Homework 3

Let f be a given real valued function in $L^2(0,1)$, $\alpha > 0$ given in $I\!R$ and q a given function in $L^\infty(0,1)$ satisfying: There exists $c_0 > 0$ such that

$$a.e.\ x \in (0,1)\ ,\ q(x) \geq c_0. \tag{1}$$

Let V be the space

$$V = \{v \in H^1(0,1)\,;\, v(0) = 0\}. \tag{2}$$

We want to solve: Find $u \in V$ such that

$$\forall v \in V\ ,\ \int_0^1 q(x)u'(x)v'(x)\,dx + \alpha u(1)v(1) = \int_0^1 f(x)v(x)\,dx. \tag{3}$$

1. The purpose of this question is to establish that there exists a constant $\gamma > 0$ such that

$$\forall v \in V\,,\ \int_0^1 q(x)|v'(x)|^2\,dx \geq \gamma \|v\|^2_{H^1(0,1)}. \tag{4}$$

1.a. Prove that there exists a constant $\gamma_0 > 0$ such that

$$\forall w \in L^2(0,1)\,,\ \int_0^1 q(x)|w(x)|^2\,dx \geq \gamma_0 \|w\|^2_{L^2(0,1)}. \tag{5}$$

1.b. Deduce (4) from (5).

2. Prove that problem (3) has one and only one solution u in V.

3. From now on, we assume that $q \in H^1(0,1)$. Find the boundary value problem that is equivalent to problem (3). Is the solution u in $H^2(0,1)$?

4. In order to discretize (3), we choose an integer $N \geq 1$, set the mesh size $h = \frac{1}{N+1}$, and the the subdivision points $x_j = j\,h$, $0 \leq j \leq N+1$. We introduce the finite element space

$$V_h = \{v_h \in \mathcal{C}^0([0,1])\,;\, v_h|_{[x_i,x_{i+1}]} \in I\!P_1, 0 \leq i \leq N\ ,\ v_h(0) = 0\}\,.$$

4. a. Is V_h a subset of V ? What is the dimension of V_h ?

b. Assuming that u is sufficiently smooth, derive an upper bound of the error $\|u_h - u\|_{H^1(0,1)}$.