Part I: Short Questions and Concept Questions

Problem 1: Spark Plug Pictured at right is a typical spark plug (for scale, the thread diameter is about 10 mm).

![Spark Plug Image]

About what voltage does your car ignition system need to generate to make a spark, if the breakdown field in a gas/air mixture is about 10 times higher than in air? For those of you unfamiliar with spark plugs, the spark is generated in the gap at the left of the top picture (top of the bottom picture). The white on the right is a ceramic which acts as an insulator between the high voltage center and the grounded outer (threaded) part.

Answer.

From the picture, if the threads have a diameter of 10 mm, then the gap is about 0.5 mm. The breakdown is 10 times higher than in air, so its $3 \times 10^7$ V/m. So to get breakdown you need a potential difference across the gap of

$$
\Delta V = Ed = (3 \times 10^7 \text{ V} \cdot \text{m}^{-1})(5 \times 10^{-4} \text{ m}) = 1.5 \times 10^4 \text{ V}
$$
Problem 2 Consider the three charges $+Q$, $+2Q$, and $-Q$, and a mathematical spherical surface (it does not physically exist) as shown in the figure below.

What is the net electric flux on the spherical surface between the charges? Briefly explain your answer.

Answer Only the positive charge $+Q$ enclosed in the surface contributes to the electric flux on that surface. Since the charge enclosed is positive, the electric flux on the surface is also positive and equal to $+Q/\varepsilon_0$. All three charges do contribute to the electric field. However if a charge is outside a closed surface then the electric flux on that surface due only to that charge is zero. The two charges $+2Q$ and $-Q$ lie outside the surface, so the electric flux on the surface due to those two charges is zero.
Problem 3: A pyramid has a square base of side $a$, and four faces which are equilateral triangles. A charge $Q$ is placed on the center of the base of the pyramid. What is the net flux of electric field emerging from one of the triangular faces of the pyramid?

1. $0$
2. $\frac{Q}{8\varepsilon_0}$
3. $\frac{Qa^2}{2\varepsilon_0}$
4. $\frac{Q}{2\varepsilon_0}$
5. Undetermined: we must know whether $Q$ is infinitesimally above or below the plane?

Answer 2: Explain your reasoning: Construct an eight faced closed surface consisting of two pyramids with the charge at the center. The total flux by Gauss’s law is just $Q/\varepsilon_0$. Since each face is identical, the flux through each face is one eighth the total flux or $Q/8\varepsilon_0$. 
Problem 4: The work done by an external agent in moving a positively charged object that starts from rest at infinity and ends at rest at the point P midway between two charges of magnitude $+Q$ and $-2Q$.

The work done by the external agent in moving the positive charge $q$ from infinity to the point P is equal to the negative of the work done by the electrostatic force. Each charged object is the same distance $d$ from the point P. The work done by the electrostatic force is equal to the potential energy difference between infinity and the point P, which is given by

$$
\Delta U = q\Delta V = q \left( k_e \frac{Q}{d} - k_e \frac{2Q}{d} \right) < 0.
$$
Problem 5

Four charges of equal magnitude (two positive and two negative) are placed on the corners of a square as pictured at left (positive charges at positions A & B, negative at positions C & D).

You decide that you want to swap charges B & C so that the charges of like sign will be on the same diagonal rather than on the same side of the square.

In order to do this, you must do…

1. … positive work
2. … negative work
3. … no net work
4. … an indeterminate amount of work, as it depends on exactly how you choose to move the charges.

Answer 2. By moving like sign charges farther apart and opposite sign charges closer together you reduce the energy of the system
**Problem 6** Two opposite charges are placed on a line as shown in the figure below.

![Diagram](image)

The charge on the right is three times the magnitude of the charge on the left. Besides infinite, where else can electric field possibly be zero?

1. between the two charges.
2. to the right of the charge on the right.
3. to the left of the charge on the left.
4. the electric field is nowhere zero.

**Answer:** 3. The electric field from the positive charge $\vec{E}_{+3q}$ and the electric field from the negative charge $\vec{E}_{-q}$ are shown for different regions with the arrow size indicative of the magnitude in the figure below.

![Diagram](image)

The only place the fields can cancel is to the left of the negative charge. For the region to the right of the charge of greater magnitude, the smaller charge is further away and so the magnitude of the field of the smaller charge can never equal the magnitude of the field of the larger charge.
Part II Analytic Problems

Problem 1 Non-uniformly charged sphere

A sphere of radius $R$ has a charge density $\rho = \rho_0 (r / R)$ where $\rho_0$ is a constant and $r$ is the distance from the center of the sphere.

a) What is the total charge inside the sphere?

Solution:

The total charge inside the sphere is the integral

$$Q = \int_{r'=0}^{r=R} \int_{r'=0}^{r=R} \rho_0 (r' / R) 4\pi r'^2 dr' = \frac{\rho_0 4\pi R^4}{4} = \rho_0 \pi R^3$$

b) Find the electric field everywhere (both inside and outside the sphere).

Solution:

There are two regions of space: region I: $r < R$, and region II: $r > R$ so we apply Gauss’ Law to each region to find the electric field.

For region I: $r < R$, we choose a sphere of radius $r$ as our Gaussian surface. Then, the electric flux through this closed surface is

$$\oint \oint \mathbf{E}_I \cdot d\mathbf{A} = E_I \cdot 4\pi r^2.$$

Since the charge distribution is non-uniform, we will need to integrate the charge density to find the charge enclosed in our Gaussian surface. In the integral below we use the integration variable $r'$ in order to distinguish it from the radius $r$ of the Gaussian sphere.

$$\frac{Q_{enc}}{\epsilon_0} = \frac{1}{\epsilon_0} \int_{r'=0}^{r'=r} \rho_0 4\pi r'^2 dr' = \frac{1}{\epsilon_0} \int_{r'=0}^{r'=r} \rho_0 (r' / R) 4\pi r'^2 dr' = \frac{\rho_0 4\pi r^4}{4 R \epsilon_0} = \frac{\rho_0 \pi r^4}{R \epsilon_0}.$$

Notice that the integration is primed and the radius of the Gaussian sphere appears as a limit of the integral.
Recall that Gauss’s Law equates electric flux to charge enclosed:

\[ \iiint \mathbf{E} \cdot d\mathbf{A} = \frac{Q_{\text{enc}}}{\varepsilon_0}. \]

So we substitute the two calculations above into Gauss’s Law to arrive at:

\[ E_I \cdot 4\pi r^2 = \frac{\rho_0 \pi r^4}{4\varepsilon_0}. \]

We can solve this equation for the electric field

\[ \mathbf{E}_I = E_I \hat{r} = \frac{\rho_0 r^2}{4\varepsilon_0}, \quad 0 < r < R. \]

The electric field points radially outward and has magnitude \[ |\mathbf{E}_I| = \frac{\rho_0 r^2}{4\varepsilon_0}, \quad 0 < r < R. \]

For region II: \( r > R \): we choose the same spherical Gaussian surface of radius \( r > R \), and the electric flux has the same form

\[ \iiint \mathbf{E}_II \cdot d\mathbf{A} = E_{II} \cdot 4\pi r^2. \]

All the charge is now enclosed, \( Q_{\text{enc}} = Q = \rho_0 \pi R^3 \), so the right hand side of Gauss’s Law becomes

\[ \frac{Q_{\text{enc}}}{\varepsilon_0} = \frac{Q}{\varepsilon_0} = \frac{\rho_0 \pi R^3}{\varepsilon_0}. \]

Then Gauss’s Law becomes

\[ E_{II} \cdot 4\pi r^2 = \frac{\rho_0 \pi R^3}{\varepsilon_0}. \]

We can solve this equation for the electric field
\[ \vec{E}_\text{in} = E_\text{in} \hat{r} = \frac{\rho_0 R^3}{4\varepsilon_0 r^2} \hat{r}, \quad r > R. \]

In this region of space, the electric field points radially outward and has magnitude \(|\vec{E}_\text{in}| = \frac{\rho_0 R^3}{4\varepsilon_0 r^2}, \quad r > R\), so it falls off as \(1/r^2\) as we expect since outside the charge distribution, the sphere acts as if all the charge were concentrated at the origin.
Problem 2 Electric field and force

A positively charged wire is bent into a semicircle of radius $R$, as shown in the figure below.

![Diagram of a positively charged wire bent into a semicircle](image)

The total charge on the semicircle is $Q$. However, the charge per unit length along the semicircle is non-uniform and given by $\lambda = \lambda_0 \cos \theta$.

(a) What is the relationship between $\lambda_0$, $R$ and $Q$?

(b) If a particle with a charge $q$ is placed at the origin, what is the total force on the particle? Show all your work including setting up and integrating any necessary integrals.

Answer.

(a) In order to find a relation between $\lambda_0$, $R$ and $Q$ it is necessary to integrate the charge density $\lambda$ because the charge distribution is non-uniform

$$Q = \int_{\text{wire}} \lambda ds = \int_{\theta=-\pi/2}^{\theta=\pi/2} \lambda_0 \cos \theta' Rd\theta' = R\lambda_0 \sin \theta' \Big|_{\theta=-\pi/2}^{\theta=\pi/2} = 2R\lambda_0.$$ 

(b) The force on the charged particle at the center $P$ of the semicircle is given by

$$\mathbf{F}(P) = q\mathbf{E}(P).$$

The electric field at the center $P$ of the semicircle is given by

$$\mathbf{E}(P) = \frac{1}{4\pi\varepsilon_0} \int_{\text{wire}} \frac{\lambda ds}{r^2} \mathbf{r}.$$ 

The unit vector, $\hat{r}$, located at the field point, is directed from the source to the field point and in Cartesian coordinates is given by

$$\hat{r} = -\sin \theta' \hat{i} - \cos \theta' \hat{j}.$$
Therefore the electric field at the center $P$ of the semicircle is given by

$$
\mathbf{E}(P) = \frac{1}{4\pi\varepsilon_0} \int \frac{\mathbf{r}}{r^2} \cdot \hat{\mathbf{r}} = \frac{1}{4\pi\varepsilon_0} \int_{\theta'=-\pi/2}^{\theta'=-\pi/2} \frac{\lambda_0 \cos \theta' R d\theta'}{R^2} (\sin \theta' \hat{\mathbf{i}} - \cos \theta' \hat{\mathbf{j}}).
$$

There are two separate integrals for the $x$- and $y$-components. The $x$-component of the electric field at the center $P$ of the semicircle is given by

$$
E_x(P) = -\frac{1}{4\pi\varepsilon_0} \int_{\theta'=-\pi/2}^{\theta'=-\pi/2} \frac{\lambda_0 \cos \theta' \sin \theta' d\theta'}{R} = \frac{\lambda_0 \cos^2 \theta'}{8\pi\varepsilon_0 R} \bigg|_{\theta'=-\pi/2}^{\theta'=-\pi/2} = 0.
$$

We expected this result by the symmetry of the charge distribution about the $y$-axis.

The $y$-component of the electric field at the center $P$ of the semicircle is given by

$$
E_y(P) = -\frac{1}{4\pi\varepsilon_0} \int_{\theta'=-\pi/2}^{\theta'=-\pi/2} \frac{\lambda_0 \cos^2 \theta' d\theta'}{R} = -\frac{1}{4\pi\varepsilon_0} \int_{\theta'=-\pi/2}^{\theta'=-\pi/2} \frac{\lambda_0 (1 + \cos 2\theta') d\theta'}{2R}
$$

$$
= -\frac{\lambda_0}{8\pi\varepsilon_0 R} \bigg|_{\theta'=-\pi/2}^{\theta'=-\pi/2} = -\frac{\lambda_0}{16\pi\varepsilon_0 R}.
$$

Therefore the force on the charged particle at the point $P$ is given by

$$
\mathbf{F}(P) = q \mathbf{E}(P) = -\frac{q\lambda_0}{8\pi\varepsilon_0 R} \hat{\mathbf{j}}.
$$
Problem 3: Charged slab and sheets

An infinite slab of charge carrying a uniform volume charge density $\rho$ has its boundaries located at $x = -2$ meters and $x = +2$ meters. It is infinite in the $y$ direction and in the $z$ direction (out of the page). Two similarly infinite charge sheets (zero thickness) are located at $x = -6$ meters and $x = +6$ meters, with uniform surface charge densities $\sigma_1$ and $\sigma_2$ respectively.

In the accessible regions you’ve measured the electric field to be:

$$\vec{E}(x) = \begin{cases} \vec{0} & x < -6 \text{ m} \\ (10 \text{ N} \cdot \text{C}^{-1})\hat{i} & -6 \text{ m} < x < -2 \text{ m} \\ (-10 \text{ N} \cdot \text{C}^{-1})\hat{i} & 2 \text{ m} < x < 6 \text{ m} \\ \vec{0} & x > 6 \text{ m} \end{cases}$$

(a) What is the charge density $\rho$ of the slab?

(b) Find the two surface charge densities $\sigma_1$ and $\sigma_2$ of the left and right charged sheets.

Solution:

(a) We begin by choosing a Gaussian surface shown in the figure below.

Because we are given the magnitude and direction of the electric field on both endcaps with
\( E \equiv |\mathbf{E}_L| = |\mathbf{E}_R| \), we can calculate the flux on the closed surface (note it is negative because the electric fields point into the Gaussian surface).

\[
\iiint_{S} \mathbf{E} \cdot d\mathbf{A} = \mathbf{E}_L \cdot \mathbf{A}_L + \mathbf{E}_R \cdot \mathbf{A}_R = -2 |E|A.
\]

The charge enclosed is \( Q_{enc} = \rho Ad \). Therefore Gauss’s Law becomes

\[
-2EA = \frac{\rho Ad}{\varepsilon_0}
\]

The charge density is then

\[
\rho = -\frac{2E}{d\varepsilon_0} = -\frac{2 \times 10 \text{ N} \cdot \text{C}^{-1}}{(4 \text{ m})} \frac{1}{(4\pi)(9 \times 10^9 \text{ N} \cdot \text{m}^2 \cdot \text{C}^{-2})} = -4.4 \times 10^{-11} \text{ C} \cdot \text{m}^{-3}
\]

(b) Find the two surface charge densities \( \sigma_1 \) and \( \sigma_2 \) of the left and right charged sheets.

Using Gauss’s law with the Gaussian pillboxes indicated in the figure above. For the sheet on the left, the charge enclosed is \( Q_{enc} = \sigma_1 A \), the flux is \( \iiint_{S} \mathbf{E} \cdot d\mathbf{A} = EA \), and so Gauss’s Law becomes

\[
EA = \frac{\sigma_1 A}{\varepsilon_0}
\]

Therefore the surface charge density on the left is

\[
\sigma_1 = E\varepsilon_0 = (10 \text{ N} \cdot \text{C}^{-1}) \frac{1}{(4\pi)(9 \times 10^9 \text{ N} \cdot \text{m}^2 \cdot \text{C}^{-2})} = 8.9 \times 10^{-11} \text{ C} \cdot \text{m}^{-2}.
\]

In a similar manner,

\[
\sigma_2 = E\varepsilon_0 = (10 \text{ N} \cdot \text{C}^{-1}) \frac{1}{(4\pi)(9 \times 10^9 \text{ N} \cdot \text{m}^2 \cdot \text{C}^{-2})} = 8.9 \times 10^{-11} \text{ C} \cdot \text{m}^{-2}
\]
Problem 4 Concentric charged cylinders

A very long uniformly charged solid cylinder (length $L$ and radius $a$) carrying a positive charge $+q$ is surrounded by a thin uniformly charged cylindrical shell (length $L$ and radius $b$) with negative charge $-q$, as shown in the figure. You may ignore edge effects. Find a vector expression for the electric field in each of the regions (i) $r < a$, (ii) $a < r < b$, and (iii) $r > b$.

Solution:

(i) $r < a$: Choose a Gaussian surface shown in the figure below.

We use a Gaussian cylinder of length $l$ and radius $r < a$. Then, the flux is

$$ \iiint \mathbf{E} \cdot d\mathbf{A} = E2\pi rl. $$

Because the charge density is uniform, the charge enclosed is given by

$$ Q_{enc} = \int_{volume\ enclosed} \rho dV = \rho \pi r^2 l. $$

So Gauss’ Law becomes

$$ E2\pi rl = \frac{\rho \pi r^2 l}{\varepsilon_0} \Rightarrow \mathbf{E}(r) = \frac{\rho r}{2\varepsilon_0} \hat{r} : r < a $$

where $\hat{r}$ is an unit vector pointing perpendicular to the axis of symmetry, in the direction of increasing $r$. 

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(ii) $a < r < b$: Choose a Gaussian surface shown in the figure below.

![Gaussian Cylinder](image)

We use a Gaussian cylinder of length $l$ and radius $a < r < b$. Then, the flux is again

$$\iint \mathbf{E} \cdot d\mathbf{A} = E2\pi rl.$$  

Because the charge density is uniform, the charge enclosed is given by

$$Q_{\text{enc}} = \int \rho dV = \rho \pi a^2 l.$$  

So Gauss’ Law becomes

$$E2\pi rl = \frac{\rho \pi r^2 l}{\varepsilon_0} \Rightarrow \mathbf{E}(r) = \frac{\rho a^2}{2\varepsilon_0 r} \hat{r} ; a < r < b.$$  

(iii) $r > b$: Choose a Gaussian surface shown in the figure below.

![Gaussian Cylinder](image)

We use a Gaussian cylinder of length $l$ and radius $r > b$. Then, the flux is again

$$\iint \mathbf{E} \cdot d\mathbf{A} = E2\pi rl.$$
The inner cylinder has charge per unit length \( \lambda_{\text{inner}} = q / L \) and the outer shell has charge per unit length \( \lambda_{\text{outer}} = -q / L \), thus \( \lambda_{\text{inner}} = -\lambda_{\text{outer}} \). Therefore the charge enclosed by the Gaussian surface is zero:

\[
Q_{\text{enc}} = (\lambda_{\text{inner}} + \lambda_{\text{outer}}) l = 0.
\]

So by Gauss’ Law the electric field in the region \( r > b \) must also be zero

\[
E 2\pi rl = 0 \Rightarrow \vec{E} = \vec{0}; r > b.
\]

Combining our results, we have that the electric field is given by

\[
\vec{E}(r) = \begin{cases} 
\frac{\rho r}{2\varepsilon_0} \hat{r}; & r < a \\
\frac{\rho a^2}{2\varepsilon_0} \frac{1}{r} \hat{r}; & a < r < b \\
\vec{0}; & r > b
\end{cases}
\]
**Problem 5 Electric field and potential of charged objects**

Two charges $+q$ and $-q$ lie along the y-axis and are separated by a distance $2d$ as shown in the figure.

(a) Calculate the total electric field $\vec{E}$ at position $A$, a distance $a$ from the y-axis. Indicate its direction on the sketch (draw an arrow).

**Solution:** See figure for arrow at point $A$.

(b) Calculate the total electric field $\vec{E}$ at position $B$, a distance $b$ from the x-axis. Indicate its direction on the sketch (draw an arrow).
Solution: See figure above for arrow at point B.

\[ \vec{E}_B = k_e \left( -\frac{q}{(b-d)^2} + \frac{q}{(b+d)^2} \right) \hat{j} = \frac{1}{4\pi \varepsilon_0} \left( -\frac{q}{(b-d)^2} + \frac{q}{(b+d)^2} \right) \hat{j} \]

(c) Find the electric potential \( V \) at position \( A \) with the electric potential zero at infinity.

\[ V(A) - V(\infty) = V(A) = -k_e \frac{q}{(a^2 + d^2)^{1/2}} + k_e \frac{q}{(a^2 + d^2)^{1/2}} = 0 \]

(d) Find the electric potential \( V \) at position \( B \).

\[ V(B) - V(\infty) = V(B) = k_e \left( +\frac{q}{b-d} - \frac{q}{b+d} \right) = \frac{1}{4\pi \varepsilon_0} \left( +\frac{q}{b-d} - \frac{q}{b+d} \right) \]

(e) A positively charged dust particle with mass \( m \) and charge \(+q\) is released from rest at point B. In what direction will it accelerate (circle the correct answer)?

LEFT  RIGHT  UP  DOWN  It Won’t Accelerate

Solution: The positively charged dust particle will accelerate down because the charge \(+q\) is closer to the dust particle and hence exerts a greater repulsive force than the negative charge \(-q\), which exerts a smaller attractive force because it is further away from the dust particle.

(f) Again, assuming the positively charged dust particle was released from rest, what will its speed be after it has traveled a distance \( s \) from its original position at point B? HINT: Don’t try to simplify any fractions

Solution: Assuming that there are no other forces acting on the dust particle, the change in potential energy of the dust particle as it moves from point B to a point a distance \( s \) from B down the y-axis, is given by

\[ \Delta U \equiv U(\text{final}) - U(B) = -q(V(\text{final}) - V(B)) \]

The final electric potential is given by

\[ V(\text{final}) = k_e \left( +\frac{q}{b+s-d} - \frac{q}{b+s+d} \right) \]

So the change in potential energy is
\[ \Delta U = +q \left( k_e \left( + \frac{q}{b+s-d} - \frac{q}{b+s+d} \right) - k_e \left( + \frac{q}{b-d} - \frac{q}{b+d} \right) \right) \]

From conservation of energy

\[ \Delta K + \Delta U = 0. \]

Since it was released from rest at point B, \( K(B) = 0 \), so the change in kinetic energy is

\[ \Delta K \equiv K(\text{final}) = \frac{1}{2} m v_f^2 \]

From conservation of energy \( \Delta K + \Delta U = 0 \) implies that \( \Delta K = -\Delta U \), so

\[ \frac{1}{2} m v_f^2 = -\Delta U = -q \left( k_e \left( + \frac{q}{b+s-d} - \frac{q}{b+s+d} \right) - k_e \left( + \frac{q}{b-d} - \frac{q}{b+d} \right) \right) \]

The final speed is thus

\[ v_f = \sqrt{ \frac{-2qk_e}{m} \left( + \frac{q}{b+s-d} - \frac{q}{b+s+d} \right) \left( + \frac{q}{b-d} - \frac{q}{b+d} \right) } \]
Problem 6: Electric field and electric potential of a non-uniformly charged rod

A rod of length \( L \) lies along the \( x \)-axis with its left end at the origin. The rod has a non-uniform charge density \( \lambda = \alpha x \), where \( \alpha \) is a positive constant.

(a) Express the total charge \( Q \) on the rod in terms of \( \alpha \) and \( L \).

(b) Calculate the electric field at point \( P \), shown in the Figure. Take the limit \( d \gg L \). What does the electric field look like in this limit? Is this what you expect? Explain. Hint: the following integral may be useful:

\[
\int \frac{x \, dx}{(x+a)^2} = \frac{a}{x+a} + \ln(x+a)
\]

\[
\ln(1+x) = x - \frac{x^2}{2} + \cdots \quad \text{for small } x
\]

(c) Calculate the electric potential at \( P \). Take the limit \( d \gg L \). What does the electric potential look like in this limit? Is this what you expect? Explain. Hint: the following integral may be useful:

\[
\int \frac{x \, dx}{x+a} = x - a \ln(x+a)
\]

Solution

(a) Since \( \lambda = dq \, / \, dx = \alpha x \), the total charge may be obtained by integrating over the entire length of the rod:

\[
Q = \int_0^L dq = \int_0^L \alpha x \, dx = \frac{\alpha L^2}{2}
\]  \hspace{1cm} (0.1)

(b) Consider an infinitesimal charge element \( dq = \lambda \, dx' = \alpha x' \, dx' \) located at a distance \( x' \) from the origin (see figure). The electric field at a point \( P \) due to this element is given by
\[ d\vec{E} = k_e \frac{dq}{r^2} (-\hat{i}) = k_e \frac{\alpha x' dx'}{(x' + d)^2} (-\hat{i}) \]  

where \( r = x' + d \) is the distance between the charge element and \( P \). The electric field points in the negative direction. Integrating over \( d\vec{E} \), we obtain

\[ \vec{E} = k_e \int_0^L \frac{\alpha x' dx'}{(x' + d)^2} (-\hat{i}) = k_e \alpha \left[ \frac{d}{x' + d} + \ln(x' + d) \right] \bigg|_0^L (-\hat{i}) \]

\[ = k_e \alpha \left[ \ln \left(1 + \frac{L}{d}\right) - \frac{L}{L + d} \right] (-\hat{i}) \]  

For \( d \gg L \), the ratio \( L / d \) is small, and Taylor-series expansion yields

\[ \ln \left(1 + \frac{L}{d}\right) = \frac{L}{d} - \frac{1}{2} \frac{L^2}{d^2} + \cdots \]  

and

\[ \frac{L}{L + d} = \frac{L}{d(1 + L/d)} = \frac{L}{d} - \frac{L^2}{d^2} + \cdots \]  

In this limit, the electric field becomes

\[ \vec{E} = k_e \alpha \left( \frac{L}{d} - \frac{1}{2} \frac{L^2}{d^2} - \frac{L}{d} + \frac{L}{d^2} \right) (-\hat{i}) = k_e \alpha \left( \frac{L}{d} \right)^2 (-\hat{i}) = k_e \frac{Q}{d^2} (-\hat{i}) \]  

which is the electric field due to a point charge \( Q \) at a distance \( d \) away from \( P \). This “point-charge” limit is precisely what is expected when \( d \gg L \).

(c) The contribution to the electric potential at \( P \) by a differential charge element \( dq \) is

\[ dV = k_e \frac{dq}{r} = k_e \frac{\alpha x' dx'}{x' + d} \]  

Integrating over the entire length yields

\[ V = k_e \int_0^L \frac{\alpha x' dx'}{x' + d} = k_e \alpha \left[ x' - d \ln(x' + d) \right]_0^L = k_e \alpha \left[ L - d \ln \left(1 + \frac{L}{d}\right) \right] \]
In the limit \( d \gg L \), we have

\[
V = k_e \alpha \left[ L - d \left( \frac{L}{d} - \frac{1}{2} \left( \frac{L}{d} \right)^2 \right) \right] = k_e \alpha \frac{1}{2} \frac{L^2}{d} = k_e \frac{Q}{d}
\]

(0.9)

which is the potential due to a point charge \( Q \) at distance \( d \), as expected.