Outline

- Rayleigh Quotient
- Generalized Rayleigh Quotient
- Perturbation of Eigenvalue and Eigenvector

Rayleigh Quotient

- **Definition of Rayleigh Quotient**

$$R(x, A) = \frac{x^H A x}{x^H x}$$

- **Properties on the Rayleigh Quotient**

$$R(\alpha x, \beta A) = \beta R(x, A)$$

$$R(x, A - \alpha I) = R(x, A) - \alpha$$

$$x \perp \left(A - R(x, A) I\right) x$$

$$\left\| \left(A - R(x, A) I\right) x \right\| \le \left\| \left(A - \mu I\right) x \right\|$$

- **Hermitian Matrix A**

$$\lambda_{\min}(A) \le R(x, A) \le \lambda_{\max}(A)$$

Rayleigh Quotient

- **Properties on the Rayleigh Quotient**

$$\left\|\left(A-R(x,A)I\right)x\right\|\le\left\|\left(A-\mu I\right)x\right\|$$

$$\left\|\left(A-R(x,A)I\right)x\right\|^2=\min_{\mu}\left\|\left(A-\mu I\right)x\right\|^2$$

- **Hermitian Matrix A**

$$\left.\begin{array}{ll}\max & x^H Ax \\ s.t. & x^H x=1\end{array}\right\}\Rightarrow L(\lambda)=x^H Ax-\lambda x^H x\Rightarrow\frac{\partial L(\lambda)}{\partial x}=Ax-\lambda x$$

$$\lambda_{\min}(A)\le x^H Ax\le\lambda_{\max}(A)$$

Numerical Range

- **Numerical Range of a Hermitian Matrix A**

$$W(A) = \left\{ \frac{x^H A x}{x^H x} : x \neq 0 \right\}$$

- **Properties on Numerical Range**

$$W(A+B) \subseteq W(A) + W(B)$$

$$W(A+\beta I) = W(A) + \beta$$

Max-Min Theorem

- **Consider Eigenvalues Ranking**

$$\lambda_1 \le \lambda_2 \le \ldots \le \lambda_n$$

- **Min-Max Theorem**

$$\lambda_k = \min_U \left\{ \max_x \{ R_A(x) \mid x \in U,\ x \ne 0 \} \mid \dim(U) = k \right\}$$

$$\lambda_k = \max_U \left\{ \min_x \{ R_A(x) \mid x \in U,\ x \ne 0 \} \mid \dim(U) = n - k + 1 \right\}$$

- **Min-Max Theorem**

$$\lambda_n = \max_x \{ R_A(x),\ x \ne 0 \}$$

$$\lambda_1 = \min_x \{ R_A(x),\ x \ne 0 \}$$

Generalized Rayleigh Quotient

- **Definition of Generalized Rayleigh Quotient**

$$R(x) = \frac{x^H A x}{x^H B x}$$

B is positive-definite matrix

- **Properties on the Rayleigh Quotient**

$$R(x, A) = \frac{\tilde{x}^H \left(B^{-1/2}\right)^H A \left(B^{-1/2}\right) \tilde{x}}{\tilde{x}^H \tilde{x}}$$

- **Eigenvalue of $B^{-1}A$**

Applications on the Rayleigh Quotient

- **Receiver-side Beamforming**
  - **Desired signal x_s, interference I_n, noise v**

$$y = x_s + \sum_{n=1}^{N} I_n + v$$

  - **Linear receiver**

$$w^T y = w^T x_s + \sum_{n=1}^{N} w^T I_n + w^T v$$

  - **SINR**

$$SINR = \frac{w^H E\left[x_s x_s^H\right] w}{w^H \left(E\left[\sum_{n=1}^{N} I_n I_n^H\right] + \sigma^2 I \right) w}$$

Perturbation of Eigenvalues

- **Bauer-Fike Theorem**
- **Consider the eigenvalue decomposition of A**
  - **$A = V\Sigma V^{-1}$**
  - **The eigenvalue decomposition of A + ΔA, μ be the eigenvalues, we have that there exists λ(A), such that**

$$|\lambda(A) - \mu| \le \|V\|_p \|V^{-1}\|_p \|\Delta A\|_p$$

Perturbation of Eigenvalues

- **Proof: we have the following**

$$0 = \left|A + \Delta A - \mu I\right| = \left|\Sigma - \mu I\right| \left|(\Sigma - \mu I)^{-1} V^{-1} \Delta A V + I\right|$$

We have that -1 is an eigenvalue of the following matrix

Then we have the following $(\Sigma - \mu I)^{-1} V^{-1} \Delta A V$

$$\left\|(\Sigma - \mu I)^{-1}\right\| \left\|V^{-1}\right\| \left\|\Delta A\right\| \left\|V\right\| \geq 1$$

which leads to the result

$$\left\|(\Sigma - \mu I)^{-1}\right\| = \left(\min_{\lambda(A)} \left|\lambda(A) - \mu\right|\right)^{-1}$$

$$\Rightarrow \min_{\lambda(A)} \left|\lambda(A) - \mu\right| \leq \left\|V^{-1}\right\| \left\|\Delta A\right\| \left\|V\right\|$$

Perturbation of Eigenvector

- **Discussion: Eigenvectors can be sufficiently small for sufficiently small perturbation of matrix?**