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Comments

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Improved result for the accuracy of Coulomb's law: A review of the Williams, Faller, and Hill experiment

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About 15 years ago Williams, Faller, and Hill (WFH) carried out a modern version of the classical null experiment which tests Coulomb's law. Their high-precision measurements of the voltage furnish the most stringent upper limit for the parameter δ, which measures a (possible) deviation from the inverse-square form of Coulomb's law. We show that the experiment of WFH is actually about three times more sensitive to the parameter δ than they supposed by carrying out a careful analysis of the geometrical factor involved in the interpretation of their experiment. The new upper limit for δ is $(1.0\pm1.2)\times10^{-16}$.

I. INTRODUCTION

Experimental tests of the accuracy of Coulomb's law have enjoyed a long and interesting history, as summarized in Table I. Most of the experiments have used the principle established in the classic experiment of Cavendish, where a search for a charge or potential difference inside a charged, closed conductor is carried out. The experimenters have often interpreted their results as a means of setting an upper limit for a parameter δ, which is introduced as a (possible) violation of Coulomb's law in the scale invariant form $r^{-2+\delta}$. During the 200 yr period described by Table I the sensitivities of the experiments have increased so much that the upper limit for the absolute value of δ has been pushed down by about 14 orders of magnitude. Interest in experiments that improve the accuracy with which Coulomb's law is known to be true will undoubtedly continue since any positive result for δ would have profound consequences for the structure of Maxwell's theory of electricity and magnetism and the theories based on this.[1]

The smallest value for the upper limit has been obtained by Williams, Faller, and Hill[2] (WFH) and it is the result usually quoted in the textbooks.[3] In order to obtain an estimate of the sensitivity of their experiment to the parameter δ, they used the geometrical factor,

$$M(a,b)=\frac{1}{2}\left[\frac{a}{b}\ln\left(\frac{a+b}{a-b}\right)-\ln\left(\frac{4a^2}{a^2-b^2}\right)\right], \tag{1}$$

TABLE I. Summary of experimental tests of Coulomb's law.

Experimenter (date)	Apparatus or geometry	Upper limit for the parameter δ
Robison (1769)	Gravitational torque on a pivot arm	0.06
Cavendish (1773)	Two concentric metal spheres	0.02
Coulomb (1785)	Torsion balance	0.04
Maxwell (1873)	Two concentric spheres	$\frac{1}{21\,600}$
Plimpton and Lawton (1938)	Two concentric spheres	2×10^{-9}
Cochran (1967)	Concentric cubical conductors	9.2×10^{-12}
Bartlett, Goldhagen, and Phillips (1970)	Five concentric spheres	1.3×10^{-13}
Williams, Faller, and Hill (1971)	Five concentric icosahedrons	$(2.7\pm3.1)\times10^{-16}$

which was derived by Maxwell[4] for two concentric spheres of radii a and b. Since the apparatus of WFH consisted of five concentric conductors, it is apparent that a more refined calculation should be done in order to obtain an accurate result for the sensitivity of the experiment. In particular, it is important that the contributions of all sources of the (possible) anomalous internal fields be included. (Maxwell considered only a single charged source.) Indeed, as we show below, the appropriate generalization of Eq. (1) is the same as that derived by Bartlett, Goldhagen, and Phillips[5] (BGP), whose experiment was also based on five concentric conductors. The correct geometrical factor is about three times larger than that calculated by WFH. This factor of 3 leads to an improvement in sensitivity to the parameter δ by the same amount.

II. THE GEOMETRICAL FACTOR

Our derivation for the geometrical factor for an arrangement of four concentric spheres begins with the exact expression for the potential at distance r from a sphere of radius a containing a uniformly distributed charge (Ref. 6) Q,

$$V(r)=\frac{Q}{2ar(1-\delta^2)}[(r+a)^{1+\delta}-|r-a|^{1+\delta}]\ , \tag{2}$$

where the use of absolute value sign extends the validity of Eq. (2) to the entire range of r. Since both the BGP and WFH experiments were done with alternating currents, the outermost two spheres are assumed to have equal but opposite charges at all times during the experiment. Thus, using superposition, we calculate the potential inside a pair of spheres of radii r_4 (charge Q) and r_3 (charge $-Q$) with $r_4>r_3$, which is given by

$$V(r)=\frac{Q}{2r_4r(1-\delta^2)}[(r_4+r)^{1+\delta}-(r_4-r)^{1+\delta}] -\frac{Q}{2r_3r(1-\delta^2)}[(r_3+r)^{1+\delta}-(r_3-r)^{1+\delta}]\ . \tag{3}$$

The measurement of the induced voltage signal is carried out interior to both of the charged spheres. Thus, we require the expression for the potential difference between a pair of pick points interior to both spheres, that is

$$V(r_2)-V(r_1)=\frac{Q}{2r_4r_2(1-\delta^2)}[(r_4+r_2)^{1+\delta}-(r_4-r_2)^{1+\delta}]-\frac{Q}{2r_4r_1(1-\delta^2)}[(r_4+r_1)^{1+\delta}-(r_4-r_1)^{1+\delta}]$$
$$-\frac{Q}{2r_3r_2(1-\delta^2)}[(r_3+r_2)^{1+\delta}-(r_3-r_2)^{1+\delta}]+\frac{Q}{2r_3r_1(1-\delta^2)}[(r_3+r_1)^{1+\delta}-(r_3-r_1)^{1+\delta}]\ . \tag{4}$$

In order to obtain a simplified expression, it is necessary to expand Eq. (4) in powers of δ using $x^{1+\delta}=x(1+\delta\ln x)$. After judiciously combining the various logarithmic terms, we obtain a first-order expression for the potential difference. This may be written as a ratio,

$$\frac{V(r_2)-V(r_1)}{V(r_3)-V(r_4)}\cong\frac{\delta r_4}{r_4-r_3}[M(r_3,r_2)-M(r_3,r_1)] -\frac{\delta r_3}{r_4-r_3}[M(r_4,r_2)-M(r_4,r_1)]\ , \tag{5}$$

where we have used the lowest-order expression for the capacitance of the two external spheres to eliminate the unknown charge Q. The geometrical factor M in Eq. (5) is the Maxwellian factor defined in Eq. (1) above. From the form of Eq. (5) it is apparent that a single Maxwellian factor is not adequate for an accurate calculation of sensitivity of the four conductor experiments, but that the correct expression involves a combination of four such factors. Our result for the potential ratio agrees with that calculated by BGP in Ref. 5.

III. RESULTS

The experiments of BGP and WFH used five concentric conductors. However, the middle conductor was not an active element in either experiment; its function was to screen out stray electric fields that leaked into the interior of the charged outer pair. Thus, the four-conductor analysis of the previous section is sufficient. WFH maintained a potential difference of 10 kV between their outermost pair of spheres at a frequency of 4 MHz. Using a phase-sensitive detector they searched for a potential difference between the innermost pair of spheres. Their result for the induced signal is $(6.4\pm7.3)\times10^{-13}$ V peak to peak, which is of course consistent with zero. WFH used Eq. (1) to take into account the geometrical factors. Substituting $b=1.6a$ there yields $|M|=0.232$ and leads to their value listed in Table I.

To apply the considerations of Sec. II to the concentric icosahedrons requires some geometrical interpretation. It seems reasonable that one could choose an effective radius to incorporate the differences between the spheres and the icosahedrons. For the time being we follow the considerations of WFH, who were confronted with a similar problem in analyzing their experiment to obtain an upper limit for the photon's rest mass.[7] Their value for the ratio of the effective radius to the length L of the triangular sides depended upon how they determined the effective radius.[8] If the icosahedron is to have the same surface area as the sphere, then

$$R=0.83L\ . \tag{6}$$

If instead the radius of the inscribed sphere is to be used, then

$$R=0.76L\ . \tag{7}$$

WFH suggested that uncertainties in the definition of the effective radius might lead to a difference of 10–20% in the determination of an upper limit to the photon rest mass. Scale invariance makes the problem of geometrical interpretation much simpler for the calculation of the sensitivity to δ. Since all of the factors that appear in Eq. (5) depend on the ratio of the radii, one obtains the same result for the sensitivity using either Eqs. (6) or (7) or any other linear relationship between the effective radius and L that is the same for all conductors.

The lengths of the sides of the four icosahedrons are $23\frac{1}{2}$, 37, $37\frac{1}{4}$, and 50 in., respectively. Using the surface area criterion to define the effective radii leads to the values $r_1 = 49.54$ cm, $r_2 = 78.00$ cm, $r_3 = 78.53$ cm, and $r_4 = 105.41$ cm. The geometrical factor F, which includes all of the expressions on the right-hand side of Eq. (5) except the common factor δ is thus given by $|F| = 0.629$. And the sensitivity of the experiment of WFH to δ can be expressed

$$\left|\frac{V(r_2) - V(r_1)}{V(r_3) - V(r_4)}\right| = |F|\,|\delta| = 0.629\delta \ . \tag{8}$$

The new value of δ determined from Eq. (8) is

$$\delta = (1.0 \pm 1.2) \times 10^{-16} \ , \tag{9}$$

which is a factor of about $2\frac{1}{2}$ times smaller than that published by WFH. The experiment of WFH is about $2\frac{1}{2}$ times more sensitive to the parameter δ than they thought. Equation (9) represents the most stringent upper limit for delta to date.

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[1]An alternative interpretation of Coulomb's law experiments is as a means of setting an upper limit for the photon rest mass. It is possible to reconcile such an interpretation with the special theory of relativity as pointed out by A. S. Goldhaber and N. M. Nieto, Rev. Mod. Phys. **43**, 277 (1971).

[2]E. R. Williams, J. E. Faller, and H. A. Hill, Phys. Rev. Lett. **26**, 721 (1971).

[3]See, for example, J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1975); or R. Resnick and D. Halliday, *Physics, Part II* (Wiley, New York, 1978).

[4]J. C. Maxwell, *A Treatise on Electricity and Magnetism* (Dover, New York, 1954), Vol. I.

[5]D. F. Bartlett, P. E. Goldhagen, and E. A. Phillips, Phys. Rev. D **2**, 483 (1970).

[6]L. P. Fulcher and M. A. Telljohann, Am. J. Phys. **44**, 366 (1976).

[7]Incidentally, the analysis of WFH for the sensitivity of their experiment to a possible photon rest mass is correct.

[8]E. R. Williams, Ph.D. thesis, Wesleyan University, 1970 (unpublished).