立方体电2^n极子在n维空间的电势分布

——电磁学小论文答辩

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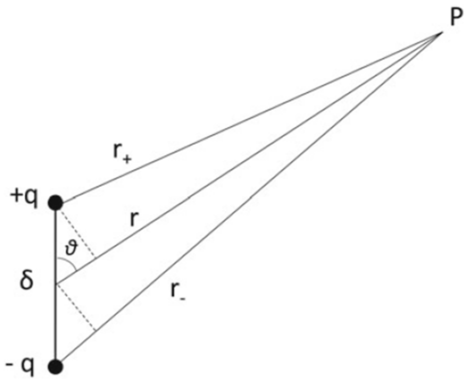
引言

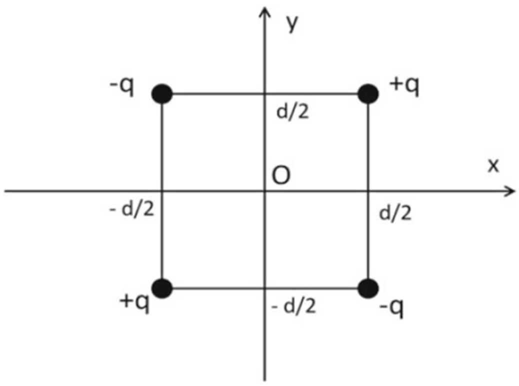
- 电势的笛卡尔多极展开

$$\Phi(r) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r} + \frac{p \cdot r}{r^3} + \frac{1}{2r^5}\sum_{\alpha=1}^{3}\sum_{\beta=1}^{3} Q_{\alpha\beta} r_\alpha r_\beta + \cdots\right)$$

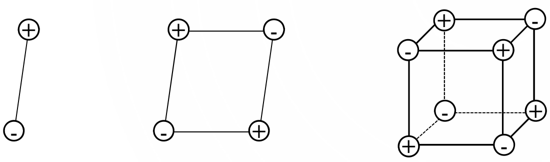
从左到右分别为单极矩、偶极矩、四极矩……

（P68例2.3电偶极子、P432习题1.16条形四极子、1.17面电四极子）





- 立方体电八极子电势分布？



16，32……？

- 传统方法十分繁琐怎么办？

  回归直角坐标，利用对称性，计算十分简洁。

- 立方体电2^n极子在n维空间中的电势分布？（怎么定义电势？）

  上述方法同样奏效。

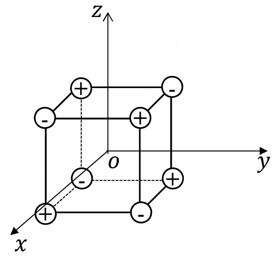
r 是考察点到原点的距离；

l 是相邻两电荷之间的距离；

始终假设$r \gg l$

$$U = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{r}$$

先忽略常系数，令$q = 1$，$l = 2$，建系如图。



$$\left(\frac{1}{\sqrt{(x-1)^2+(y-1)^2+(z-1)^2}}+\frac{1}{\sqrt{(x+1)^2+(y+1)^2+(z-1)^2}}+\frac{1}{\sqrt{(x-1)^2+(y+1)^2+(z+1)^2}}+\frac{1}{\sqrt{(x+1)^2+(y-1)^2+(z+1)^2}}\right)-$$
$$\left(\frac{1}{\sqrt{(x+1)^2+(y+1)^2+(z+1)^2}}+\frac{1}{\sqrt{(x+1)^2+(y-1)^2+(z-1)^2}}+\frac{1}{\sqrt{(x-1)^2+(y+1)^2+(z-1)^2}}+\frac{1}{\sqrt{(x-1)^2+(y+1)^2+(z-1)^2}}\right)$$

$$\left(\frac{1}{\sqrt{(x-1)^2+(y-1)^2+(z-1)^2}}+\frac{1}{\sqrt{(x+1)^2+(y+1)^2+(z-1)^2}}+\frac{1}{\sqrt{(x-1)^2+(y+1)^2+(z+1)^2}}+\frac{1}{\sqrt{(x+1)^2+(y-1)^2+(z+1)^2}}\right)-$$
$$\left(\frac{1}{\sqrt{(x+1)^2+(y+1)^2+(z+1)^2}}+\frac{1}{\sqrt{(x+1)^2+(y-1)^2+(z-1)^2}}+\frac{1}{\sqrt{(x-1)^2+(y+1)^2+(z-1)^2}}+\frac{1}{\sqrt{(x-1)^2+(y+1)^2+(z-1)^2}}\right)$$

利用对称性将上式写成如下紧凑形式：

$$\boxed{\sum_{\varepsilon_x,\varepsilon_y,\varepsilon_z\in\{\pm1\}} \frac{\varepsilon_x\varepsilon_y\varepsilon_z}{\sqrt{(x-\varepsilon_x)^2+(y-\varepsilon_y)^2+(z-\varepsilon_z)^2}}}$$

$$= \sum_{\varepsilon_x,\varepsilon_y,\varepsilon_z\in\{\pm1\}} \frac{\varepsilon_x\varepsilon_y\varepsilon_z}{\sqrt{x^2+y^2+z^2}\cdot\left(1+\cancel{\frac{3}{x^2+y^2+z^2}}-\frac{2(x\varepsilon_x+y\varepsilon_y+z\varepsilon_z)}{x^2+y^2+z^2}\right)^{\frac{1}{2}}}$$

记分母括号中第二项为$S_{\varepsilon_x,\varepsilon_y,\varepsilon_z}$, 则$|S_{\varepsilon_x,\varepsilon_y,\varepsilon_z}| \leq \frac{6\cdot\max\{|x|,|y|,|z|\}}{x^2+y^2+z^2} \leq \frac{6}{\sqrt{x^2+y^2+z^2}} \ll 1$. 可以泰勒展开。

$$\boxed{\frac{1}{\sqrt{1-x}} = \sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}(-x)^n = \sum_{n=0}^{\infty}\frac{(2n-1)!!}{2^n\cdot n!}x^n, \qquad (|x|<1)}$$

$$\sum_{\varepsilon_x,\varepsilon_y,\varepsilon_z\in\{\pm1\}} \frac{\varepsilon_x\varepsilon_y\varepsilon_z}{\sqrt{x^2+y^2+z^2}} \cdot \left(1+\frac{1}{2}S_{\varepsilon_x,\varepsilon_y,\varepsilon_z}+\frac{3}{8}S_{\varepsilon_x,\varepsilon_y,\varepsilon_z}{}^2+\frac{5}{16}S_{\varepsilon_x,\varepsilon_y,\varepsilon_z}{}^3+\cancel{o(S_{\varepsilon_x,\varepsilon_y,\varepsilon_z}{}^3)}\right)$$

别急着展开，稍微观察即知上式是个纸老虎，求和之后只有一项非零。

$$\sum_{\varepsilon_x,\varepsilon_y,\varepsilon_z\in\{\pm1\}} \varepsilon_x{}^1\varepsilon_y{}^?\varepsilon_z{}^?x^?y^?z^? = \sum_{\varepsilon_x\in\{\pm1\}}\varepsilon_x{}^1 \sum_{\varepsilon_y,\varepsilon_z\in\{\pm1\}} \varepsilon_y{}^?\varepsilon_z{}^?x^?y^?z^? = (1+(-1)) \sum_{\varepsilon_y,\varepsilon_z\in\{\pm1\}} \varepsilon_y{}^?\varepsilon_z{}^?x^?y^?z^? = 0\,!$$

$$\sum_{\varepsilon_x,\varepsilon_y,\varepsilon_z\in\{\pm1\}} \varepsilon_x\varepsilon_y\varepsilon_z x\varepsilon_x y\varepsilon_y z\varepsilon_z = \sum_{\varepsilon_x,\varepsilon_y,\varepsilon_z\in\{\pm1\}} xyz = 8xyz \qquad (a+b+c)^3 = 6abc+\cdots$$

$$\boxed{S_{\varepsilon_x,\varepsilon_y,\varepsilon_z} = \frac{2(x\varepsilon_x+y\varepsilon_y+z\varepsilon_z)}{x^2+y^2+z^2}}$$

$$\sum_{\varepsilon_x,\varepsilon_y,\varepsilon_z\in\{\pm1\}} \frac{\varepsilon_x\varepsilon_y\varepsilon_z}{\sqrt{x^2+y^2+z^2}} \cdot \left(1+\frac{1}{2}S_{\varepsilon_x,\varepsilon_y,\varepsilon_z}+\frac{3}{8}S_{\varepsilon_x,\varepsilon_y,\varepsilon_z}{}^2+\frac{5}{16}S_{\varepsilon_x,\varepsilon_y,\varepsilon_z}{}^3\right)$$

$$= \sum_{\varepsilon_x,\varepsilon_y,\varepsilon_z\in\{\pm1\}} \frac{1}{\sqrt{x^2+y^2+z^2}^7} \cdot \frac{5}{16} \cdot 8 \cdot 6 \cdot xyz = \frac{120 \cdot xyz}{\sqrt{x^2+y^2+z^2}^7}$$

$$\boxed{\sum_{\varepsilon_x,\varepsilon_y,\varepsilon_z\in\{\pm1\}} \frac{\varepsilon_x\varepsilon_y\varepsilon_z}{\sqrt{(x-\varepsilon_x)^2+(y-\varepsilon_y)^2+(z-\varepsilon_z)^2}} \approx \frac{120\cdot xyz}{\sqrt{x^2+y^2+z^2}^7}}$$

$$\mathrm{U} = \frac{q}{4\pi\varepsilon_0}\cdot \sum_{\varepsilon_x,\varepsilon_y,\varepsilon_z\in\{\pm1\}} \frac{\varepsilon_x\varepsilon_y\varepsilon_z}{\sqrt{\left(x-\varepsilon_x\cdot\frac{l}{2}\right)^2+\left(y-\varepsilon_y\cdot\frac{l}{2}\right)^2+\left(z-\varepsilon_z\cdot\frac{l}{2}\right)^2}}$$

$$= \frac{q}{4\pi\varepsilon_0}\cdot\frac{2}{l}\cdot \sum_{\varepsilon_x,\varepsilon_y,\varepsilon_z\in\{\pm1\}} \frac{\varepsilon_x\varepsilon_y\varepsilon_z}{\sqrt{\left(x\cdot\frac{2}{l}-\varepsilon_x\right)^2+\left(y\cdot\frac{2}{l}-\varepsilon_y\right)^2+\left(z\cdot\frac{2}{l}-\varepsilon_z\right)^2}}$$

$$\approx \frac{q}{4\pi\varepsilon_0}\cdot\frac{2}{l}\cdot\frac{120\cdot x\cdot\frac{2}{l}\cdot y\cdot\frac{2}{l}\cdot z\cdot\frac{2}{l}}{\sqrt{(x\cdot\frac{2}{l})^2+(y\cdot\frac{2}{l})^2+(z\cdot\frac{2}{l})^2}^7} = \frac{q}{4\pi\varepsilon_0}\cdot\frac{15l^3xyz}{\sqrt{x^2+y^2+z^2}^7}$$

至此三维情况解决

$$U = \frac{q}{4\pi\varepsilon_0}\cdot\frac{15l^3xyz}{\sqrt{x^2+y^2+z^2}^7}$$ (直角坐标)

$$U = \frac{q}{4\pi\varepsilon_0}\cdot\frac{15l^3\cos\alpha\cos\beta\cos\gamma}{r^4}$$ (立体角)

$$U = \frac{q}{4\pi\varepsilon_0}\cdot\frac{15l^3\sin\varphi\cos\varphi\sin\theta(\cos\theta)^2}{r^4}$$ (极坐标)

立方体电2^n极子在n维空间中的电势分布

- 定义（n维空间中的立方体电2^n极子）2^n个等量电荷置于n维立方体的2^n个顶点，电性与坐标分量乘积一致。
- 定义(n维空间中的电势) $U_{n,m} = k\frac{q}{r^m}$ ($m = n - 2$ 为库仑势).
- 例. n=4, 坐标分量绝对值都是1.

正：$(1,1,1,1)$, $(-1,-1,1,1)$, $(-1,1,-1,1)$, $(-1,1,1,-1)$, $(1,-1,-1,1)$, $(1,-1,1,-1)$, $(1,1,-1,-1)$, $(-1,-1,-1,-1)$;

负：$(-1,1,1,1)$, $(1,-1,1,1)$, $(1,1,-1,1)$, $(1,1,1,-1)$, $(-1,-1,-1,1)$, $(-1,-1,1,-1)$, $(-1,1,-1,-1)$, $(1,-1,-1,-1)$;

$$\sum_{\substack{|\varepsilon_i|=1\\ i\in[n]}} \frac{\prod_{i=1}^n \varepsilon_i}{\left(\sqrt{\sum_{i=1}^n (x_i-\varepsilon_i)^2}\right)^m} = \sum_{\substack{|\varepsilon_i|=1\\ i\in[n]}} \frac{\prod_{i=1}^n \varepsilon_i}{\left(\sum_{i=1}^n {x_i}^2 - 2\sum_{i=1}^n x_i\varepsilon_i + n\right)^{\frac{m}{2}}}$$

$$\approx \sum_{\substack{|\varepsilon_i|=1\\ i\in[n]}} \frac{\prod_{i=1}^n \varepsilon_i}{\left(\sum_{i=1}^n {x_i}^2\right)^{\frac{m}{2}}}\left(1 - 2\frac{\sum_{i=1}^n x_i\varepsilon_i}{\sum_{i=1}^n {x_i}^2}\right)^{-\frac{m}{2}} = \sum_{\substack{|\varepsilon_i|=1\\ i\in[n]}} \frac{\prod_{i=1}^n \varepsilon_i}{\left(\sum_{i=1}^n {x_i}^2\right)^{\frac{m}{2}}}\left(1 - S(\vec{\varepsilon})\right)^{-\frac{m}{2}}$$

$$\approx \sum_{\substack{|\varepsilon_i|=1\\ i\in[n]}} \frac{\prod_{i=1}^n \varepsilon_i}{\left(\sum_{i=1}^n {x_i}^2\right)^{\frac{m}{2}}}\sum_{k=0}^{n}\binom{-\frac{m}{2}}{k}(-1)^k S(\vec{\varepsilon})^k = \sum_{\substack{|\varepsilon_i|=1\\ i\in[n]}} \frac{\prod_{i=1}^n \varepsilon_i}{\left(\sum_{i=1}^n {x_i}^2\right)^{\frac{m}{2}}}\binom{-\frac{m}{2}}{n}(-1)^n S(\vec{\varepsilon})^n$$

$$= \sum_{\substack{|\varepsilon_i|=1\\ i\in[n]}} \frac{1}{\left(\sum_{i=1}^n {x_i}^2\right)^{\frac{m}{2}}}\binom{-\frac{m}{2}}{n}(-1)^n n!\,\frac{2^n}{\left(\sum_{i=1}^n {x_i}^2\right)^n}\prod_{i=1}^n x_i = 2^n\prod_{i=1}^n (m+2(i-1))\,\frac{\prod_{i=1}^n x_i}{\left(\sqrt{\sum_{i=1}^n {x_i}^2}\right)^{2n+m}}$$

$$\tilde{C}_{n,m} = 2^n\prod_{i=1}^n (m+2(i-1))$$

$$\boxed{\sum_{\substack{|\varepsilon_i|=1\\ i\in[n]}} \frac{\prod_{i=1}^{n}\varepsilon_i}{(\sqrt{\sum_{i=1}^{n}(x_i-\varepsilon_i)^2})^m} \approx \tilde{C}_{n,m}\frac{\prod_{i=1}^{n}x_i}{(\sqrt{\sum_{i=1}^{n}{x_i}^2})^{2n+m}}}$$

$$U_{n,m}(\vec{r}) = kq\sum_{\substack{|\varepsilon_i|=1\\ i\in[n]}} \frac{\prod_{i=1}^{n}\varepsilon_i}{(\sqrt{\sum_{i=1}^{n}(x_i-\varepsilon_i\frac{l}{2})^2})^m} = kq(\frac{2}{l})^m\sum_{\substack{|\varepsilon_i|=1\\ i\in[n]}} \frac{\prod_{i=1}^{n}\varepsilon_i}{(\sqrt{\sum_{i=1}^{n}(x_i\frac{2}{l}-\varepsilon_i)^2})^m}$$

$$\approx kq\left(\frac{2}{l}\right)^m\tilde{C}_{n,m}\frac{\prod_{i=1}^{n}(x_i\frac{2}{l})}{\left(\sqrt{\sum_{i=1}^{n}(x_i\frac{2}{l})^2}\right)^{2n+m}} = C_{n,m}kql^n\frac{\prod_{i=1}^{n}x_i}{(\sqrt{\sum_{i=1}^{n}{x_i}^2})^{2n+m}}$$

$$C_{n,m} = \frac{\tilde{C}_{n,m}}{2^n} = \prod_{i=1}^{n}(m+2(i-1))$$

$$\boxed{U_{n,m}(\vec{r}) = C_{n,m}kql^n\frac{\prod_{i=1}^{n}x_i}{(\sqrt{\sum_{i=1}^{n}{x_i}^2})^{2n+m}} \sim \frac{1}{r^{n+m}}}$$

$$C_{n,m} = \prod_{i=1}^{n}(m+2(i-1))$$

立方体电2^n极子在n维空间的电势分布（库伦势)

$m = n - 2$

$$U_{n,n-2}(\vec{r}) = C_{n,n-2}kql^n \frac{\prod_{i=1}^{n} x_i}{(\sqrt{\sum_{i=1}^{n} {x_i}^2})^{3n-2}} \sim \frac{1}{r^{2n-2}}$$

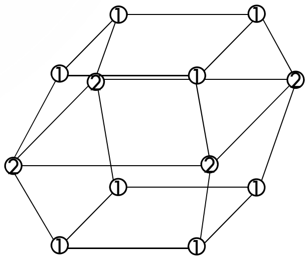
$$C_{n,n-2} = \prod_{i=1}^{n}(n - 2 + 2(i - 1))$$

如果先有m后有n呢？

$$U_{n,m}(\vec{r}) \sim \frac{1}{r^{n+m}}$$

$m = 1$

$$U_{n,1}(\vec{r}) \sim \frac{1}{r^{n+1}}$$



$m = 1, k = \frac{1}{4\pi\varepsilon_0}$

$$U_{n,1} = \frac{q}{4\pi\varepsilon_0} \cdot \frac{(2n-1)!! \cdot l^n \cdot \prod_{i=1}^{n} x_i}{\left(\sqrt{\sum_{i=1}^{n} {x_i}^2}\right)^{2n+1}}$$

- (电偶极子在所在直线电势分布) $n = 1$

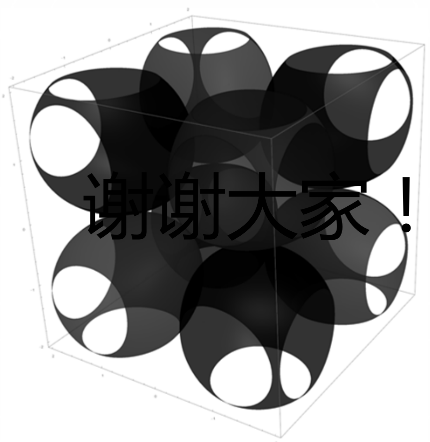
$$U_1 = \frac{q}{4\pi\varepsilon_0} \cdot \frac{lx}{|x|^3} = \frac{q}{4\pi\varepsilon_0} \cdot \frac{l \cdot sgn(x)}{r^2}$$

- (面电四极子在所在平面电势分布) $n = 2$

$$U_2 = \frac{q}{4\pi\varepsilon_0} \cdot \frac{3l^2 xy}{\sqrt{x^2+y^2}^5} = \frac{q}{4\pi\varepsilon_0} \cdot \frac{3l^2 \cos\theta \sin\theta}{r^3}$$

- (立方体电八极子三维空间电势分布) $n = 3$

$$U_3 = \frac{q}{4\pi\varepsilon_0} \cdot \frac{15l^3 xyz}{\sqrt{x^2+y^2+z^2}^7} = \frac{q}{4\pi\varepsilon_0} \cdot \frac{15l^3 \sin\varphi \cos\varphi \sin\theta (\cos\theta)^2}{r^4}$$



U[x_, y_, z_] :=

$$\left(\frac{1}{\sqrt{(x-1)^2+(y-1)^2+(z-1)^2}} + \frac{1}{\sqrt{(x+1)^2+(y+1)^2+(z-1)^2}} + \frac{1}{\sqrt{(x-1)^2+(y+1)^2+(z+1)^2}} + \frac{1}{\sqrt{(x+1)^2+(y-1)^2+(z+1)^2}}\right) -$$

$$\left(\frac{1}{\sqrt{(x+1)^2+(y+1)^2+(z+1)^2}} + \frac{1}{\sqrt{(x-1)^2+(y-1)^2+(z+1)^2}} + \frac{1}{\sqrt{(x+1)^2+(y-1)^2+(z-1)^2}} + \frac{1}{\sqrt{(x-1)^2+(y+1)^2+(z-1)^2}}\right)$$

```
ContourPlot3D[{U[x, y, z] == 0.25, U[x, y, z] == -0.25}, {x, -2, 2}, {y, -2, 2}, {z, -2, 2}, Mesh → None,
三维等高线                                                                                网格   无
 ContourStyle → {{Purple, Opacity[0.8], Specularity[Blue, 30]}, {Black, Opacity[0.8], Specularity[Red, 30]}}]
 等高线样式       紫色     不透明度        反射度        蓝色          黑色   不透明度        反射度       红色
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