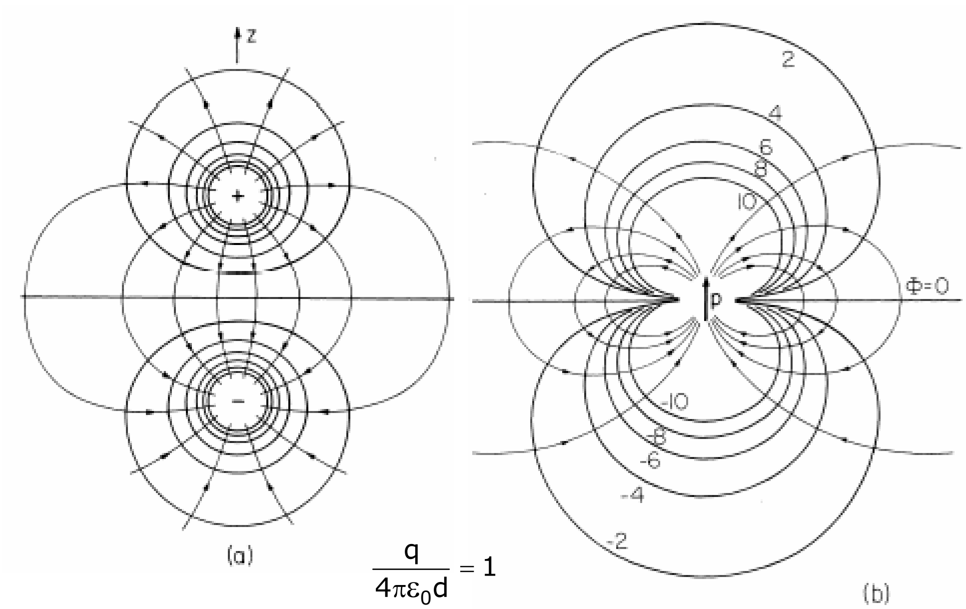
$$= \frac{p}{4\pi\varepsilon_0 r^3}\left[2\cos\theta\,\bar{i}_r + \sin\theta\,\bar{i}_\theta\right]$$

3. Field Lines: $\frac{dr}{r d\theta} = \frac{E_r}{E_\theta} = \frac{2\cos\theta}{\sin\theta} = 2\cot\theta$

$$\frac{dr}{r} = 2\cot\theta d\theta \Rightarrow \ln r = 2\ln(\sin\theta) + C$$

$$r = r_0 \sin^2\theta$$

$$r_0 = r\left(\theta = \pi/2\right)$$



$$\frac{q}{4\pi\varepsilon_0 d} = 1$$

Figure 4.4.2 (a) Cross section of equipotentials and lines of electric field intensity for the two charges of Figure 4.4.1. (b) Limit in which pair of charges form a dipole at the origin.

V. Line Current Above a Perfect Conductor

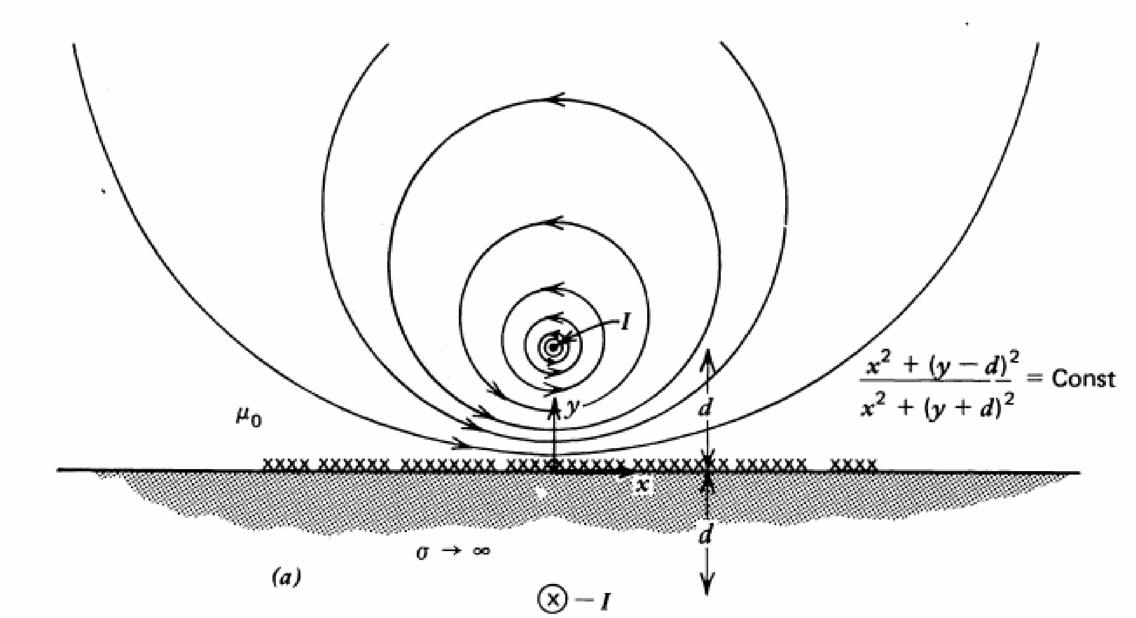


Figure 5-24 (*a*) A line current above a perfect conductor induces an oppositely directed surface current that is equivalent to a symmetrically located image line current.

$$\bar{f}_I = \bar{I} \times \left(\mu_0 \bar{H}\right) \quad \text{Newton/meter [force per unit length]}$$

$$= I\,\bar{i}_z \times \left(\mu_0 \frac{I}{4\pi d}\,\bar{i}_x\right) = \frac{\mu_0 I^2}{4\pi d}\,\bar{i}_y$$