矩阵基本知识

- 矩阵基本运算
- 线性相关性
- 矩阵初等变换
- 随机向量与随机矩阵
- 矩阵直和
- 矩阵Hardmard积
- 矩阵Kronecker积

矩阵基本运算

- 矩阵共轭转置
- 矩阵加法

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1N} \\ a_{21} & a_{22} & \dots & a_{2N} \\ \dots & \dots & \dots & \dots \\ a_{M1} & a_{M2} & \dots & a_{MN} \end{pmatrix}^H = \begin{pmatrix} a_{11}^* & a_{21}^* & \dots & a_{M1}^* \\ a_{12}^* & a_{22}^* & \dots & a_{M2}^* \\ \dots & \dots & \dots & \dots \\ a_{1N}^* & a_{2N}^* & \dots & a_{MN}^* \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1N} \\ a_{21} & a_{22} & \dots & a_{2N} \\ \dots & \dots & \dots & \dots \\ a_{M1} & a_{M2} & \dots & a_{MN} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1N} \\ b_{21} & b_{22} & \dots & b_{2N} \\ \dots & \dots & \dots & \dots \\ b_{M1} & b_{M2} & \dots & b_{MN} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \dots & a_{1N}+b_{1N} \\ a_{21}+b_{21} & a_{22}+b_{22} & \dots & a_{2N}+b_{2N} \\ \dots & \dots & \dots & \dots \\ a_{M1}+b_{M1} & a_{M2}+b_{M2} & \dots & a_{MN}+b_{MN} \end{pmatrix}$$

矩阵基本运算

- 矩阵标量乘法
- 矩阵乘法

$$\alpha\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \cdots & \cdots & \cdots & \cdots \\ a_{M1} & a_{M2} & \cdots & a_{MN} \end{pmatrix} = \begin{pmatrix} \alpha a_{11} & \alpha a_{12} & \cdots & \alpha a_{1N} \\ \alpha a_{21} & \alpha a_{22} & \cdots & \alpha a_{2N} \\ \cdots & \cdots & \cdots & \cdots \\ \alpha a_{M1} & \alpha a_{M2} & \cdots & \alpha a_{MN} \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \cdots & \cdots & \cdots & \cdots \\ a_{M1} & a_{M2} & \cdots & a_{MN} \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1N} \\ b_{21} & b_{22} & \cdots & b_{2K} \\ \cdots & \cdots & \cdots & \cdots \\ b_{N1} & b_{N2} & \cdots & b_{NK} \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^{N} a_{1k}b_{k1} & \sum_{k=1}^{N} a_{1k}b_{k2} & \cdots & \sum_{k=1}^{N} a_{1k}b_{kK} \\ \sum_{k=1}^{N} a_{2k}b_{k1} & \sum_{k=1}^{N} a_{2k}b_{k2} & \cdots & \sum_{k=1}^{N} a_{2k}b_{kK} \\ \cdots & \cdots & \cdots & \cdots \\ \sum_{k=1}^{N} a_{Mk}b_{k1} & \sum_{k=1}^{N} a_{Mk}b_{k2} & \cdots & \sum_{k=1}^{N} a_{Mk}b_{kK} \end{pmatrix}$$

矩阵基本运算

- 加法交换律：$A + B = B + A$
- 乘法结合律：$(AB)C = A(BC)$
- 乘法分配律：$(A + B)C = AC + BC$

  $A(B + C) = AB + AC$
- 逆矩阵：$AB = BA = I$；
- 幂等矩阵：$AA = A$；幂等矩阵例子？
- 幂单矩阵：$AA = I$；幂单矩阵例子？
- 幂零矩阵：$AA = 0$；

矩阵基本运算

- 幂等矩阵：AA = A；幂等矩阵例子？
  - A 可逆么？如果可逆，A = ？
  - 如果A是Hermitian矩阵，那么A = ？
- 幂单矩阵：AA = I；幂单矩阵例子？
  - A 可逆么？
  - 如果A是Hermitian矩阵，那么A = ？
- 幂零矩阵：AA = 0；
  - 非零 的幂零矩阵？

矩阵函数

- 考虑矩阵的Taylor展开

$$f(x)=\sum_{n=0}^{+\infty}\frac{f^{(n)}(0)}{n!}x^n$$

- 有如下的矩阵函数定义

$$f(A)=\sum_{n=0}^{+\infty}\frac{f^{(n)}(0)}{n!}A^n$$

- 举例：

$$e^A=\sum_{n=0}^{+\infty}\frac{A^n}{n!} \qquad e^{At}=\sum_{n=0}^{+\infty}\frac{A^n}{n!}t^n \qquad \ln(I+A)=\sum_{n=0}^{+\infty}\frac{(-1)^{n-1}}{n}A^n$$

矩阵线性相关性

- **线性无关，线性相关**

$$\sum_{n=1}^{N} x_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \cdots \\ a_{Mn} \end{pmatrix} = 0 \Leftrightarrow x_n = 0, \forall 1 \le n \le N.$$

- **线性无关，线性方程**

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \cdots & \cdots & \cdots & \cdots \\ a_{M1} & a_{M2} & \cdots & a_{MN} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_N \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_N \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 0 \end{pmatrix}$$

初等行变换

- 初等行变换
  - 交换矩阵任意两行
  - 一行乘以非零常数
  - 某行乘以非零常数，加到另外一行
- 行等价矩阵：矩阵A通过行初等变换→矩阵B
- 初等行变换的矩阵表示

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & a & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

随机向量

- 向量每个元素为随机变量

$$\bar{x} = (x_1 \quad x_2 \quad \ldots \quad x_N)^T$$

- 随机向量期望

$$E\bar{x} = (Ex_1 \quad Ex_2 \quad \ldots \quad Ex_N)^T$$

- 随机向量自相关与协方差

$$R_x = E\left[\bar{x}\bar{x}^H\right] \qquad \operatorname{cov}(\bar{x}, \bar{x}) = E\left[(\bar{x} - \mathrm{E}\bar{x})(\bar{x} - \mathrm{E}\bar{x})^H\right]$$

$$R_{xy} = E\left[\bar{x}\bar{y}^H\right] \qquad \operatorname{cov}(\bar{x}, \bar{y}) = E\left[(\bar{x} - \mathrm{E}\bar{x})(\bar{y} - \mathrm{E}\bar{y})^H\right]$$

高斯随机向量

- 高斯随机变量

$$\bar{x} = \begin{pmatrix} x_1 & x_2 & \dots & x_N \end{pmatrix}^T$$

- 随机向量期望

$$f(\bar{x}) = \frac{1}{(2\pi)^{N/2} |\Gamma|^{1/2}} e^{-\frac{(\bar{x}-\bar{\mu})^T \Gamma^{-1} (\bar{x}-\bar{\mu})}{2}}$$

- 高斯随机的线性变换，仍然为高斯随机向量

$$\bar{y} = A\bar{x} = A\begin{pmatrix} x_1 & x_2 & \dots & x_N \end{pmatrix}^T$$

随机矩阵

- 矩阵每个元素为随机变量

$$A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \cdots & \cdots & \cdots & \cdots \\ a_{M1} & a_{M2} & \cdots & a_{MN} \end{pmatrix}$$

- 随机矩阵期望
  - 每个元素的期望
- 随机矩阵举例

$$A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \cdots & \cdots & \cdots & \cdots \\ a_{M1} & a_{M2} & \cdots & a_{MN} \end{pmatrix}\in\{0,1\}^{M\times N} \quad A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \cdots & \cdots & \cdots & \cdots \\ a_{M1} & a_{M2} & \cdots & a_{MN} \end{pmatrix}\in[0,1]^{M\times N}$$

随机矩阵

- **随机矩阵的秩**
  - **随机矩阵满秩的概率？**

$$A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \cdots & \cdots & \cdots & \cdots \\ a_{M1} & a_{M2} & \cdots & a_{MN} \end{pmatrix} \in \{0,1\}^{M\times N}$$

$$A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \cdots & \cdots & \cdots & \cdots \\ a_{M1} & a_{M2} & \cdots & a_{MN} \end{pmatrix} \in GF(q)^{M\times N}$$

$$A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \cdots & \cdots & \cdots & \cdots \\ a_{M1} & a_{M2} & \cdots & a_{MN} \end{pmatrix} \in [0,1]^{M\times N}$$

矩阵直和

- **矩阵直和定义**

$$[A]_{m\times m}\oplus[B]_{n\times n}=\begin{bmatrix}A & 0\\ 0 & B\end{bmatrix}$$

- **矩阵直和性质**

$$(A\pm B)\oplus(C\pm D)=A\oplus C\pm B\oplus D$$

$$(A\oplus B)(C\oplus D)=AC\oplus BD$$

$$(A\oplus B)\oplus C=A\oplus(B\oplus C)$$

$$(A\oplus B)^{-1}=A^{-1}\oplus B^{-1}$$

矩阵Hardmard积

- 矩阵Hardmard积定义 $[A*B]_{ij} = a_{ij}b_{ij}$

$$[A*B]=\begin{pmatrix} a_{11}b_{11} & a_{12}b_{12} & \cdots & a_{1N}b_{1N} \\ a_{21}b_{21} & a_{22}b_{22} & \cdots & a_{2N}b_{2N} \\ \cdots & \cdots & \cdots & \cdots \\ a_{M1}b_{M1} & a_{M2}b_{M2} & \cdots & a_{MN}b_{MN} \end{pmatrix}$$

- 矩阵Hardmard积性质

$$A*B=B*A$$

$$(A*B)*C=A*(B*C)$$

$$A*(B\pm C)=A*B\pm A*C$$

矩阵Kronecker积

- **矩阵Kronecker积定义**

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1N}B \\ a_{21}B & a_{22}B & \cdots & a_{2N}B \\ \cdots & \cdots & \cdots & \cdots \\ a_{M1}B & a_{M2}B & \cdots & a_{MN}B \end{pmatrix}$$

- **矩阵Kronecker积性质**

$$A \otimes (B \pm C) = A \otimes B \pm A \otimes C \quad (B \pm C) \otimes A = B \otimes A \pm C \otimes A$$

$$AB \otimes CD = (A \otimes C)(B \otimes D) \qquad A^{-1} \otimes B^{-1} = (A \otimes B)^{-1}$$

$$(A \otimes B) \otimes C = A \otimes (B \otimes C)$$

$$rank(A \otimes B) = rank(A) rank(B)$$

矩阵Kronecker积

- **矩阵Kronecker积性质证明**

$$(A\otimes C)(B\otimes D)$$

$$=\begin{pmatrix} a_{11}C & a_{12}C & \cdots & a_{1N}C \\ a_{21}C & a_{22}C & \cdots & a_{2N}C \\ \cdots & \cdots & \cdots & \cdots \\ a_{M1}C & a_{M2}C & \cdots & a_{MN}C \end{pmatrix}\begin{pmatrix} b_{11}D & b_{12}D & \cdots & b_{1N}D \\ b_{21}D & b_{22}D & \cdots & b_{2N}D \\ \cdots & \cdots & \cdots & \cdots \\ b_{N1}D & b_{N2}D & \cdots & b_{NK}D \end{pmatrix}$$

$$=(AB)\otimes(\mathrm{C}\,D)$$

- **矩阵分析的语言**