8.02   **Spring 2014**

Exam One Solutions

Problem 1 (25 points): answers without work shown will not be given any credit.

Four point-like objects of charge $-2Q$, $+Q$, $-Q$, and $+2Q$ respectively are placed at the corners of a square of side a as shown in the figure below.
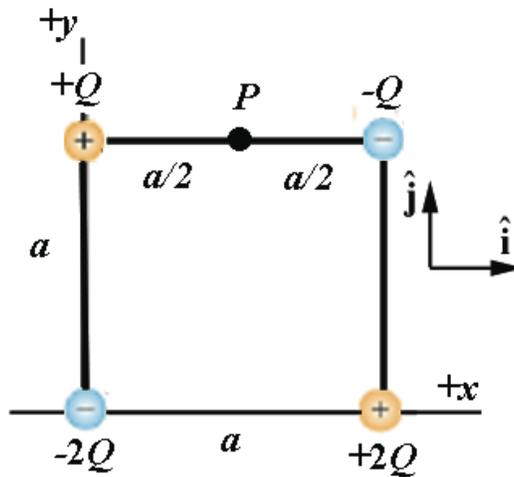

Part (a)

Determine the electric field at the point P, which has coordinates $(x_P, y_P) = (a/2, a)$. Express your answer in terms of $k = 1/4\pi\varepsilon_0$, Q, a, $\hat{\mathbf{i}}$, and $\hat{\mathbf{j}}$ as needed.

Part (b)

Choose infinity to be the zero point for the electric potential ($V(\infty) = 0$). Determine the electric potential at the point P, which has coordinates $(x_P, y_P) = (a/2, a)$. Express your answer in terms of $k = 1/4\pi\varepsilon_0$, Q, a, $\hat{\mathbf{i}}$, and $\hat{\mathbf{j}}$ as needed.

Part (c)

Determine the direction and magnitude of the electric dipole moment of this charge configuration. Express your answer in terms of $k = 1/4\pi\varepsilon_0$, Q, a, $\hat{\mathbf{i}}$, and $\hat{\mathbf{j}}$ as needed.

Problem 1

a)

$Q_1 = +Q$

$Q_2 = -Q$
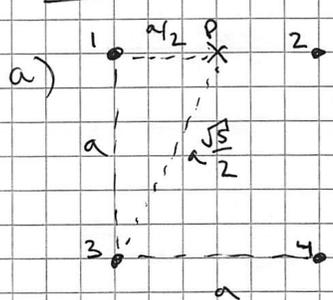
$Q_3 = -2Q$

$Q_4 = +2Q$

$\vec{r}_{1,p} = +\frac{a}{2}\hat{\imath}$

$\vec{r}_{2,p} = -\frac{a}{2}\hat{\imath}$

$\vec{r}_{3,p} = +\frac{a}{2}\hat{\imath} + a\hat{\jmath}$

$\vec{r}_{4,p} = -\frac{a}{2}\hat{\imath} + a\hat{\jmath}$

$$|\vec{r}_1| = |\vec{r}_2| = \frac{a}{2} \quad |\vec{r}_3| = |\vec{r}_4| = \frac{a\sqrt{5}}{2}$$

Using superposition

$$\vec{E} = \sum_i \vec{E}_i = \sum_i \frac{kQ_i}{\vec{r}_{i,p}^{\,2}} \hat{r}_{i,p}$$

$$= \sum_i \frac{kQ_i}{|\vec{r}_{i,p}|^3} \vec{r}_{i,p}$$
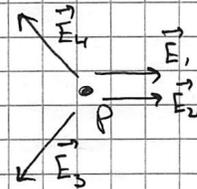
Note that $\hat{\jmath}$ components cancel, so only need $\hat{\imath}$ components

Also note that $E_1^x = E_2^x$ and $E_3^x = E_4^x$

$$E_1^x = \frac{kQ}{(a/2)^2}\hat{\imath} \qquad E_3^x = \frac{k(-2Q)}{(a\sqrt{5}/2)^3}\left(-\frac{a}{2}\right)\hat{\imath}$$

$$\vec{E}_{total} = \left(2E_1^x + 2E_3^x\right)\hat{\imath}$$

$$\boxed{= \frac{kQ}{a^2}\left(8 - \frac{16}{5\sqrt{5}}\right)\hat{\imath}}$$

b)

+Q   P   -Q

r_1 r_2

r_3 r_4

-2Q   +2Q
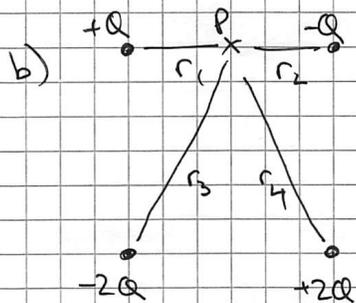
$$V(P) - \underbrace{V(\infty)}_{0} = \sum_i \frac{kQ_i}{r_i}$$

Note that $r_1 = r_2$

but $Q_1 = -Q_2$,

so $V_1 + V_2 = 0$

The same is true for V_3 and V_4.

So the potential $\boxed{V(P) = 0}$.

This makes sense, because this is a system of two dipoles, and for a single dipole, the equipotentials look like this:
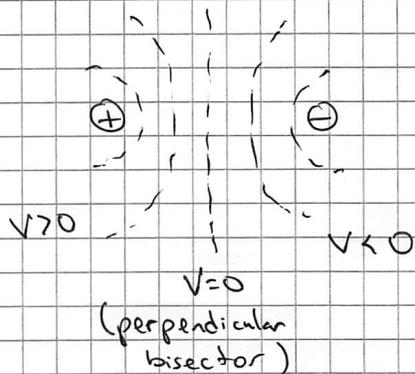
(i.e. zero potential on plane bisecting the dipoles.)

$\oplus$ $\ominus$

$V > 0$ $V < 0$

$V = 0$
(perpendicular bisector)

c)
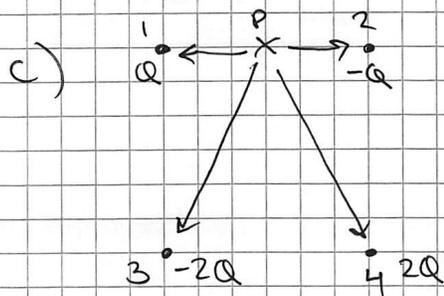
The dipole is given by

$$\vec{p} = \sum_i Q_i \, \vec{r}_{p,i}$$

We can use an arbitrary reference point, but might as well use P, since we've already calculated the vectors in part a.

Note though that: $\vec{r}_{i,p} = -\vec{r}_{p,i}$

from part a   in dipole formula.

$$\vec{p} = +Q\left(-\frac{a}{2}\hat{\imath}\right) - Q\left(+\frac{a}{2}\hat{\imath}\right)$$
$$-2Q\left(-\frac{a}{2}\hat{\imath} - a\hat{\jmath}\right) + 2Q\left(+\frac{a}{2}\hat{\imath} - a\hat{\jmath}\right)$$

Note that $\hat{\jmath}$ components cancel.

$$= \underbrace{-Qa\hat{\imath}}_{\text{top dipole}} + \underbrace{2Qa\hat{\imath}}_{\text{bottom dipole}}$$

$$\boxed{= +Qa\hat{\imath}}$$

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Problem 1 grading Scheme

a) 12 points total

Typical point deductions: (overlapping)

    −4 : thinking that top charges cancel

    −3 : unit issue

    −3 : not using $\frac{kQ}{r^3}\vec{r} = \vec{E}$ everywhere

    −3 : error in trigonometry

    −3 : sign error in $\vec{E}$ fields

    −2 : finding a $\vec{\jmath}$ component

    −2 : calculating $\vec{F}$ instead of $\vec{E}$

    −2 : random algebra mistake

    −2 : overall sign error

    −1 : transcription error on last line

b) 6 points total

Typical point deductions: (overlapping)

    −6 : completely missing the point

    −5 : thinking that V was a vector
           (but still writing $V = kQ/r$)

    −4 : Not realizing that V had a sign

    −3 : Random sign errors.

c) 7 points total

Typical point deductions.

    −5 : Thinking $\vec{p}$ is a scalar, or wrong $\vec{p}$ definition

    −4 : Knowing $\vec{p}$ is a vector, but
           doing a scalar sum

    −4 : Doing too many (combinatoric) pairings

    −3 : Error in trigonometry

    −3 : Confusing $\vec{r}$ with $\hat{r}$

    −3 : various sign errors

    −2 : Wrong $\vec{r}$ (or changing origin)

    −2 : Overall sign.

Problem 2 (25 points): answers without work shown will not be given any credit.

A stick with length L has a positive charge Q uniformly distributed on it. It lies along the x-axis between the points $x = 0$ and $x = L$. A point-like object with identical positive charge Q lies on the x-axis at the point $x = 2L$.

uniformly charged stick
of length L and charge Q

point-like object
with charge Q

$x = 0$ $x = L$ $x = a$ $x = 2L$
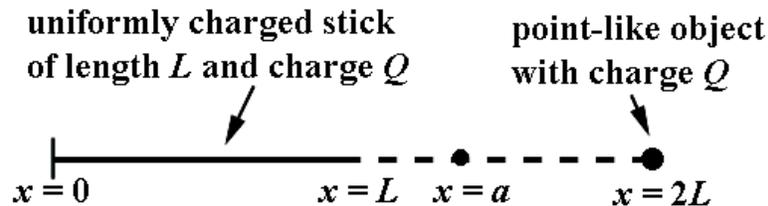
You may find the following integrals useful:

$$\int \frac{dx}{a-x} = -\ln(a-x)$$

$$\int \frac{dx}{(a-x)^2} = \frac{1}{(a-x)}$$

$$\int \frac{dx}{(x^2+y^2)^{3/2}} = \frac{1}{y^2}\frac{x}{(x^2+y^2)^{1/2}}$$

$$\int \frac{xdx}{(x^2+y^2)^{3/2}} = -\frac{1}{(x^2+y^2)^{1/2}}$$

Part (a)

Let $x = a$ be the point on the x-axis between the two objects where the electric field is zero. Find a. Express your answer in terms of $k = 1/4\pi\varepsilon_0$, Q, and L as needed.

Part (b)

Choose the zero point for the electric potential to be at infinity $V(\infty) = 0$. Determine the electric potential at the point $x = a$. **You may express your answer in terms of any of the quantities mentioned in part (a), whether or not you answered part (a); in particular in terms of $k = 1/4\pi\varepsilon_0$, Q, L, and a as needed.**

Problem 2

(a)
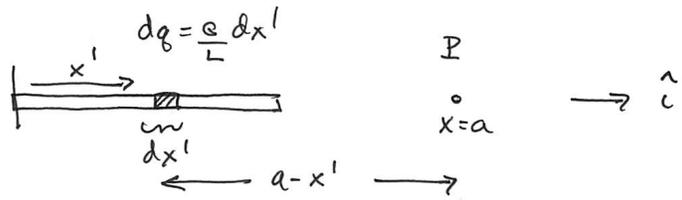
$$\vec{E}_{rod}(P) = k \int_{x'=0}^{x'=L} \frac{dq'}{r_{s,P}^2} \hat{r}_{s,P} = k \int_{x'=0}^{x'=L} \frac{\frac{Q}{L} dx'}{(a-x')^2} \hat{i}$$

$$= \frac{kQ}{L} \frac{1}{(a-x')} \Big|_{x'=0}^{x'=L} \hat{i} = \frac{kQ}{L}\left(\frac{1}{a-L} - \frac{1}{a}\right)\hat{i}$$
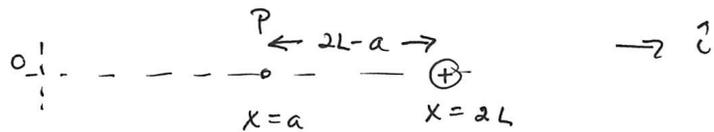
$$\vec{E}_{+Q}(P) = \frac{kQ}{(2L-a)^2}(-\hat{i})$$

$$\vec{E}(P) = \vec{E}_{rod}(P) + \vec{E}_{+Q}(P) = \vec{0}$$

$$\left(\frac{kQ}{L}\left(\frac{1}{a-L} - \frac{1}{a}\right) - \frac{kQ}{(2L-a)^2}\right)\hat{i} = \vec{0} \implies$$

$$\frac{1}{L}\left(\frac{1}{a-L} - \frac{1}{a}\right) - \frac{1}{(2L-a)^2} = 0 \implies$$

$$\frac{1}{(a-L)a} = \frac{1}{(2L-a)^2} \implies (2L-a)^2 = (a-L)a \implies$$

$$4L^2 - 4La + a^2 = a^2 - aL \implies 4L^2 = 3La \implies$$

$$\boxed{a = \frac{4L}{3}}$$

(b)

$$x' \quad \longleftarrow a-x' \longrightarrow P$$

$$dx'$$

$$V_{rod}(\infty) = 0$$

$$x = a \qquad +x$$
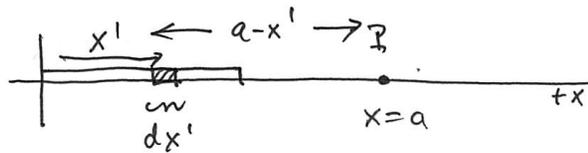
$$dq' = \frac{Q}{L} dx'$$

$$V_{rod}(P) - V_{rod}(\infty) = k\int_{rod} \frac{dq}{r_{s,P}} = k \int_{x'=0}^{x'=L} \frac{\frac{Q}{L} dx'}{(a-x')} = -\frac{kQ}{L}\ln(a-x')\Big|_{x'=0}^{x'=L}$$

$$V_{rod}(P) = -\frac{kQ}{L}\ln\left(\frac{a-L}{a}\right) = \frac{kQ}{L}\ln\left(\frac{a}{a-L}\right)$$

$$-|- \qquad \underset{x=a}{\overset{P}{\underset{\sigma}{\rule{0pt}{0pt}}}} \qquad \underset{x=2L}{\overset{+Q}{\oplus}} \qquad V_{+Q}(\infty) = 0$$

$$\longleftarrow 2L-a \longrightarrow$$
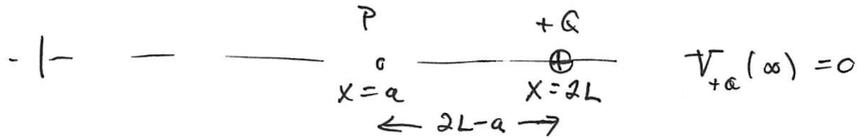
$$V_{+Q}(P) - V_{+Q}(\infty) = \frac{kQ}{2L-a} \implies V_{+Q}(P) = \frac{kQ}{2L-a}$$

$$\boxed{V(P) = V_{rod}(P) + V_{+Q}(P) = \frac{kQ}{L}\ln\left(\frac{a}{a-L}\right) + \frac{kQ}{2L-a}}$$

when $a = \frac{4L}{3}$

$$\boxed{V(P) = \frac{kQ}{L}\ln(4) + \frac{3}{2}\frac{kQ}{L} = \frac{kQ}{L}\left(\ln(4) + \frac{3}{2}\right)}$$

Problem Two   grading scheme

Part (a)   13 total points

$\vec{E}_{rod}(P)$ worth +6 points

$\vec{E}_{+a}(P)$ worth +3 points

$\vec{E}(P) = \vec{E}_{rod}(P) + \vec{E}_{+a}(P)$ + 1 point

correct integration   + 1 point

correct solution for a   +2 points

Part (b)   12 total points

$V_{rod}(P)$ + 6 points

$V_{+a}(P)$ + 3 points

correction integration   +1 point

Superposition $V_{rod}(P) + V_{+a}(P) = V(Q)$ +2 points

Problem 3 (25 points): answers without work shown will not be given any credit.

Consider a **positively** charged infinite slab with uniform volume charge density ρ_1 and thickness $4a$. This slab is oriented such that the two faces of the slab are located on the planes $x = -2a$ and $x = 2a$, respectively. A positively charged solid sphere of radius $R = 2a$ and uniform volume charge density ρ_2 has its center located at the point on the x-axis at $x = +6a$, as shown in the figure below. Let $\hat{\mathbf{i}}$ be the unit vector pointing towards the right.

positively chargd sphere of radius $R = 2a$
with uniform charge density ρ_2

positively charged infinite slab with
uniform volume charge density ρ_1
and thickness $4a$
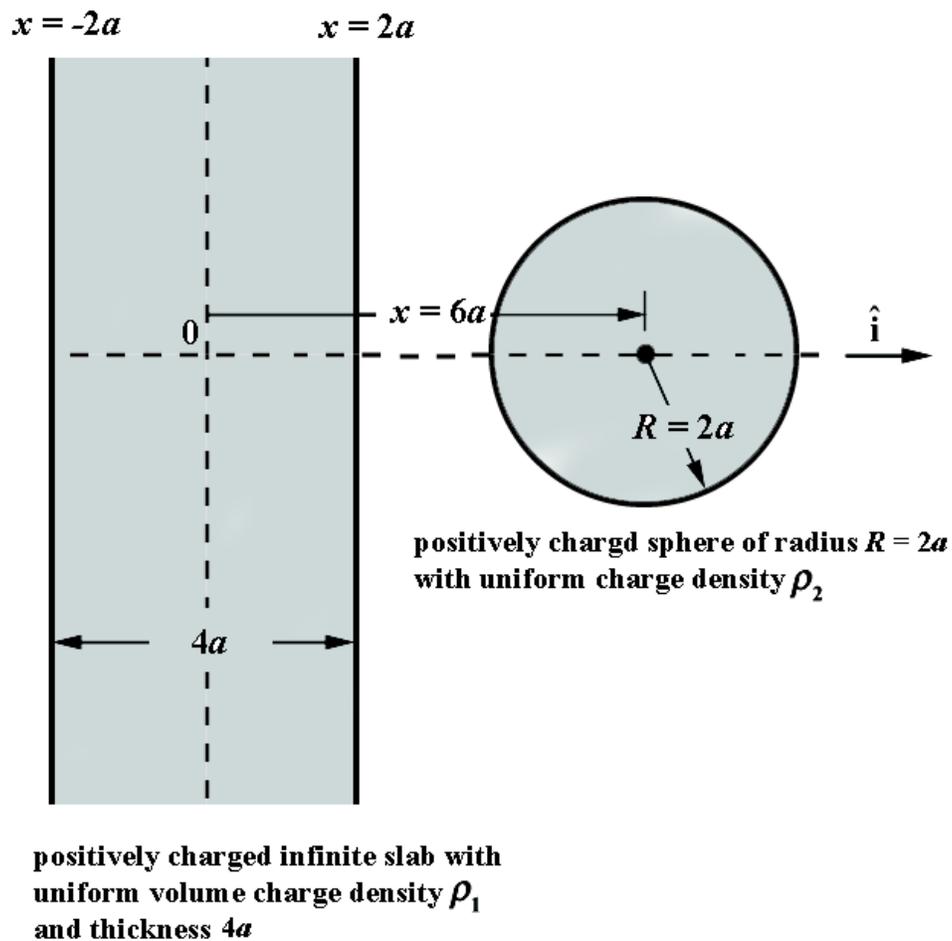
In the questions that follow you must show your work. Answers without justification will receive no credit.

Part (a)

Determine the direction and magnitude of the electric field at the point A located at $x_A = +a$. Express your answer in terms of ε_0, ρ_1, ρ_2, a, and $\hat{\mathbf{i}}$ as needed.

$x = -2a$ $x = 2a$

0

$x = 6a$ $\hat{\mathbf{i}}$

A

$R = 2a$

$x_A = a$

positively chargd sphere of radius $R = 2a$
with uniform charge density ρ_2

$4a$

positively charged infinite slab with
uniform volume charge density ρ_1
and thickness $4a$

(a) $\vec{E}_{slab}(x=0) = 0$

$0 \le x \le 2a$

$\oint \vec{E}_{slab} \cdot d\vec{a} = \dfrac{q_{enc}}{\varepsilon_o} \Rightarrow$

$E_{slab} A_{cap} = \dfrac{\rho_1 A_{cap} x}{\varepsilon_o} \Rightarrow$

$\vec{E}_{slab}(x) = \dfrac{\rho_1 x}{\varepsilon_o} \hat{\imath}$

when $x = a \Rightarrow \vec{E}_{slab} = \dfrac{\rho_1 a}{\varepsilon_o} \hat{\imath}$

$r \ge 2R$
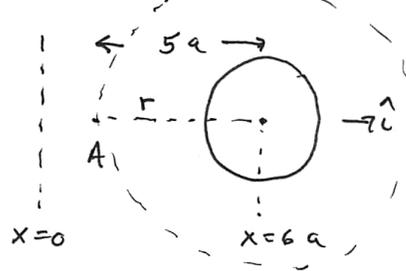
$\oint \vec{E}_{sphere} \cdot d\vec{a} = \dfrac{q_{enc}}{\varepsilon_o} \Rightarrow$

$E_{sphere} 4\pi r^2 = \dfrac{\rho_2 \frac{4}{3} \pi (2a)^3}{\varepsilon_o} \Rightarrow$

$\vec{E}_{sphere} = \dfrac{8}{3} \dfrac{\rho_2}{\varepsilon_o} \dfrac{a^3}{r^2} (-\hat{\imath}) \Rightarrow$

when $r = 5a \Rightarrow \vec{E}_{sphere}^{(A)} = -\dfrac{8}{75} \dfrac{\rho_2 a}{\varepsilon_o} \hat{\imath}$

$\vec{E}(A) = \vec{E}_{sphere}(A) + \vec{E}_{slab}(A)$

$$\boxed{\vec{E}(A) = \left(-\dfrac{8}{75} \dfrac{\rho_2 a}{\varepsilon_o} + \dfrac{\rho_1 a}{\varepsilon_o}\right) \hat{\imath} = \dfrac{a}{\varepsilon_o}\left(-\dfrac{8\rho_2}{75} + \rho_1\right)\hat{\imath}}$$

Part (b)

Determine the direction and magnitude of the electric field at the point B located at $x_B = +6a$. Express your answer in terms of ε_0, ρ_1, ρ_2, a, and $\hat{\mathbf{i}}$ as needed.

$x = -2a$ $x = 2a$

positively charged sphere of radius $R = 2a$
with uniform charge density ρ_2

positively charged infinite slab with
uniform volume charge density ρ_1
and thickness $4a$

(b)

$\vec{E}_{slab}(x=0) = 0$

$x \geq 2a$

$x=0 \quad x=2a$
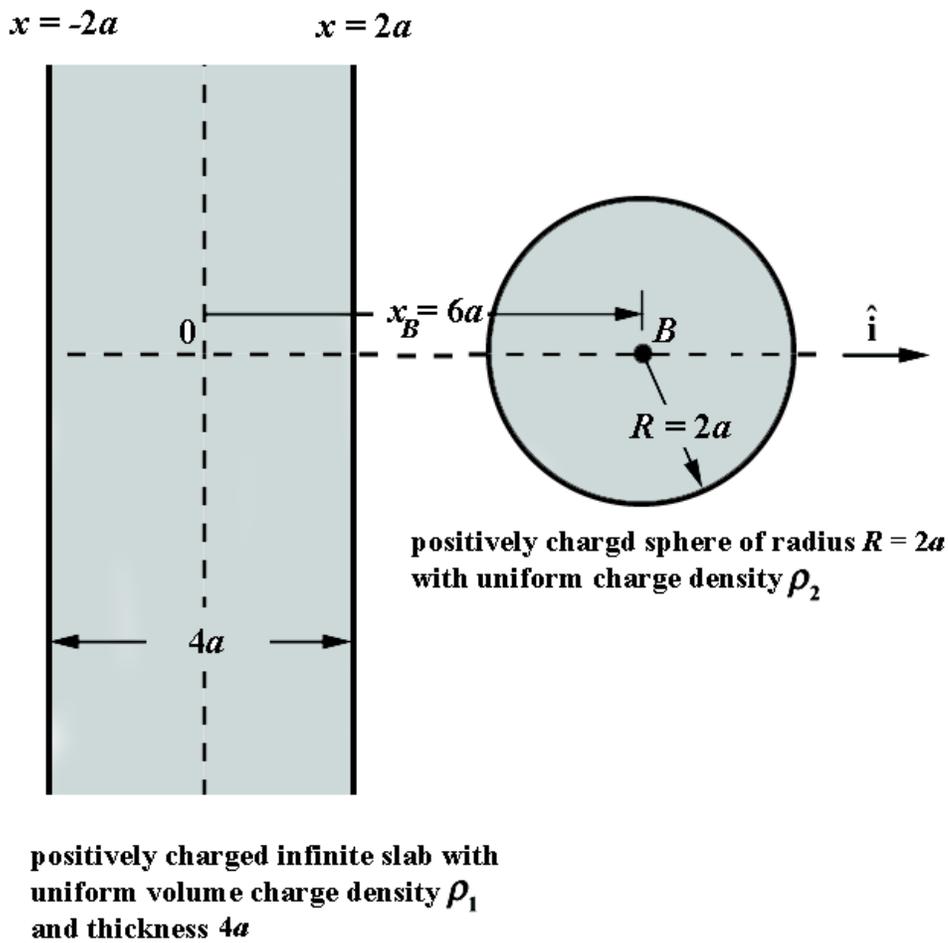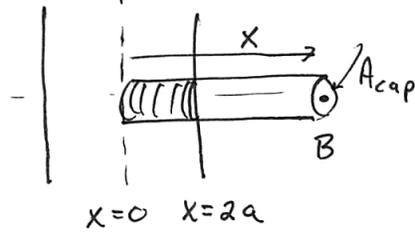
$\oint_B \vec{E}_{slab} \cdot d\vec{a} = \dfrac{q_{enc}}{\varepsilon_o} \Rightarrow$

$E_{slab} A_{cap} = \dfrac{\rho_1 A_{cap} 2a}{\varepsilon_o} \Rightarrow$

$\vec{E}_{slab}(B) = \dfrac{2\rho_1 a}{\varepsilon_o}\hat{\imath}$

$\vec{E}_{sphere}(B) = \vec{0}$

$\boxed{\vec{E}(B) = \vec{E}_{slab}(B) = \dfrac{2\rho_1 a}{\varepsilon_o}\hat{\imath}}$

Part (c)

Determine the direction and magnitude of the electric field at the point C located at $x_C = +7a$. Express your answer in terms of ε_0, ρ_1, ρ_2, a, and $\hat{\mathbf{i}}$ as needed.
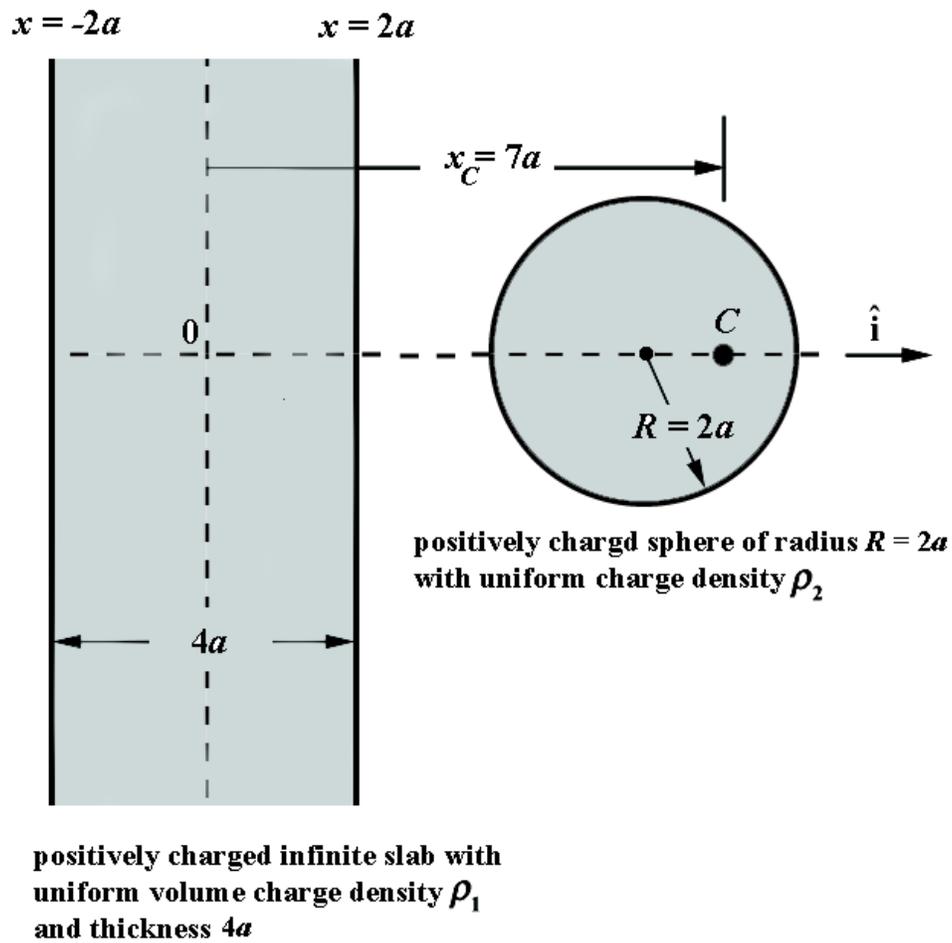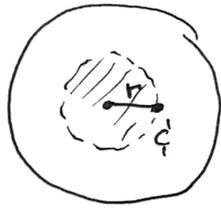
$x = -2a$ $x = 2a$

$x_C = 7a$

0 C $\hat{\mathbf{i}}$

$R = 2a$

positively charged sphere of radius $R = 2a$
with uniform charge density ρ_2

$4a$

positively charged infinite slab with
uniform volume charge density ρ_1
and thickness $4a$

(c)   r

$$\oint \vec{E}_{sphere} \cdot d\vec{a} = \frac{q_{enc}}{\varepsilon_o} \Rightarrow$$

$$\rightarrow \hat{i}$$

$$E_{sphere} \, 4\pi r^2 = \frac{\rho_2 \frac{4}{3}\pi r^3}{\varepsilon_o} \Rightarrow$$

$$\vec{E}_{sphere} = \frac{\rho_2 \, r}{3\,\varepsilon_o} \, \hat{i}$$

when $r = a \Rightarrow \vec{E}_{sphere} = \frac{\rho_2 \, R}{3\,\varepsilon_o} \, \hat{i} = \frac{\rho_2 \, a}{3\,\varepsilon_o} \, \hat{i}$

$$\vec{E}_{slab}(c) = \frac{2\rho_1 \, a}{\varepsilon_o} \, \hat{i} \qquad (\text{same as in part (b)})$$

$$\vec{E}(c) = \vec{E}_{sphere}(c) + \vec{E}_{slab}(c) = \frac{\rho_2 \, a}{3\,\varepsilon_o} \, \hat{i} + \frac{2\rho_1 \, a}{\varepsilon_o} \, \hat{i}$$

$$\boxed{\vec{E}(c) = \frac{a}{\varepsilon_o} \left(\frac{\rho_2}{3} + 2\rho_1 \right) \hat{i}}$$

Problem 3   grading scheme

part (a)   11 points

   $\vec{E}_{slab}(A)$ 4 points

   $\vec{E}_{sphere}(A)$ 4 points

   Superposition   3 points

part (b)   7 points

   $\vec{E}_{sphere}(B) = \vec{0}$ 3 points

   $\vec{E}_{slab}(B)$ 3 points

   Superposition (answer)   1 point

part (c)   7 points

   $\vec{E}_{slab}(c)$ 1 point

   $\vec{E}_{sphere}(c)$ 4 points

   Superposition   2 points

Problem 4 of 4: (25 points) Concept Questions (Parts A through E)

Part A (5 points):

Consider an infinite positively charged cylinder with uniform positive volume charge density, $\rho > 0$. An electric dipole consisting of a positive charge $+Q$ and a negative charge $-Q$ attached to opposite ends of a massless rod is oriented as shown in the figure below. The axis of the cylinder and the dipole lie in the same plane.

**positively charged infinite cylider
with uniform volume charge density** ρ
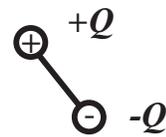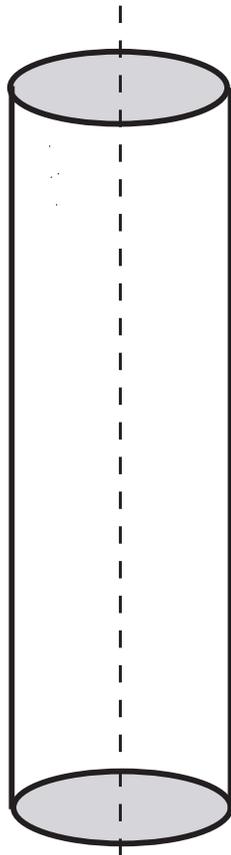
electric dipole

At the instant depicted in the figure, the electric dipole will experience

a) a force to the left and a torque trying to make it rotate clockwise.

b) a force to the left and a torque trying to make it rotate counterclockwise.

c) a force to the left and no torque.

d) a force to the right and a torque trying to make it rotate clockwise.

e) a force to the right and a torque trying to make it rotate counterclockwise.

f) a force to the right and no torque.

g) no force and a torque trying to make it rotate clockwise.

h) no force and a torque trying to make it rotate counterclockwise.

i) no force and no torque.

Answer _____d_____.

Problem 4 Part(a) $\boxed{\text{Answer } d.}$
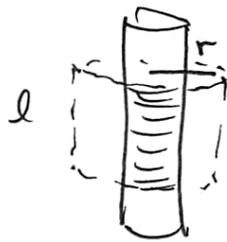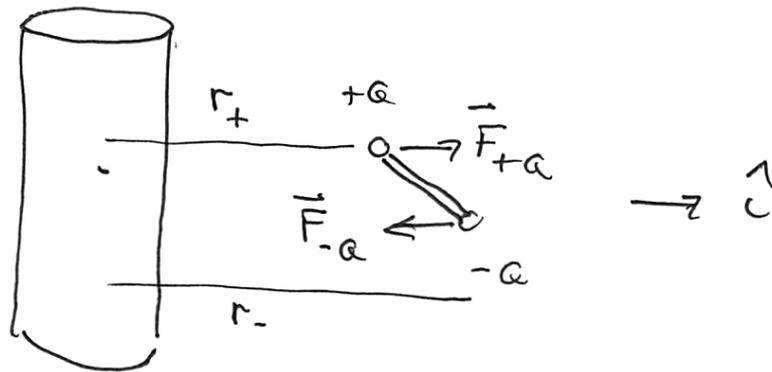
$$E_{ext} 2\pi r \ell = \frac{\lambda \ell}{\varepsilon_0} \Rightarrow \bar{E}_{ext} = \frac{\lambda}{2\pi \varepsilon_0} \frac{1}{r} \hat{\imath}$$

$$|\vec{E}_+| > |\vec{E}_-| \Rightarrow |\vec{F}_+| > |\vec{F}_-|$$

$$\vec{F} = \vec{F}_+ + \vec{F}_- \quad \text{points to right}$$

$$\vec{\tau}_{dipole} = \vec{P} \times \vec{E}_{ext}$$
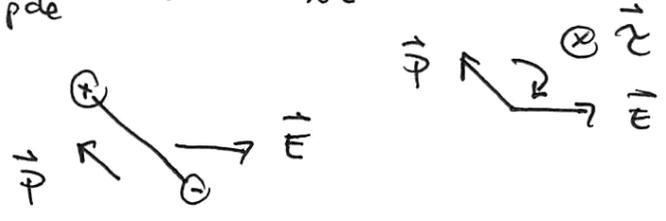
torque is into plane of figure $\Rightarrow$

clockwise rotation

Part B (5 points):

Consider a uniformly negatively charged sphere of charge $-Q$ and radius R with its center located at the origin. A point-like object with positive charge $+4Q$ is placed directly above the center of the sphere at the point $z_{4Q} = 2R$. At which point(s) shown in the figure below is the electric field due to both charged sphere and point charge zero?

$+z$

● A $z_A = 3R$

$+4Q$ ◯ $z_{4Q} = 2R$

● B $z_B = R$

uniform negatively charged sphere with total charge $-Q$ and radius R

● C $z_C = 0$

● D $z_D = -R$

● E $z_E = -2R$
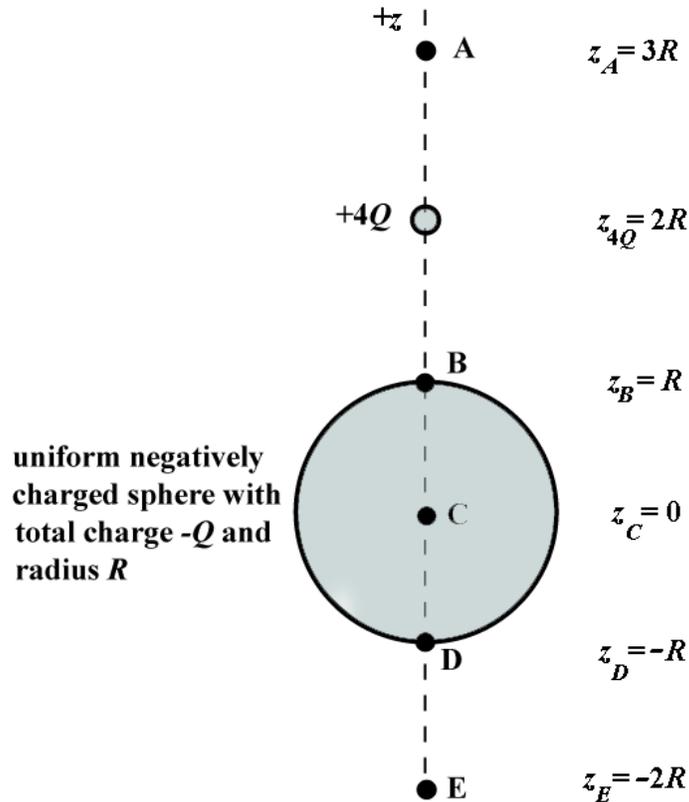
Answer(s): _____E_____

$\vec{E}_{sphere}$ outside is equal to electric field of point charge located at center

$\vec{E}(E) - \vec{E}_{sphere}(E) + \vec{E}_{+4Q}(E)$

$= +\dfrac{kQ}{(2R)^2}\hat{k} - \dfrac{k4Q}{(4R)^2}\hat{k} = \vec{0}$

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Part C (5 points):
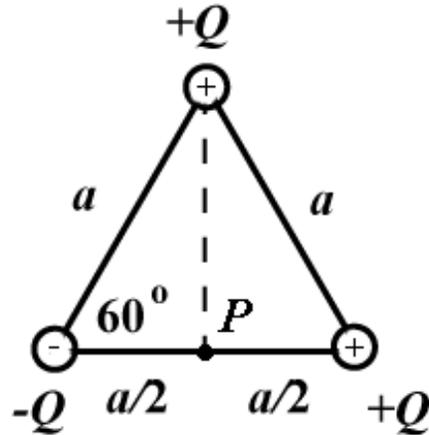
Consider the set of three point-like charged objects that are placed on the vertices of an equilateral triangle with side a. A positively charged point-like object of charge $+2Q$ is brought in from infinity and placed at the point P, shown in the figure above. The change in the electric potential energy of the system of four charged objects is equal to

1. $\dfrac{1}{4\pi\varepsilon_0}\dfrac{8}{3}\dfrac{Q^2}{a^2}+\dfrac{1}{4\pi\varepsilon_0}8\dfrac{Q^2}{a^2}$

2. $\dfrac{1}{4\pi\varepsilon_0}\dfrac{4}{\sqrt{3}}\dfrac{Q^2}{a}+\dfrac{1}{4\pi\varepsilon_0}8\dfrac{Q^2}{a}$

3. $\dfrac{1}{4\pi\varepsilon_0}\dfrac{8}{3}\dfrac{Q^2}{a^2}$

4. $\dfrac{1}{4\pi\varepsilon_0}\dfrac{4}{\sqrt{3}}\dfrac{Q^2}{a}$

5. 0

Answer: _____4_____

Problem 4|c)

$+G$

a

$\frac{\sqrt{3}}{2}a$

$-Q$

$a/2$ P $-Q$
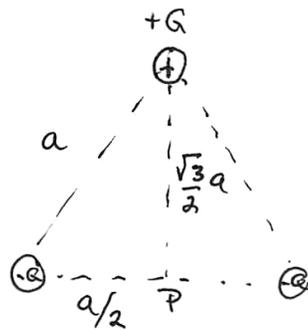
$$V_{system}(P) - V_{system}(\infty) = +\frac{kG}{\frac{\sqrt{3}}{2}a} + \frac{kQ}{\frac{a}{2}} - \frac{kQ}{\frac{a}{2}} = \frac{2kQ}{\sqrt{3}\,a}$$

$$U_{+2a}(P) - U_{+2a}(\infty) = (2G)\left(V_{system}(P) - V_{system}(\infty)\right)$$

$$\boxed{U_{+2a}(P) - U_{+2a}(\infty) = \frac{4kQ^2}{\sqrt{3}\,a}}$$

Part D (5 points):

Below we show the grass seeds representation of either the electric field or electric potential of two point charges. Which of the following statement(s) is correct?
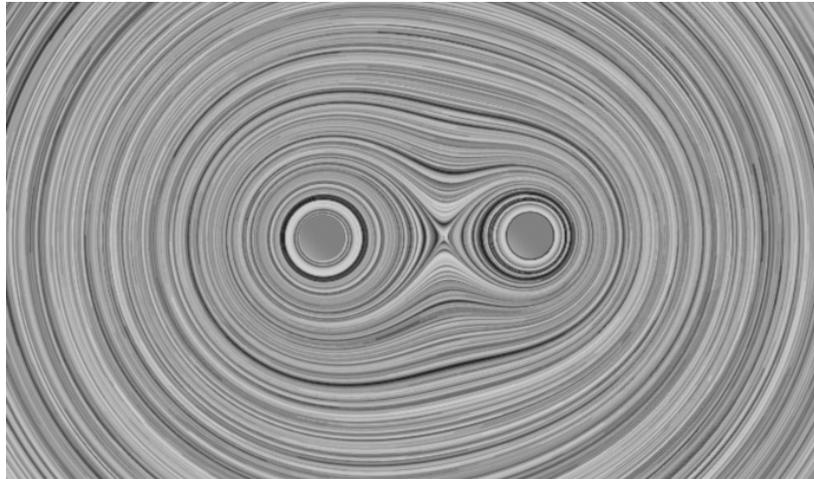

a) Both charges have the same sign, equal magnitude, and the grass seeds representation is the electric field.

b) Both charges have the same sign, equal magnitude, and the grass seeds representation is the electric potential.

c) The charges have opposite signs, equal magnitude, and the grass seeds representation is the electric field.

d) The charges have opposite signs, equal magnitude, and the grass seeds representation is the electric potential.

e) Both charges have the same sign, unequal magnitude, and the grass seeds representation is the electric field.

f) Both charges have the same sign, unequal magnitude, and the grass seeds representation is the electric potential.

g) The charges have opposite signs, unequal magnitude, and the grass seeds representation is the electric field.

h) The charges have opposite signs, unequal magnitude, and the grass seeds representation is the electric potential.

Answer(s): ____f_____

Grass seeds represent equipotential lines because equipotential lines near point-like charged objects are circles (electric field lines begin and end on point charges or got to infinity). The charged objects have the same sign of charge with unequal magnitudes. If the charges were opposite signs then there would be a zero point between them, and the equipotential corresponding to the zero point would extend in a perpendicular direction to a line passing through the two charged objects. If the charged objects had equal magnitude then the equipotential lines would be symmetric about an axis passing perpendicular to a line passing through the two charged objects.

Part E (5 points):

Consider two oppositely charged infinite plates with uniform area charge densities $+\sigma$ on the upper plate and $-\sigma$ on the lower plate, respectively. A closed Gaussian surface is shown in figure below that passes through the upper plate.
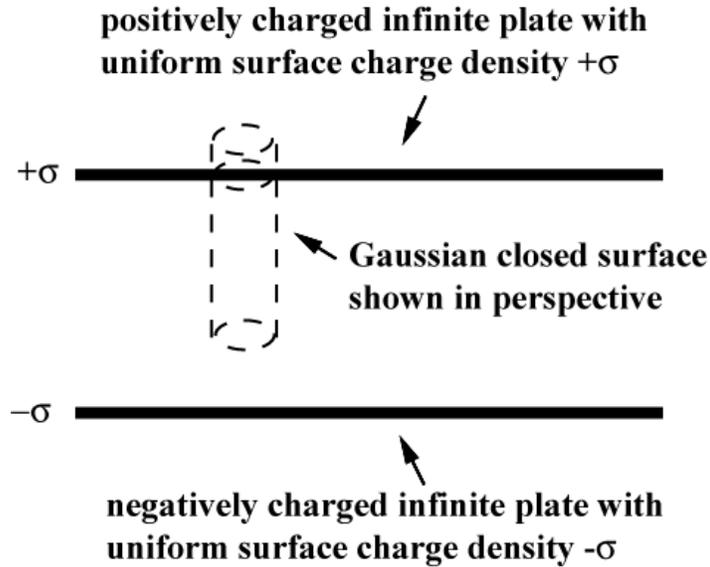
Which of the following statement(s) is correct?

1. The flux on both endcaps of the Gaussian surface is zero.

2. Only the flux on the endcap of the Gaussian surface between the plates is zero.

3. The electric field between the plates is only due to the charge enclosed in the volume of the Gaussian surface that intersects the upper plate.

4. The electric field between the plates is due to all the charges on the two infinite plates.

5. None of the above.

Answer(s): _____4_____

The electric field is due to all the charges on both plates. The flux on the endcap between the plates is non-zero.