Homework #2

Please do the problems as marked below!

Problems

1.3 Construct a finite-dimensional subspace V_h of V consisting of functions which are quadratic on each subinterval I_j of a partition of $I=(0, 1)$. How can one choose the parameters to describe such functions? Find the corresponding basis functions. Then formulate a finite element method for (*D*) using the space V_h and write down the corresponding linear system of equations in case of a uniform partition.

1.4 Formulate a difference method for (*D*) and compare with (1.6).

● (1.5) Consider the boundary value problem

$$\frac{d^4u}{dx^4}=f, \qquad 0<x<1, \qquad (1.7)$$
$$u(0)=u'(0)=u(1)=u'(1)=0.$$

Here u represents e g the deflection of a clamped beam subject to a transversal force with intensity f (see Fig 1.5).

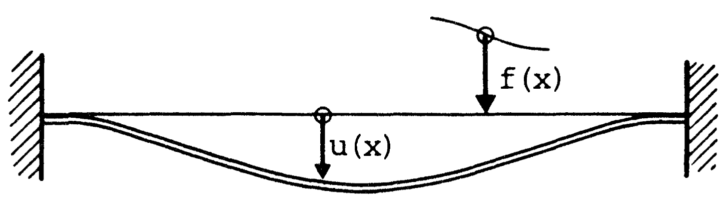

Fig 1.5

(a) In mechanics this beam problem would naturally be formulated as follows:

(1.8a) $M=u''$, $\quad 0<x<1$,
(1.8b) $M''=f$, $\quad 0<x<1$,
(1.8c) $u(0)=u'(0)=u(1)=u'(1)=0$.

What does here the quanity M represent and what is the mechanical interpretation of (1.8a–c)?

(b) Show that the problem (1.7) can be given the following variational formulation: Find $u \in W$ such that

$$(u'', v'') = (f, v) \quad \forall v \in W,$$

where $W = \{v:\ v$ and v' are continuous on $[0,1]$, v'' is piecewise continuous and $v(0) = v'(0) = v(1) = v'(1) = 0\}$.

(c) For $I = [a, b]$ an interval, define

$$P_3(I) = \{v:\ v \text{ is a polynomial of degree} \leq 3 \text{ on } I, \text{ ie, } v \text{ has the form } v(x) = a_3x^3 + a_2x^2 + a_1x + a_0,\ x \in I \text{ where } a_i \in R\}.$$

Show that $v \in P_3(I)$ is uniquely determined by the values $v(a)$, $v'(a)$, $v(b)$, $v'(b)$. Find the corresponding basis functions (the basis function corresponding to the value $v(a)$ is the cubic polynomial v such that $v(a) = 1$, $v'(a) = 0$, $v(b) = v'(b) = 0$, etc).

(d) Starting from (c) construct a finite-dimensional subspace W_h of W consisting of piecewise cubic functions. Specify suitable parameters to describe the functions in W_h and determine the corresponding basis functions.

(e) Formulate a finite element method for (1.7) based on the space W_h. Find the corresponding linear system of equations in the case of a uniform partition. Determine the solution in e g the case of two intervals and f constant. Compare with the exact solution.

homework #2 (continued)

Do all the problems on this page!

Consider the following two-points boundary value problem in a bounded interval with periodic boundary conditions:

$$-u''(x) + Bu(x) = f(x) \text{ in } (a,b) \tag{1}$$

$$u(a) = u(b)\ ,\ u'(a) = u'(b), \tag{2}$$

where $-\infty < a < b < \infty$, $B \geq 0$ is a given constant and f is a given function in $L^2(a,b)$. We look for u in the space

$$H^1_p(a,b) = \{v \in H^1(a,b)\,;\, v(a) = v(b)\},$$

equipped with the norm $\|\cdot\|_{H^1(a,b)}$.

1.a. Prove that $H^1_p(a,b)$ is a closed subspace of $H^1(a,b)$, i.e. prove that if $(v_n)_{n\geq 1}$ is a sequence of functions of $H^1_p(a,b)$ such that

$$\lim_{n\to\infty} \|v_n - v\|_{H^1(a,b)} = 0,$$

for some $v \in H^1(a,b)$, then v belongs to $H^1_p(a,b)$.

b. Do the constant functions belong to $H^1_p(a,b)$?

c. Is $\mathcal{D}(a,b)$ contained in $H^1(a,b)$?

Hint--Recall a useful result: The functions of $H^1(a,b)$ are continuous in $[a,b]$.

2. Show that (1), (2) with the condition $u \in H^1(a,b)$, is equivalent to the variational formulation: Find $u \in H^1_p(a,b)$ such that

$$\forall v \in H^1_p(a,b)\,,\ \int_a^b u'(x)v'(x)dx + B\int_a^b u(x)v(x)dx = \int_a^b f(x)v(x)dx. \tag{3}$$